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T-COLORING OF SIERPINSKI-LIKE GRAPHS

P. Sivagami[1], Indra Rajasingh[2]

[1]Research and Development Centre
Bharathiar University
Coimbatore, INDIA

[1]Department of Mathematics
Jeppiaar Engineering College
Chennai, INDIA

[2]School of Advanced Sciences
VIT University
Chennai, INDIA

Abstract: For a given finite set T of non-negative integers including zero, a proper vertex coloring is called a T-coloring if the distance of the colors of adjacent vertices is not an element of T. The T-span of T-coloring is the difference between the largest and smallest colors and the T-span of G is the minimum span over all T-colorings of G. In this paper, we compute T-span and T-edge span of Sierpinski-like graphs.

AMS Subject Classification: 05C15, 05C38, 05C70, 05C76

Key Words: T-coloring, T-span, T-edge span, sierpinski torus, sierpinski rhombus, sierpinski gasket torus, sierpinski gasket rhombus, extended sierpinski graphs.

1. Introduction

Hale's graph theoretic formulation of the channel assignment problem is as follows [4]: Let V be the set of transmitters $\{x_1, x_2, \ldots, x_n\}$ and G be a graph in which $V(G) = V$ and there is an edge between transmitters x_i and x_j if and only if they interfere. Let T be a subset of the non-negative integers, containing zero, which represents the forbidden set. Such a T will be called a T-set. For a given graph G and a T-set T, a *T-coloring* of G is a function f

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from $V(G)$ to the set of non-negative integers such that $(x, y) \in E(G)$ implies $|f(x) - f(y)| \notin T$. The *order* of a T-coloring f of G denoted by $\chi_T^f(G)$ is the number of distinct values of $f(x)$, $x \in V(G)$. The *span* of a T-coloring f of G, $sp_T^f(G) = \max_{x,y \in V(G)} |f(x) - f(y)|$. The *edge span* of a T-coloring f of G, $esp_T^f(G) = \max_{(x,y) \in E(G)} |f(x) - f(y)|$. The *$T$-chromatic number* $\chi_T(G)$ is defined as $\chi_T(G) = \min \chi_T^f(G)$ where the minimum is taken over all T-colorings f of G. Similarly the *T-span* $sp_T(G)$ is defined as $sp_T(G) = \min sp_T^f(G)$ and *T-edge span* $esp_T(G)$ is defined as $esp_T(G) = \min esp_T^f(G)$, where the minimum is taken over all T-colorings f of G. If $T = \{0\}$, then the T-coloring of G is the same as a proper coloring of G. In this case, $sp_T(G) = \chi(G) - 1$ where $\chi(G)$ is the chromatic number of G. In general the problem of finding $sp_T(G)$ is NP-complete.

The T-coloring problem has been studied for over two decades. It was first introduced by Hale [4] who formulated several frequency assignment problems in graph-theoretic terms. Cozzens and Roberts [1] formulated some criteria for efficient T-colorings and obtained some basic results in that regard. Cozzens and Wang [2], Raychaudhuri [11,12] and Tesman [15,16] extended the work of Cozzens and Roberts [1] by considering various frequency interference constraints. The number of facts concerning its computational complexity has been discovered in [12]. T-span for graphs has been studied extensively in [3,9,10,13]. However, compared to the T-span case, there are relatively few known results concerning the T-edge span of graphs [1,5]. In this paper we compute T-span and T-edge span of T-coloring for Sierpinski-like graphs.

2. *T*-Coloring of Sierpinski-Like Graphs

In the case of radio frequency assignment, the forbidden T-sets can be very complex and difficult to model. We focus on a special family of T-sets called the *k multiple of s sets* which has the form $T = \{0, s, 2s, \ldots, ks\} \cup S$, where $s, k \geq 1$ and $S \subseteq \{s+1, s+2, \ldots, ks-1\}$. The k multiple of s sets has been studied by Raychaudhuri [11,12]. When $s = 1$, the set $T = \{0, 1, 2, \ldots, k\}$ is also called a *k-initial set*. Some practical forbidden sets, such as those that arise in Ultra High Frequency television problem are very similar to k multiple of s sets [12,15]. In this paper, we consider $T = \{0, s, s+1, s+2, \ldots, 2s, \ldots, ks-1, ks\}$.

Lemma 2.1. [1] *If T is k-initial, then for all graphs G, $sp_T(G) = sp_T(K_{\chi(G)}) = (k+1)(\chi(G) - 1)$.*

Lemma 2.2. [1] *If G is weakly γ perfect, then for all sets T, $esp_T(G) =$*

$sp_T(G) = sp_T(K_{\chi(G)})$.

Lemma 2.3. [1] *For all graphs G and sets T, (1) $\chi_T(G) = \chi(G)$, (2) $\chi(G) - 1 \leq esp_T(G) \leq sp_T(G)$, (3) $sp_T(K_{\omega(G)}) \leq esp_T(G) \leq sp_T(K_{\chi(G)})$.*

Lemma 2.4. [12] *If T is a k multiple of s set, then for all graphs G,*

$$sp_T(G) = sp_T(K_{\chi(G)}) = \begin{cases} st + skt - sk - 1, & \text{if } \chi(G) = st; \\ st + skt + m - 1, & \text{if } \chi(G) = st + m,\ 1 \leq m \leq s-1. \end{cases}$$

Lemma 2.5. [6] *If T is a k multiple of s set and $\chi(G) \leq s$, then $sp_T(G) = esp_T(G) = \chi(G) - 1$.*

2.1. *T*-Coloring of Sierpinski Graphs

A Sierpinski triangle is a type of crystal that regularly repeats. It is possible to coax short strands of artificial DNA to spontaneously assemble into a Sierpinski triangle [7].

Definition 2.6. The Sierpinski graphs $S(n, 3)$, $n \geq 1$, are defined in following way [8]: $V(S(n, 3)) = \{1, 2, 3\}^n$, two different vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ being adjacent if and only if there exist an $h \in \{1, ..., n\}$ such that (i) $u_t = v_t$, for $t = 1, ..., h - 1$; (ii) $u_h \neq v_h$; and (iii) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, ..., n$.

Let us divide $S(n, 3)$ into different levels such that level 1 denoted by i=1, level 2 denoted by i=2 and so on. In $S(n, 3)$ there are 2^n levels.

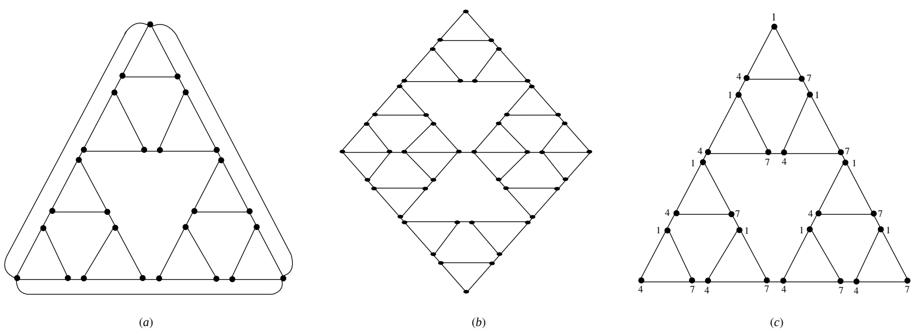

Figure 1(a): $ST(3, 3)$, (b): $SR(3, 3)$, (c): Labeling of $S(3, 3)$ when $T = \{0, 1, 2\}$

Definition 2.7. The Sierpinski Torus $ST(n, 3)$, $n \geq 1$, is obtained from $S(n, 3)$ by adding edges with the extreme vertices $\langle 1...1 \rangle$ and $\langle 2...2 \rangle$, $\langle 2...2 \rangle$ and $\langle 3...3 \rangle$, $\langle 3...3 \rangle$ and $\langle 1...1 \rangle$. See Figure 1(a).

Definition 2.8. The Sierpinski Rhombus $SR(n, 3)$, is obtained by identifying the edges in two Sierpinski graphs $S(n, 3)$ along one of their side. See Figure 1(b).

We give an algorithm for T-coloring of Sierpinski graphs

Algorithm for T-coloring of Sierpinski graphs

Input: Sierpinski graphs $S(n, 3)$ of dimension n.

Label the vertices of $S(n, 3)$ as follows:

Case 1: s=1

When level i is odd, label the vertices as 1. Label the vertices as $k + 2$ and $2k + 3$ alternately when level i is even. See Figure 1(c).

Case 2: $s = 2$

When level i is odd, label the vertices as 1. Label the vertices as 2 and $2k + 3$ alternately when level i is even.

Case 3: $s \geq 3$ When level i is odd, label the vertices as 1. Label the vertices as 2 and 3 alternately when level i is even.

Output: T-coloring of Sierpinski graphs $S(n, 3)$.

Proof of correctness: Since the distance of the colors of adjacent vertices in level i and i+1 is not an element of T, the function yields a T-coloring function.

This algorithm gives the following result.

Theorem 2.9. *If G is the Sierpinski graphs $S(n, 3)$ of dimension n and T, a k multiple of s sets, then*

$$esp_T(G) = sp_T(G) = \begin{cases} 2k + 2, & \text{if } s = 1,\ s = 2; \\ 2, & \text{if } s \geq 3. \end{cases}$$

Similarly, we have an algorithm for Sierpinski torus $ST(n, 3)$ and Sierpinski rhombus $SR(n, 3)$. Hence we obtain the following results.

Corollary 2.10. *If G is the Sierpinski Torus $ST(n, 3)$ of dimension n and T, a k multiple of s sets, then*

$$esp_T(G) = sp_T(G) = \begin{cases} 2k + 2, & \text{if } s = 1,\ s = 2; \\ 2, & \text{if } s \geq 3. \end{cases}$$

Corollary 2.11. *If G is the Sierpinski Rhombus $SR(n, 3)$ of dimension n and T, a k multiple of s sets, then*

$$esp_T(G) = sp_T(G) = \begin{cases} 2k + 2, & \text{if } s = 1,\ s = 2; \\ 2, & \text{if } s \geq 3. \end{cases}$$

2.2. T-Coloring of Sierpinski Gasket

The most important families of Sierpinski graph is Sierpinski gasket. These graphs were introduced already in 1944 by Scorer, Grundy and Smith [14]. Among others, the Sierpinski gasket graphs play an important role in dynamic systems and probability as well as in psychology.

Definition 2.12. Sierpinski gasket graph S_n is obtained from $S(n,3)$ by contracting every edge of $S(n,3)$ that lies in no triangle.

To construct S_{n+1} from S_n we add one downward facing triangle in each of the 3^{n-1} upward facing triangles of S_n. See Figure 2(a).

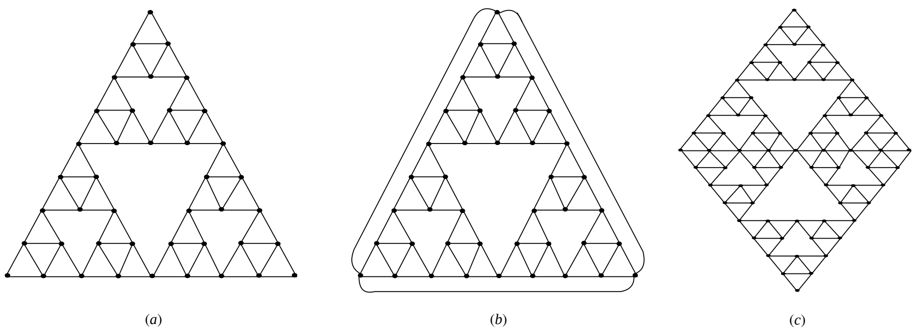

Figure 2(a): Sierpinski gasket S_4, (b): ST_4, (c): SR_4

Definition 2.13. The Sierpinski Gasket Torus ST_n, is obtained from S_n by adding edges with the extreme vertices. See Figure 2(b).

Definition 2.14. The Sierpinski Gasket Rhombus SR_n, is obtained by identifying the edges in two Sierpinski Gasket graphs S_n along one of their side. See Figure 2(c).

We give an algorithm for T-coloring of Sierpinski gasket.

Algorithm for T-coloring of Sierpinski Gasket

Input: Sierpinski gasket S_n of dimension n

Label the vertices of S_n as follows:

Case 1: s=1

First label the vertices of S_1 with 1, $k+2$, $2k+3$. Then label the vertices of S_2 as follows: After inserting one downward facing triangle in each upward facing triangle of S_1, each added vertex is assigned a previously used label different from the labels of the vertices of S_1 adjacent to it. Repeat the process and label S_3, S_4 and so on.

Case 2: $s = 2$

First label the vertices of S_1 with 1, 2, $2k+3$. Then label the vertices of S_2 as follows: After inserting one downward facing triangle in each upward facing triangle of S_1, each added vertex is assigned a previously used label different from the labels of the vertices of S_1 adjacent to it. Repeat the process and label S_3, S_4 and so on.

Case 3: $s \geq 3$

First label the vertices of S_1 with 1, 2, 3. Then label the vertices of S_2 as follows: After inserting one downward facing triangle in each upward facing triangle of S_1, each added vertex is assigned a previously used label different from the labels of the vertices of S_1 adjacent to it. Repeat the process and label S_3, S_4 and so on.

Output: T-coloring of Sierpinski gasket S_n. This algorithm gives the following result.

Theorem 2.15. *If G is the Sierpinski gasket S_n of dimension n and T, a k multiple of s sets, then*

$$esp_T(G)=sp_T(G)=\begin{cases} 2k+2, & \text{if } s=1,\ s=2; \\ 2, & \text{if } s \geq 3. \end{cases}$$

Similarly, we have an algorithm for Sierpinski gasket torus ST_n and Sierpinski gasket rhombus SR_n. Hence we obtain the same results.

2.3. T-Coloring of Extended Sierpinski graphs $S^+(n,k)$

The extended Sierpinski graphs $S^+(n,k)$ and $S^{++}(n,k)$ were introduced in [8]. The graph $S^+(n,k)$, $n \geq 1$, $k \geq 1$, is obtained from $S(n,k)$ by adding a new vertex w, called the *special vertex* of $S^+(n,k)$, and edges joining w with all extreme vertices of $S(n,k)$. These edges will be called the additional edges of $S^+(n,k)$. See Figure 3(a).

The graphs $S^{++}(n,k)$, $n \geq 1$, $k \geq 1$, are defined as follows: For $n=1$ we set $S^{++}(1,k)=K_{k+1}$. Suppose that $n \geq 2$, $S^{++}(n,k)$ is the graph obtained from the disjoint union of $k+1$ copies of $S(n-1,k)$ in which the extreme vertices in distinct copies of $S(n-1,k)$ are connected as the complete graph K_{k+1}. See Figure 3(b).

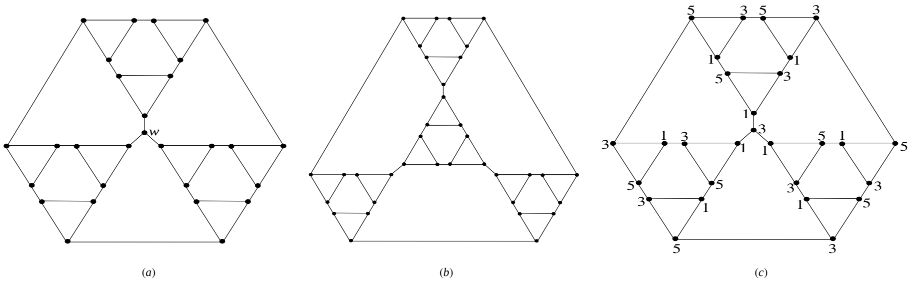

Figure 5(a): $S^+(3,3)$, (b): $S^{++}(3,3)$, (c): Labeling of $S^+(3,3)$

We give an algorithm for T-coloring of extended Sierpinski graphs.

Algorithm for T-coloring of extended Sierpinski graph $S^+(n,3)$

Input: Extended Sierpinski graph $S^+(n,3)$ of dimension n

Case 1: $s = 1$

The special vertex w is labeled as $k+2$. The adjacent vertices of w in $S(n-1,3)$ considered as level i=1 and when level i is odd, label the vertices as 1. When level i is even label the vertices as $k+2$ and $2k+3$ alternately. See Figure 3(c).

Case 2: $s = 2$

The special vertex w is labeled as 2. The adjacent vertices of w in $S(n-1,3)$ considered as level $i = 1$ and when level i is odd, label the vertices as 1. When level i is even label the vertices as 2 and $2k+3$ alternately.

Case 3: $s \geq 3$

The special vertex w is labeled as 2. The adjacent vertices of w in $S(n-1,3)$ considered as level $i = 1$ and when level i is odd, label the vertices as 1. When level i is even label the vertices as 2 and 3 alternately.

Output: T-coloring of extended Sierpinski graphs $S^+(n,3)$.

This algorithm gives the following result.

Theorem 2.16. *If G is the extended Sierpinski graph $S^+(n,3)$ of dimension $n \geq 2$ and T, a k multiple of s sets, then*

$$esp_T(G)=sp_T(G)=\begin{cases} 2k+2, & \text{if } s=1,\ s=2; \\ 2, & \text{if } s\geq 3. \end{cases}$$

Similarly, we have an algorithm for extended Sierpinski graph $S^{++}(n,3)$. Hence we obtain the following results.

Theorem 2.17. *If G is the extended Sierpinski graph $S^{++}(n,3)$ of dimension $n \geq 2$ and T, a k multiple of s sets, then*

$$esp_T(G)=sp_T(G)=\begin{cases} 2k+2, & \text{if } s=1,\ s=2; \\ 2, & \text{if } s\geq 3. \end{cases}$$

3. Conclusion

In this paper we have obtained the T-span and T-edge span for the sierpinski-like graphs. Finding T-span and T-edge span for other interconnection networks such as cube connected cycle, hexagonal network are under investigation and are quite challenging.

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