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ALUTHGE AND ∗-ALUTHGE TRANSFORMATION OF POWERS OF N-CLASS $A(K)$ OPERATORS

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Abstract: In this paper, we introduced and study a new class of operators, we call it powers of N-class $A(k)$ operators. We give a characterization of such an operator and investigate the Aluthge transformation and ∗- Aluthge transformation of such a class of operators and other related results.

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Key Words: aluthge transformation, ∗-aluthge transformation, N-class $A(k)$ operators

1. Introduction

Let H be an infinite dimensional complex (sepearble) Hilbert space and $B(H)$ denote the algebra of all bounded linear operators on H. An operator T is said to be hyponormal if $T^*T \geq TT^*$. An operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, $p \in (0, \infty)$. An operator T belongs to class $A(k)$ if $|T|^2 \leq \left(T^* |T|^{2k} T\right)^{\frac{1}{k+1}}$. Furuta [4] introduced class $A(k)$ and absolute k-paranormal operators. In this paper, we study the Aluthge transformation of powers of N-class $A(k)$ operators and $*-$Aluthge transformation of powers of N-class $A(k)$ operators. Also, we defined a new class of operators powers of N-class $A(k)$ operators $|T|^{2p} \leq N \left(T^* |T|^{2k} T\right)^{\frac{p}{k+1}}$ for a fixed $N > 0$.

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2. Aluthge Transformation of Powers of N-Class $A(k)$ Operators

Aluthge [1] has defined an transformation $\widetilde{T}$ of T as $\widetilde{T} = |T|^{\frac{1}{2}} \cup |T|^{\frac{1}{2}}$, where $\widetilde{T}$ is the Aluthge transformation, $T = \cup |T|$ is the polar decomposition of T, where $\cup$ is a partial isometric operator.

Definition 1. Let $T = \cup |T|$ be the polar decomposition of an operator T. Then Aluthge transformation of T is defined as follows $\widetilde{T} = |T|^{\frac{1}{2}} \cup |T|^{\frac{1}{2}}$ and its adjoint is $\left(\widetilde{T}\right)^* = |T|^{\frac{1}{2}} \cup^* |T|^{\frac{1}{2}}$.

Theorem 2. *Let $T \in B(H)$. If T is powers of N-class $A(k)$, then* $\|T^p x\| \leq N \left\| \left(T^* \left(T^*T\right)^k T\right) \right\|^{\frac{p}{k+1}} \|x\|$.

Theorem 3. *T is powers of N-class $A(k)$ operator for $N > 0$ if and only if* $|T^*|^{2p} \leq N \left(|T^*| \, |T|^{2k} \, |T^*|\right)^{\frac{p}{k+1}}$.

Proof. If T is powers of N-class $A(k)$ operator then

$$|T|^{2p} \leq N \left(T^* \, |T|^{2k} \, T\right)^{\frac{p}{k+1}},$$

for a fixed $N > 0$,

$$\begin{aligned} N \left(|T^*| \, |T|^{2k} \, |T^*|\right)^{\frac{p}{k+1}} &= N \left[(TT^*)^{\frac{1}{2}} \, |T|^{2k} \, (TT^*)^{\frac{1}{2}}\right]^{\frac{p}{k+1}} \\ &= N \left(T^* \, |T|^{2k} \, T\right)^{\frac{p}{k+1}} \\ &\geq |T|^{2p} = U \, |T|^{2p} \, U^* = |T^*|^{2p} \end{aligned}$$

Therefore $|T^*|^{2p} \leq N \left(|T^*| \, |T|^{2k} \, |T^*|\right)^{\frac{p}{k+1}}$ is proved. □

Theorem 4. *Let $T = \cup |T| \in B(H)$ be the polar decomposition of T. Then T belongs to powers of N-class $A(k)$ operator for $N > 0$ if and only if* $|T^*|^{2p} \leq N \left(|T^*| \, |T|^{2k} \, |T^*|\right)^{\frac{p}{k+1}}$.

Proof. Let $T = \cup |T| \in B(H)$ be the polar decomposition of T. Suppose that T is powers of N-class $A(k)$ operator then,

$$N \left[|T^*| \, |T|^{2k} \, |T^*|\right]^{\frac{p}{k+1}} = N \left[(TT^*)^{\frac{1}{2}} \, |T|^{2k} \, (TT^*)^{\frac{1}{2}}\right]^{\frac{p}{k+1}}$$

$$= N\left[(\cup|T|\cup^*|T^*|)^{\frac{1}{2}}|T|^{2k}(\cup|T|\cup^*|T^*|)^{\frac{1}{2}}\right]^{\frac{p}{k+1}}$$
$$= N\left(T^*|T|^{2k}T\right)^{\frac{p}{k+1}}$$
$$\geq |T|^{2p} = \cup|T|^{2p}\cup^* = |T^*|^{2p}$$

Hence $N\left(|T^*|\,|T|^{2k}\,|T^*|\right)^{\frac{p}{k+1}} \geq |T^*|^{2p}$.
Conversely suppose that

$$N\left(T^*|T|^{2k}T\right)^{\frac{p}{k+1}} = N\left(\cup^*|T^*|\,|T|^{2k}\cup|T|\right)^{\frac{p}{k+1}}$$
$$= N\left(\cup^*|T^*|\,|T|^{2k}\,|T^*|\cup\right)^{\frac{p}{k+1}}$$
$$\geq \cup^*|T^*|^{2p}\cup = |T|^{2p}$$

Hence T is powers of N-class $A(k)$ operators. □

Theorem 5. *If T is powers of N-class $A(k)$ operators then T^* is also powers of N-class $A(k)$ operators.*

Theorem 6. *Let T is powers of N-class $A(k)$ operator and S is a self adjoint operator on H. If $T^* = T$ then $\|T^p x\| \leq N\left\|\left(T^*|T|^{2k}T\right)\right\|^{\frac{p}{k+1}}\|x\|$.*

Theorem 7. *If T is powers of N-class $A(k)$ operator then T is p-hyponormal.*

Proof. By assumptions T is powers of N-class $A(k)$ operator.

$$|T|^{2p} \leq N\left(T^*|T|^{2k}T\right)^{\frac{p}{k+1}}$$
$$T^{*p}T^p \leq N\left(T^*T^{*k}T^kT\right)^{\frac{p}{k+1}}$$
$$T^pT^{*p} \leq N\left((T^*T)^{k+1}\right)^{\frac{p}{k+1}}$$
$$T^pT^{*p} \leq NT^{*p}T^p$$

This shows that T is p-hyponormal. □

Theorem 8. *Let $T = \cup|T|$ be the polar decomposition of p-hyponormal for $0 < p \leq 1$ with $N(T) = N(T^*)$. Then $\widetilde{T} = |T|^q\cup|T|^q$ is $\frac{1}{2}\left(1+\frac{p}{q}\right)$ -hyponormal for any q such that $q \geq p$.*

Theorem 9. *Let $T = \cup |T|$ be the polar decomposition of powers of N-class $A(k)$ operator for $N > 0$ then $\widetilde{T} = |T|^{\frac{1}{2}} \cup |T|^{\frac{1}{2}}$ is $\left(p + \frac{1}{2}\right)$ powers of N-class $A(k)$ operators for $0 < p \leq 1$.*

Proof. Let T be the powers of N-class $A(k)$ operator then

$$\begin{aligned}
|T|^{2p} &\leq N\left(T^* |T|^{2k} T\right)^{\frac{p}{k+1}} \\
|T|^{2p} &\leq (T^*T)^{p\left(p+\frac{1}{2}\right)} \\
&= (\cup^* |T^*| \cup |T|)^{p\left(p+\frac{1}{2}\right)} \\
&= \left(\cup^* |T^*|^{\frac{1}{2}} |T^*|^{\frac{1}{2}} \cup |T|^{\frac{1}{2}} |T|^{\frac{1}{2}}\right)^{p\left(p+\frac{1}{2}\right)} \\
&= \left(\widetilde{T^*}\widetilde{T}\right)^{p\left(p+\frac{1}{2}\right)} \\
&= \left|\widetilde{T}\right|^{2p\left(p+\frac{1}{2}\right)}
\end{aligned}$$

$$\begin{aligned}
\left[N\left(T^* |T|^{2k} T\right)^{\frac{p}{k+1}}\right]^{\left(p+\frac{1}{2}\right)} &= \left[N\left(T^* T^{*k} T^k T\right)^{\frac{p}{k+1}}\right]^{\left(p+\frac{1}{2}\right)} \\
&= \left[N\left(\cup^* |T^*| \cup^{*k} |T^*|^k \cup^k |T|^k \cup |T|\right)^{\frac{p}{k+1}}\right]^{\left(p+\frac{1}{2}\right)} \\
&= \left[N\left(\widetilde{T^*} |T|^{\frac{k}{2}} \cup^{*k} |T|^{\frac{k}{2}} |T|^{\frac{k}{2}} \cup^k |T|^{\frac{k}{2}} \widetilde{T}\right)^{\frac{p}{k+1}}\right]^{\left(p+\frac{1}{2}\right)} \\
&= \left[N\left(\widetilde{T^*} |T|^{2k} \widetilde{T}\right)^{\frac{p}{k+1}}\right]^{\left(p+\frac{1}{2}\right)}
\end{aligned}$$

Hence $\left|\widetilde{T}\right|^{2p\left(p+\frac{1}{2}\right)} \leq \left[N\left(\widetilde{T^*} |T|^{2k} \widetilde{T}\right)^{\frac{p}{k+1}}\right]^{\left(p+\frac{1}{2}\right)}$. □

3. $*-$ Aluthge Transformation of Powers of N-Class $A(k)$ Operators

Yamazaki [6], [7] has defined $*-$ Aluthge transformation and have discussed some properties of $*-$ Aluthge transformation. In this section, we have given the definition of $*-$ Aluthge transformation and we have discussed some of its results.

Definition 10. An operator $T \in B(H)$. Let $T^* = \cup^* |T^*|$ be the polar decomposition of T^*. Then $*-$ Aluthge transformation is defined as $\widetilde{T}^{(*)} = \left(\widetilde{T^*}\right)^* = |T^*|^{\frac{1}{2}} \cup |T^*|^{\frac{1}{2}}$ and its adjoint is $\left(\widetilde{T}^{(*)}\right)^* = |T^*|^{\frac{1}{2}} \cup^* |T^*|^{\frac{1}{2}}$.

Theorem 11. *Let $T \in B(H)$, then $\widetilde{T}^{(*)}$ is p-hyponormal.*

Proof. Let $T \in B(H)$ then

$$\begin{aligned} |T|^{2p} &\leq N\left(T^* |T|^{2k} T\right)^{\frac{p}{k+1}} \\ |T|^{2p} &= \left(\cup^* |T^*| \cup |T|\right)^p \\ &= \left(|T|^{\frac{1}{2}} \cup |T^*|^{\frac{1}{2}} |T^*|^{\frac{1}{2}} \cup^* |T|^{\frac{1}{2}}\right)^p \\ &= \left(\widetilde{T}^{(*)} \left(\widetilde{T}^{(*)}\right)^*\right)^p \end{aligned}$$

$$\begin{aligned} N\left(T^* |T|^{2k} T\right)^{\frac{p}{k+1}} &= N\left(\cup^* |T^*| \cup^{*k} |T^*|^k \cup^k |T|^k \cup |T|\right)^{\frac{p}{k+1}} \\ &= N\Big(|T^*|^{\frac{1}{2}} \cup^* |T^*|^{\frac{1}{2}} |T^*|^{\frac{k}{2}} \cup^{*k} |T^*|^{\frac{k}{2}} |T^*|^{\frac{k}{2}} \\ &\qquad \cup^k |T^*|^{\frac{k}{2}} |T^*|^{\frac{1}{2}} \cup |T^*|^{\frac{1}{2}}\Big)^{\frac{p}{k+1}} \\ &= N\left\{\left(\widetilde{T}^{(*)}\right)^* \left[\left(\widetilde{T}^{(*)}\right)^*\right]^k \left(\widetilde{T}^{(*)}\right)^k \widetilde{T}^{(*)}\right\}^{\frac{p}{k+1}} \\ &= N\left[\left(\widetilde{T}^{(*)}\right)^* \widetilde{T}^{(*)}\right]^p \end{aligned}$$

Hence $N\left(\left(\widetilde{T}^{(*)}\right)^* \widetilde{T}^{(*)}\right)^p \geq \left(\widetilde{T}^{(*)} \left(\widetilde{T}^{(*)}\right)^*\right)^p$. □

Theorem 12. *If $T \in B(H)$ is powers of N-class $A(k)$ operator then $\left(\widetilde{T}^{(*)}\right)^*$ is p-hyponormal.*

Theorem 13. *If $T \in B(H)$ is powers of N-class $A(k)$ operator then $\widetilde{T}$ is p-hyponormal.*

Theorem 14. *If $T \in B(H)$ is powers of N-class $A(k)$ operator then $\widetilde{T^*}$ is p-hyponormal.*

Theorem 15. *Let $T \in B(H)$, then $\widetilde{T}$ is p-hyponormal $\Leftrightarrow$ $\widetilde{T}^{(*)}$ is p-hyponormal.*

Theorem 16. *Let $T \in B(H)$, then $\widetilde{T^*}$ is p-hyponormal $\Leftrightarrow$ $\left(\widetilde{T}^{(*)}\right)^*$ is p-hyponormal.*

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, *Integral equations operator theory*, **13** (1990), 307 - 315.

[2] A. Aluthge and D. Wang, Powers of p-hyponormal operators, *J. Inequal. Appl.*, **3** (1997), 279 - 284.

[3] B.P. Duggal, I.H. Jeon and I.H. Kim, Weyl's theorem in the class of algebraically p-hyponormal operators, *Math. PraceMat.* , **40** (2000), 49 - 56.

[4] T. Furuta, Invitation to linear operator, *Taylor and Francis* , (2001).

[5] Takayuki Furuta, Generalized Aluthge transformation on p-hyponormal operators, *Proceedings of the American Mathematical Society*, **10** (1996), 3071 - 3075.

[6] T. Yamazaki, On operators of class $A(k)$ operators including p-hyponormal and log-hyponormal operators, *Math. Inequal. Appl.*, **3** (2000), 97 - 104.

[7] T. Yamazaki, Parallelisms between Aluthge transformation and powers of operators, *Acta Sci. Math(Szeged),***67** (2001), 801 - 820.

[8] T. Yoshino, The p-hyponormality of the Aluthge transform, *Interdiscip. Inform Sci.,* **3** (1997), 91 - 93.