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OSCILLATIONS OF NONLINEAR IMPULSIVE NEUTRAL FUNCTIONAL HYPERBOLIC EQUATIONS WITH DAMPING

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Abstract: In this paper, we are investigated oscillatory behavior of solutions of nonlinear impulsive neutral functional hyperbolic equations with damping term using Riccati technique and integral averaging method. A sufficient condition for oscillation of the solutions of nonlinear impulsive neutral functional hyperbolic equations with damping term is obtained.

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Key Words: oscillation, hyperbolic equation, Riccati method, impulsive, neutral

1. Introduction

Oscillatory properties of partial differential equations are very important both in theory and application. The developing theory of partial functional differential equations has been applied to many fields, such as biology, chemistry, engineering, theoretical physics, population growth, control theory, and so on (see [20]). The oscillation of partial functional differential equations has been studied by many authors (see [1, 5, 8, 9, 16, 17, 18, 19]).

The first paper on impulsive partial differential equation was published in 1991 (see [6]). For an excellent exposition of this paper and its applications (see [3, 4, 10]).

Recently, the theory of impulsive partial differential equations with delay has been investigated by many authors (see [2, 7, 11, 12, 13, 14, 15, 21, 22, 23]), especially impulsive partial differential equations of neutral type.

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In recent years, there has been much research activity concerning the oscillation theory of nonlinear hyperbolic equations with functional arguments by employing Riccati technique. Riccati techniques were used to obtain various oscillation results. Recently, [22] derived oscillation criteria by using Riccati inequality.

In this paper, we study the oscillation of nonlinear impulsive neutral functional hyperbolic equation

$$\left.\begin{array}{l}\frac{\partial}{\partial t}\left(r(t)\frac{\partial}{\partial t}\left[u(x,t)+g(t)u(x,\tau(t))\right]\right)+p(t)\frac{\partial}{\partial t}\left[u(x,t)+g(t)u(x,\tau(t))\right]\\ -\sum\limits_{i=1}^{n} b_i(t)\Delta u(x,\sigma_i(t))+\sum\limits_{j=1}^{m} q_j(x,t)f_j(u(x,\delta_j(t)))\\ =a(t)\Delta u(x,t), \qquad t\neq t_k, \quad (x,t)\in\Omega\times(0,+\infty)\equiv G\\ u(x,t_k^+)-u(x,t_k^-)= \ \alpha_k u(x,t_k),\\ u_t(x,t_k^+)-u_t(x,t_k^-)= \ \beta_k u_t(x,t_k), \qquad t=t_k, \quad k=1,2,....\end{array}\right\}$$

where Ω is bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in R^N. Equation (1) is supplemented by one of the following boundary conditions, namely

$$u(x,t)= \ 0, \qquad (x,t)\in\partial\Omega\times[0,+\infty) \tag{1}$$

$$\frac{\partial u}{\partial\gamma}+\mu(x,t)u(x,t)= \ 0, \qquad (x,t)\in\partial\Omega\times[0,+\infty) \tag{2}$$

where γ is the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t)\in C(\partial\Omega\times[0,+\infty);[0,+\infty))$. Throughout this paper, we assume that the following conditions hold:

(H1) $r(t)\in C^1\left([0,+\infty);(0,+\infty)\right)$, $g(t)\in C^2\left([0,+\infty);(0,+\infty)\right)$, $0\leq g(t)<1$, $\tau(t)\in C^2\left([0,+\infty);R\right)$, $\lim\limits_{t\to+\infty}\tau(t)=+\infty$ and $p(t)\in C\left([0,+\infty);(0,+\infty)\right)$.

(H2) $a(t),b_i(t)\in PC\left([0,+\infty);[0,+\infty)\right)$, $i\in I_n=\{1,2,...,n\}$ $q_j(x,t)\in C(\bar{G};[0,+\infty))$, $j\in I_m=\{1,2,...,m\}$,$q_j(t)=\min\limits_{x\in\bar{\Omega}}q_j(x,t)$, where PC denotes the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,...$ and left continuous at $t=t_k, u(x,t_k)=u(x,{t_k}^-), u_t(x,t_k)=u_t(x,t_k^-)$, $k=1,2,....$

(H3) $\sigma_i(t)\in C([0,+\infty);R)$, $\lim\limits_{t\to+\infty}\sigma_i(t)=+\infty$, $i\in I_n$. $\delta_j(t)\in C([0,+\infty);R)$, $\lim\limits_{t\to+\infty}\delta_j(t)=+\infty$, $j\in I_m$.

(H4) $f_j(u)\in C\left(R;R\right)$ are convex on $[0,+\infty)$ and $\frac{f_j(u)}{u}\geq c_j$ for $u\neq 0, j\in I_m$. $\alpha_k>-1,\beta_k>-1,\alpha_k<\beta_k, 0<t_1<...<t_k<...,$ $\lim\limits_{t\to+\infty}t_k=+\infty$, $k=1,2,....$

(H5) For some $l\in I_m$, there exist a constant δ such that $\delta_l^t=1$ $\int_{t_0}^{\infty}e^{-\tilde{R}(t)}dt=\infty$, where $\tilde{R}(t)=\int_{t_0}^{t}\left(\frac{r'(s)+p(s)}{r(s)}\right)ds$.

Definition 1. By a solution u of problem (1)-(3) we mean a function $u \in C^2\left(\bar{\Omega} \times [t_{-1}, +\infty)\right) \cap C^1\left(\bar{\Omega} \times [\hat{t}_{-1}, +\infty)\right) \cap C\left(\bar{\Omega} \times [\tilde{t}_{-1}, +\infty)\right)$ which satisfies problem (1)-(3), where $t_{-1}= \min\left\{0, \min_{1\le i\le n}\left\{\inf_{t\ge 0}\sigma_i(t)\right\}\right\}$, $\hat{t}_{-1}= \min\left\{0, \inf_{t\ge 0}\tau(t)\right\}$, $\tilde{t}_{-1}= \min\left\{0, \min_{1\le j\le m}\left\{\inf_{t\ge 0}\delta_j(t)\right\}\right\}$

Definition 2. The solution u of problem (1)-(3) is said to be nonoscillatory in domain Ω if it is either eventually positive (or) eventually negative. Otherwise, it is called oscillatory.

Definition 3. We say that functions (H_1, H_2) belong to a function class $\mathcal{H}$, denoted by $(H_1, H_2) \in \mathcal{H}$, if $(H_1, H_2) \in C(D; [0, +\infty))$ satisfy $H_l(t,t) = 0, \quad H_l(t,s) > 0 \quad (l = 1, 2)$ for $t > s$ where $D = \{(t,s) : 0 < s \le t < +\infty\}$. Moreover, the partial derivatives $\frac{\partial H_1}{\partial t}$ and $\frac{\partial H_2}{\partial s}$ exist on D such that

$$\frac{\partial H_1}{\partial t}(s,t) = h_1(s,t)H_1(s,t) \text{ and } \frac{\partial H_2}{\partial s}(t,s) = -h_2(t,s)H_2(t,s),$$

where $h_1, h_2 \in C_{loc}(D; R)$.

2. Main Results

In this section we establish sufficient conditions for the oscillation of all solutions of the problem for (1). It is known that the first eigenvalue λ_0 of the eigenvalue problem

$$\begin{aligned} \Delta\omega(x) + \lambda\omega(x) &= 0 \quad \text{in } \Omega \\ \omega(x) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in Ω. With each solution $u(x,t)$ of the problem (1) and (2) [(or) (1) and (3)] we associate the functions $V(t)$ and $\tilde{V}(t)$ defined by $V(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx$, $\tilde{V}(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$ and $Z(t) = V(t) + g(t)V(\tau(t))$ where $K_\Phi = \left(\int_\Omega \Phi(x)dx\right)^{-1}$ and $|\Omega| = \int_\Omega dx$.

For the main theorem of this paper, we need following lemmas.

Lemma 4. *If the functional impulsive differential inequality*

$$\left.\begin{aligned} &(r(t)Z'(t))' + p(t)Z'(t) + (1-g(t))\textstyle\sum_{j=1}^m c_j q_j(t)Z(\delta_j(t)) \le 0, \quad t \ne t_k \\ &Z(t_k^+) \le (1+\alpha_k)Z(t_k) \\ &Z'(t_k^+) \le (1+\beta_k)Z'(t_k), \quad k = 1, 2, \cdots \end{aligned}\right\} \qquad (3)$$

has no eventually positive solution, then every solution u of the problem (1) and (2) is oscillatory in G.

Proof. Suppose to the contrary that true is a nonoscillatory solution u of the problem (1) and (2). Without loss of generality we may assume that there exists a $T > 0$, $t_0 > T$ such that $u(x,t) > 0$, $u(x,\sigma_i(t)) > 0$ and $u(x,\delta_j(t)) > 0$ in $\Omega \times [t_0,+\infty)$, $i \in I_n$, $j \in I_m$. For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \cdots$ multiplying both sides of equation (1) by $K_\Phi \Phi(x) > 0$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{array}{l} \frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)K_\Phi\Phi(x)dx + g(t)\int_\Omega u(x,\tau(t))K_\Phi\Phi(x)dx\right)\right] \\ +p(t)\frac{d}{dt}\left(\int_\Omega u(x,t)K_\Phi\Phi(x)dx + g(t)\int_\Omega u(x,\tau(t))K_\Phi\Phi(x)dx\right) \\ -\sum_{i=1}^{n} b_i(t)\int_\Omega \Delta u(x,\sigma_i(t))K_\Phi\Phi(x)dx + \sum_{j=1}^{m}\int_\Omega q_j(x,t)f_j(u(x,\delta_j(t)))K_\Phi\Phi(x)dx \\ = a(t)\int_\Omega \Delta u(x,t)K_\Phi\Phi(x)dx \end{array}\right\} \tag{4}$$

From Green's formula and boundary condition (2), it follows that

$$K_\Phi \int_\Omega \Delta u(x,t)\Phi(x)dx = -\lambda_0 V(t) \leq 0, \tag{5}$$

and for $i \in I_n$,

$$K_\Phi \int_\Omega \Delta u(x,\sigma_i(t))\Phi(x)dx = -\lambda_0 V(\sigma_i(t)) \leq 0. \tag{6}$$

From $(H2), (H4)$ and Jensen's inequality, it follows that

$$K_\Phi \int_\Omega q_j(x,t)f_j(u(x,\delta_j(t)))\Phi(x)dx \geq q_j(t)c_jV(\delta_j(t)), \quad j \in I_m. \tag{7}$$

In view (4)-(7), we obtain

$$\left(r(t)\left(V(t)+g(t)V(\tau(t))\right)'\right)' + p(t)\left(V(t)+g(t)V(\tau(t))\right)' + \sum_{j=1}^{m} q_j(t)c_jV(\delta_j(t)) \leq 0, \quad t \neq t_k. \tag{8}$$

Let $Z(t) = V(t) + g(t)V(\tau(t))$. Then $Z(t) > 0$, $Z(t) \geq V(t)$. By the inequality (9), $Z''(t) \leq 0$, $t \geq t_0$ and it's easy to obtain

$$Z'(t) \geq 0, \quad t \geq t_0. \tag{9}$$

In fact, if the inequality (10) does not hold, there exists $t_1 \geq t_0$ satisfying $Z'(t_1) < 0$. Because $Z'(t)$ is decreasing, then $Z(t) - Z(t_1) = \int_{t_1}^{t} Z'(s)ds \leq \int_{t_1}^{t} Z'(t_1)ds = Z'(t_1)(t - t_1)$, and $\lim_{t\to+\infty} Z(t) = -\infty$, which contradicts, $z(t) > 0$, so (10) holds. From (9), we have $(r(t)Z'(t))' + p(t)Z'(t) + \sum_{j=1}^{m} q_j(t)c_jV(\delta_j(t)) \leq 0$. Because $V(t) = Z(t) - g(t)V(\tau(t))$, $V(\tau(t)) = Z(\tau(t)) - g(\tau(t))V(\tau^2(t))$, $V(t) \geq (1-g(t))Z(t)$ and $j \in I_m$, $V(\delta_j(t)) \geq (1-g(t))Z(\delta_j(t))$. Hence, we obtain

$$(r(t)Z'(t))' + p(t)Z'(t) + (1-g(t))\sum_{j=1}^{m} q_j(t)c_jZ(\delta_j(t)) \leq 0, \quad t \geq t_0, \quad t \neq t_k.$$

For $t \geq t_0$, $t = t_k$, $k = 1, 2, \cdots$ multiplying (1) by $K_\Phi \Phi(x) > 0$ and integrating with respect to x over the domain Ω, we obtain

$$K_\Phi \int_\Omega u(x, t_k^+)\Phi(x)dx - K_\Phi \int_\Omega u(x, t_k^-)\Phi(x)dx = \alpha_k \int_\Omega u(x, t_k) K_\Phi \Phi(x)dx$$
$$V(t_k^+) = (1 + \alpha_k)V(t_k)$$

Because, $V(t_k^+) = Z(t_k^+) - g(t_k^+)V(\tau(t_k^+))$, $V(\tau(t_k^+)) = Z(\tau(t_k^+)) - g(\tau(t_k^+))V(\tau^2(t_k^+))$ $V(t_k^+) \geq Z(t_k) - g(t)Z(\tau(t_k))$, $Z(t_k^+) \leq (1+\alpha_k)Z(t_k)$ similarly, $K_\Phi \int_\Omega u_t(x, t_k^+)\Phi(x)dx - K_\Phi \int_\Omega u_t(x, t_k^-)\Phi(x)dx = \beta_k \int_\Omega u_t(x, t_k) K_\Phi \Phi(x)dx$ $V'(t_k^+) = (1 + \beta_k)V'(t_k)$ hence, $Z'(t_k^+) \leq (1 + \beta_k)Z'(t_k)$. Therefore $Z(t)$ is an eventually positive solution of (4). This contradicts the hypothesis and completes the proof. □

Lemma 5. *If the functional impulsive differential inequality (4) has no eventually positive solution, then every solution u of the problem (1) and (3) is oscillatory in G.*

Proof. Suppose to the contrary that there is a nonoscillatory solution u of the problem (1) and (3). Without loss of generality we may assume that there exists a $T > 0$ $t_0 > T$ such that $u(x, t) > 0$, $u(x, \tau(t)) > 0$, $u(x, \sigma_i(t)) > 0$ and $u(x, \delta_j(t)) > 0$ in $\Omega \times [t_0, +\infty)$, $i \in I_n$, $j \in I_m$. For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \cdots$ dividing both sides of equation (1) by $|\Omega|$ and integrating with respect to x over the domain Ω, we have

$$\left.\begin{aligned} &\tfrac{d}{dt}\left[r(t)\tfrac{d}{dt}\left(\tfrac{1}{|\Omega|}\int_\Omega u(x,t)dx + \tfrac{1}{|\Omega|}g(t)\int_\Omega u(x,\tau(t))dx\right)\right] \\ &+p(t)\tfrac{d}{dt}\left(\tfrac{1}{|\Omega|}\int_\Omega u(x,t)dx + \tfrac{1}{|\Omega|}g(t)\int_\Omega u(x,\tau(t))dx\right) - \sum_{i=1}^{n} b_i(t)\tfrac{1}{|\Omega|}\int_\Omega \Delta u(x,\sigma_i(t))dx \\ &+ \sum_{j=1}^{m}\int_\Omega q_j(x,t)\tfrac{1}{|\Omega|}f_j(u(x,\delta_j(t)))dx = a(t)\tfrac{1}{|\Omega|}\int_\Omega \Delta u(x,t)dx \end{aligned}\right\} \tag{10}$$

From Green's formula and boundary condition (3), it follows that

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial\gamma}dS = -\int_{\partial\Omega} \mu(x,t)u(x,t)dS \leq 0, \quad t \geq t_0 \tag{11}$$

and for $i \in I_n$

$$\int_\Omega \Delta u(x,\sigma_i(t))dx = \int_{\partial\Omega} \frac{\partial u}{\partial\gamma}(x,\sigma_i(t))dS = -\int_{\partial\Omega} \mu(x,\sigma_i(t))u(x,\sigma_i(t))dS \leq 0, \quad t \geq t_0, \tag{12}$$

where dS is the surface element on $\partial\Omega$. Also, from $(H2)$, $(H4)$ and Jensen's inequality, we obtain

$$\int_\Omega q_j(x,t)\frac{1}{|\Omega|}f_j(u(x,\delta_j(t)))dx \geq q_j(t)c_j\tilde{V}(t). \tag{13}$$

In view (10)-(13), (11) yield

$$\left(r(t)\left(\tilde{V}(t)+g(t)\tilde{V}(\tau(t))\right)'\right)'+p(t)\left(\tilde{V}(t)+g(t)\tilde{V}(\tau(t))\right)'$$
$$+\sum_{j=1}^{m}q_j(t)c_j(\tilde{V}(\delta_j(t)))\leq 0,\ t\geq t_0,\ t\neq t_k.$$

Rest of the proof is similar to that of Lemma 1 and hence the details are omitted. □

The following theorems are main results of the paper.

Theorem 6. *Assume that* $(H5)$ *holds. If for each* $T>0$ *there exists a* $(H_1,H_2)\in\mathcal{H}$ *and* $a,b,c\in R$ *such that* $T\leq a<c<b$ *and*

$$\frac{1}{H_1(c,a)}\int_a^c\prod_{t_1\leq t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_1(s,a)\left(B(s)-\frac{1}{4}\lambda_1^2(s,a)\rho(s)r(\delta_l(s))\right)\psi(s)ds$$
$$+\frac{1}{H_2(b,c)}\int_c^b\prod_{t_1\leq t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_2(b,s)\left(B(s)-\frac{1}{4}\lambda_2^2(b,s)\rho(s)r(\delta_l(s))\right)\psi(s)ds>0,\tag{14}$$

then (4) has no eventually positive solution, where $\psi(t)\in C^1((T_0,+\infty),(0,+\infty))$ *and* $\rho(t)\in C'((T_0,+\infty),[0,+\infty))$ *for some* $T_0>0$ *and*

$$\lambda_1(s,t)=\frac{\psi'(s)}{\psi(s)}-A(s)+h_1(s,t),\qquad \lambda_2(t,s)=\frac{\psi'(s)}{\psi(s)}-A(s)-h_2(t,s)$$

where $A(t)=\frac{p(t)}{r(t)}-\frac{\rho'(t)}{\rho(t)}$ *and* $B(t)=c_l(1-g(t))\rho(t)q_l(t)$.

Proof. Suppose that $Z(t)$ is a positive solution of (4) on $[t_0,+\infty)$ for some $t_0>0$. From (4) there exists a $l\in I_m$ such that

$$\left.\begin{array}{l}(r(t)Z'(t))'+p(t)Z'(t)\leq -q_l(t)c_l(1-g(t))Z(\delta_l(t)),\quad t\neq t_k\\ Z(t_k^+)\leq(1+\alpha_k)Z(t_k)\\ Z'(t_k^+)\leq(1+\beta_k)Z'(t_k),\quad t=t_k,\quad k=1,2,...\end{array}\right\}\tag{15}$$

which can be rewritten as $r(t)Z''(t)+(r'(t)+p(t))Z'(t)\leq 0$, $t\geq t_0$, $t\neq t_k$. Since

$$r(t)\left(exp\Big(\int_{t_0}^t\frac{r'(s)+p(s)}{r(s)}ds\Big)Z'(t)\right)'=exp\left(\int_{t_0}^t\frac{r'(s)+p(s)}{r(s)}ds\right)$$
$$(r(t)Z''(t)+(r'(t)+p(t))Z'(t))\leq 0$$

we see that

$$\left(\exp\left(\int_{t_0}^t\frac{r'(s)+p(s)}{r(s)}ds\right)Z'(t)\right)'\leq 0.\tag{16}$$

Define

$$\begin{aligned} W(t) =& \ \rho(t)\frac{r(t)Z'(t)}{Z(\delta_l(t))} \\ W'(t) =& \ \frac{(\rho(t)r(t)Z'(t))' Z(\delta_l(t)) - (Z(\delta_l(t)))' \rho(t)r(t)Z'(t)}{(Z(\delta_l(t)))^2} \\ =& \ -\frac{\rho(t)p(t)Z'(t)}{Z(\delta_l(t))} - q_l(t)c_l(1-g(t))\rho(t) + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{Z'(\delta_l(t))}{Z(\delta_l(t))}W(t). \end{aligned}$$

From $(r(t)Z'(t))' \leq 0$ for $t \geq t_0$ we have $r(\delta(t))Z'(\delta_l(t)) \geq r(t)Z'(t)$ and above equation, we obtain that

$$\begin{aligned} & W'(t) + A(t)W(t) + \frac{W^2(t)}{\rho(t)r(\delta_l(t))} \leq -B(t), \quad t \neq t_k \\ & W(t_k^+) = \ \rho(t_k^+)\frac{r(t_k^+)Z'(t_k^+)}{Z(\delta_l(t_k^+))} \leq \frac{1+\beta_k}{1+\alpha_k}\frac{Z'(t_k)}{Z(\delta_l(t_k))}r(t_k)\rho(t_k) \\ & W(t_k^+) \leq \ \frac{1+\beta_k}{1+\alpha_k}W(t_k), \quad k = 1, 2, \cdots \end{aligned}$$

Define $U(t) = \prod_{t_1 \leq t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} W(t)$. In fact, $W(t)$ is continuous on each interval $(t_k, t_{k+1}]$ and in view of $W(t_k^+) \leq \frac{1+\beta_k}{1+\alpha_k}W(t_k)$, it follows that for $t \geq t_1$

$$U(t_k^+) = \prod_{t_1 \leq t_j < t_{k-1}} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} W(t_k^+) \leq U(t_k)$$

and for all $t \geq t_1$

$$U(t_k^-) = \prod_{t_1 \leq t_j < t_{k-1}} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} W(t_k^-) \leq U(t_k)$$

which implies that $U(t)$ is continuous on $[t_1, +\infty)$

$$\begin{aligned} & U'(t) + \prod_{t_1 \leq t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right) \frac{U^2(t)}{\rho(t)r(\delta_l(t))} + A(t)U(t) + \prod_{t_1 \leq t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} B(t) \\ & = \prod_{t_1 \leq t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} \left(W'(t) + \frac{W^2(t)}{\rho(t)r(\delta_l(t))} + W(t)A(t) + B(t)\right) \leq 0. \end{aligned}$$

That is

$$U'(t) + \prod_{t_1 \leq t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right) \frac{U^2(t)}{\rho(t)r(\delta_l(t))} + A(t)U(t) \leq -\prod_{t_1 \leq t_k < t} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} B(t). \tag{17}$$

Multiplying (18) by $\psi(s)$, we obtain

$$\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}B(s)\psi(s)\le -U'(s)\psi(s)-A(s)U(s)\psi(s)$$

$$-\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)\frac{U^2(s)}{\rho(s)r(\delta_l(s))}\psi(s) \qquad (18)$$

Multiplying (19) by $H_2(t,s)$ and integrating over $[c,t]$ for $t\in[c,b)$, we have

$$\int_c^t\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_2(t,s)B(s)\psi(s)ds\le-\int_c^t H_2(t,s)U'(s)\psi(s)ds$$

$$-\int_c^t H_2(t,s)A(s)U(s)\psi(s)ds-\int_c^t H_2(t,s)\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)\frac{U^2(s)}{\rho(s)r(\delta_l(s))}\psi(s)ds$$

$$=H_2(t,c)U(c)\psi(c)+\frac{1}{4}\int_c^t\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_2(t,s)\psi(s)\lambda_2^2(t,s)\rho(s)r(\delta_l(s))ds$$

$$-\int_c^t H_2(t,s)\left\{\sqrt{\frac{\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)}{\rho(s)r(\delta_l(s))}}U(s)-\frac{1}{2}\lambda_2(t,s)\sqrt{\frac{\rho(s)r(\delta_l(s))}{\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)}}\right\}^2\psi(s)ds$$

$$\int_c^t\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_2(t,s)B(s)\psi(s)ds\le H_2(t,c)U(c)\psi(c)$$

$$+\frac{1}{4}\int_c^t\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_2(t,s)\psi(s)\lambda_2^2(t,s)\rho(s)r(\delta_l(s))ds$$

and so

$$\frac{1}{H_2(t,c)}\int_c^t\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_2(t,s)\left(B(s)-\frac{1}{4}\lambda_2^2(t,s)\rho(s)r(\delta_l(s))\right)\psi(s)ds$$

$$\le\psi(c)U(c).$$

Letting $t\to b^-$ in the above inequality, we obtain

$$\frac{1}{H_2(b,c)}\int_c^b\prod_{t_1\le t_k<s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1}H_2(b,s)\left(B(s)-\frac{1}{4}\lambda_2^2(b,s)\rho(s)r(\delta_l(s))\right)\psi(s)ds$$

$$\le\psi(c)U(c). \quad (19)$$

On the other hand, multiplying (19) by $H_1(s,t)$ and integrating over $[t,c]$ for $t \in (a,c]$, we obtain

$$\int_t^c \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,t)B(s)\psi(s)ds \le -\int_t^c H_1(s,t)U'(s)\psi(s)ds$$

$$-\int_t^c H_1(s,t)A(s)U(s)\psi(s)ds - \int_t^c H_1(s,t) \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right) \frac{U^2(s)}{\rho(s)r(\delta_l(s))}\psi(s)ds$$

$$= -H_1(c,t)U(c)\psi(c) + \frac{1}{4}\int_t^c \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,t)\psi(s)\lambda_1^2(s,t)\rho(s)r(\delta_l(s))ds$$

$$-\int_t^c H_1(s,t)\left\{\sqrt{\frac{\prod_{t_1 \le t_k < s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)}{\rho(s)r(\delta_l(s))}}U(s) - \frac{1}{2}\lambda_1(s,t)\sqrt{\frac{\rho(s)r(\delta_l(s))}{\prod_{t_1 \le t_k < s}\left(\frac{1+\beta_k}{1+\alpha_k}\right)}}\right\}^2 \psi(s)ds$$

$$\int_t^c \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,t)B(s)\psi(s)ds \le -H_1(c,t)U(c)\psi(c)$$

$$+\frac{1}{4}\int_t^c \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,t)\psi(s)\lambda_1^2(s,t)\rho(s)r(\delta_l(s))\psi(s)ds$$

and so

$$\frac{1}{H_1(c,t)}\int_t^c \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,t)\left(B(s) - \frac{1}{4}\lambda_1^2(s,t)\rho(s)r(\delta_l(s))\right)\psi(s)ds$$
$$\le -\psi(c)U(c).$$

Letting $t \to a^+$ in the above inequality, we obtain

$$\frac{1}{H_1(c,a)}\int_a^c \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,a)\left(B(s) - \frac{1}{4}\lambda_1^2(s,a)\rho(s)r(\delta_l(s))\right)\psi(s)ds$$
$$\le -\psi(c)U(c).$$

Adding (20) and (21), we easily obtain the following

$$\frac{1}{H_1(c,a)}\int_a^c \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,a)\left(B(s) - \frac{1}{4}\lambda_1^2(s,a)\rho(s)r(\delta_l(s))\right)\psi(s)ds$$
$$+\frac{1}{H_2(b,c)}\int_c^b \prod_{t_1 \le t_k < s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_2(b,s)\left(B(s) - \frac{1}{4}\lambda_2^2(b,s)\rho(s)r(\delta_l(s))\right)\psi(s)ds \le 0,$$

which contradicts the condition (15). This proof is complete. □

Theorem 7. *For some functions $(H_1, H_2) \in \mathcal{H}$, each $T > 0$, if*

$$\limsup_{t\to+\infty} \int_T^t \prod_{t_1\le t_k<s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_1(s,T) \times \left(B(s) - \frac{1}{4}\lambda_1^2(s,T)\rho(s)r(\delta_l(s))\right)\psi(s)ds > 0 \quad (20)$$

and

$$\limsup_{t\to+\infty} \int_T^t \prod_{t_1\le t_k<s} \left(\frac{1+\beta_k}{1+\alpha_k}\right)^{-1} H_2(T,s) \times \left(B(s) - \frac{1}{4}\lambda_2^2(T,s)\rho(s)r(\delta_l(s))\right)\psi(s)ds > 0, \quad (21)$$

then (4) has non eventually positive solution, where $\psi(t) \in C^1((T_0, +\infty), (0, +\infty))$ for some $T_0 > 0$.

Proof. The proof of the theorem can be found in [24]. The conclusion come from Theorem 6, and the proof is complete. □

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