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ON DYNAMIC COLORING OF FAN GRAPHS

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Abstract: An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring. In this paper, we obtain the r-dynamic chromatic number ($r = \delta$) of middle, total, central and line graph of fan graphs $F_{1,n}$. The metric relationship between r-dynamic chromatic number and b-chromatic number of fan graphs are also discussed.

AMS Subject Classification: 05C69

Key Words: r-dynamic coloring, b-coloring, fan graph, middle graph, total graph, central graph and line graph

1. Introduction

Throughout this paper all graphs are finite and simple. The r-dynamic chromatic number was first introduced by Montgomery [12]. In case $r = 2$, it is called dynamic chromatic number.

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An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that

- If $uv \in E(G)$, then $c(u) \neq c(v)$, and
- For each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$.

The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring.

The r-dynamic graph coloring model can be motivated as follows [8]. Imagine that students in a class are to be placed in groups. To encourage communication, friends must be in different groups, and each student should encounter many groups among his or her friends. Requiring all friends of each student to be in different groups splinters the students into too many groups. Instead, we specify a threshold r; the friends of a student with d friends must represent at least $\min\{r, d\}$ groups. We ask how many groups are needed.

The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number in the papers [1, 2, 3, 9].

There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. Below, we state some of the know results.

For a graph G with $\Delta(G) \geq 3$, it was proved that $\chi_d(G) \leq \Delta(G) + 1$ [9]. An upper bound for the d-dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in [5]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2\log_2 \alpha(G) + 3$.

Ali Taherkhani[13] has given an upper bound for $\chi_2(G)$ in terms of chromatic number, maximum degree and minimum degree

$$\chi_2(G) - \chi(G) \leq \left\lceil e\frac{\Delta}{\delta} log\left(2e\left(\Delta^2+1\right)\right)\right\rceil.$$

In [11], it has been proved that computing $\chi_d(G)$, where G is a 3-regular graph, is an NP-complete problem. Furthermore, in [10] it is shown that it is NP-complete to determine whether there exists a 3-dynamic coloring for a claw free graph with the maximum degree 3.

An interesting question to study is how evolve coloring parameters with the topology of the graphs. Some studies focus on power graphs. For such graph the number of edges grows rapidly.

In this article, the authors consider r to be equal to $\delta(G)$ (the minimum degree of G) and find the $\delta(G)$-dynamic chromatic number for G being the middle, total, central and line graph of a fan graph.

2. Preliminaries

The *middle graph* [4] of G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is the vertex and the other is an edge incident with it and it is denoted by $M(G)$.

The *total graph* [4] of G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G.

The *central graph* [14] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The *line graph* [6] of G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.

A *fan graph* $F_{m,n}$ [15] is defined as the graph join $\overline{K_m} + P_n$, where $\overline{K_m}$ is the empty graph on m vertices and P_n is the path graph on n vertices. In paricular, when $m = 1$ the graph $F_{1,n}$ is called the *fan graph of order n*.

The *b-chromatic number* $\varphi(G)$ [7] of a graph G is the largest positive integer k such that G admits a proper k-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-coloring. This concept of b-chromatic number was introduced in 1999 by Irving and Manlove [7], who proved that determining $\varphi(G)$ is NP-hard in general and polynomial for trees.

3. Main Results

Theorem 1. *Let $n \geq 7$. The δ-dynamic chromatic number of the middle graph of a fan graph of order n is $\chi_\delta(M(F_{1,n})) = n + 1$.*

Proof. Let

$$V(F_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$$

and let

$$V(M(F_{1,n})) = \{v, v_1, v_2, \ldots, v_n\} \cup \{e_1, e_2, \ldots, e_n\} \cup \{u_1, u_2, \ldots, u_{n-1}\},$$

where u_i is the vertex of $M(F_{1,n})$ corresponding to the edge $v_i v_{i+1}$ of $F_{1,n}$ $(1 \leq i \leq n-1)$ and e_i is the vertex of $M(F_{1,n})$ corresponding to the edge vv_i of $F_{1,n}$ $(1 \leq i \leq n)$. By the definition of middle graph, the vertices v and $\{e_i : (1 \leq i \leq n)\}$ form a clique of order $n+1$ in $M(F_{1,n})$. Therefore, $\chi_\delta(M(F_{1,n})) \geq n+1$.

Consider the following $n+1$-coloring of $M(F_{1,n})$:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v. For $1 \leq i \leq n-1$, assign to vertex u_i one of the allowed colors - such color exists, because $5 \leq deg(u_i) \leq 6$. For $1 \leq i \leq n$, if any, assign to vertex v_i one of the allowed colors - such color exists, because $2 \leq deg(v_i) \leq 3$. Also, $\delta(G) = 2$. An easy check shows that $N(u)$ contains an induced clique of order 3, for every $u \in V(M(F_{1,n}))$. Thus, this coloring is a 2-dynamic coloring. Hence, $\chi_\delta(M(F_{1,n})) \leq n+1$. Therefore, $\chi_\delta(M(F_{1,n})) = n+1$, $\forall\, n \geq 7$. □

Theorem 2. *Let $n \geq 9$. The δ-dynamic chromatic number of the total graph of a fan graph of order n is $\chi_\delta(T(F_{1,n})) = n+1$.*

Proof. Let

$$V(F_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$$

and let

$$V(T(F_{1,n})) = \{v, v_1, v_2, \ldots, v_n\} \cup \{e_1, e_2, \ldots, e_n\} \cup \{u_1, u_2, \ldots, u_{n-1}\},$$

where u_i is the vertex of $T(F_{1,n})$ corresponding to the edge v_iv_{i+1} of $F_{1,n}$ $(1 \leq)$ $(i \leq n-1)$ and e_i is the vertex of $T(F_{1,n})$ corresponding to the edge vv_i of $F_{1,n}$ $(1 \leq i \leq n)$. By the definition of total graph, the vertices v and $\{e_i : (1 \leq i \leq n)\}$ form a clique of order $n+1$ in $T(F_{1,n})$. Therefore, $\chi_\delta(T(F_{1,n})) \geq n+1$.

Consider the following $n+1$-coloring of $T(F_{1,n})$:
For $1 \leq i \leq n$, assign color c_i to e_i and assign color c_{n+1} to v. For $2 \leq i \leq n-2$, assign color c_{i+1} to v_i and assign the colors c_n, c_1, c_2 to v_1, v_{n-1}, v_n respectively. For $1 \leq i \leq n-1$, assign to vertex u_i one of the allowed colors-such color exists, because $5 \leq deg(u_i) \leq 6$. An easy check shows that this coloring is a 4-dynamic coloring. Hence, $\chi_\delta(T(F_{1,n})) \leq n+1$.Therefore, $\chi_\delta(T(F_{1,n})) = n+1$, $\forall\, n \geq 9$. □

Theorem 3. *Let $n \geq 4$. The δ-dynamic chromatic number of the central graph of a fan graph of order n is $\chi_\delta(C(F_{1,n})) = n+1$.*

Proof. Let

$$V(F_{1,n}) = \{v, v_1, v_2, \ldots, v_n\} \cup \{e_1, e_2, \ldots, e_n\} \cup \{u_1, u_2, \ldots, u_{n-1}\},$$

where u_i is the vertex corresponding to the edge v_iv_{i+1} of $F_{1,n}$ $(1 \leq i \leq n-1)$ and e_i is the vertex corresponding to the edge vv_i of $F_{1,n}$ $(1 \leq i \leq n)$. Clearly, the graph induced by $\{v_{2i} : \ i = 1, 2, \ldots, n/2\}$ is a complete graph. Thus, a proper coloring assign at least $n/2$ colors to them. The same happens with the subgraph induced by $\{v_{2i-1} : i = 1, 2, \ldots, n/2\}$. Moreover, if we are considering a δ-dynamic coloring when n is odd v_n should have a different color from v_{2i-1}, $i = 1, 2, \ldots, n/2$, because v_{n-1} and v_n are the only neighbors of u_{n-1}, and v_{n-1} is adjacent to v_{2i-1}, $i = 2, \ldots, n/2$. A similar reasoning also shows that in a δ-dynamic coloring, the colors assigned to odd vertices should be different to the colors assigned to even vertices and that all of them should be different from the color assigned to v. Thus, $\chi_\delta(C(F_{1,n})) \geq n + 1$.

Consider the following $n + 1$-coloring of $C(F_{1,n})$:
For $1 \leq i \leq n$, assign the color c_i to v_i and assign the color c_{n+1} to v. For $1 \leq i \leq n-1$, assign to vertex u_i and for $1 \leq i \leq n$ assign to vertex e_i one of the allowed colors - such color exists, because $\deg(u_i) = \deg(e_i) = 2$. An easy check shows that this coloring is a δ-dynamic coloring. Hence, $\chi_\delta(C(F_{1,n})) \leq n + 1$. Therefore, $\chi_\delta(C(F_{1,n})) = n + 1$. □

Theorem 4. *Let $n \geq 6$. The δ-dynamic chromatic number of the line graph of a fan graph of order n is $\chi_\delta(L(F_{1,n})) = n$.*

Proof. Let

$$V(L(F_{1,n})) = \{e_1, e_2, \ldots, e_n\} \cup \{u_1, u_2, \ldots, u_{n-1}\},$$

where u_i is the vertex corresponding to the edge $v_i v_{i+1}$ of $F_{1,n}$ $(1 \leq i \leq n-1)$ and e_i is the vertex corresponding to the edge vv_i of $F_{1,n}$ $(1 \leq i \leq n)$. By the definition of line graph, $\{e_1, e_2, \ldots, e_n\}$ form a clique of order K_n in $L(F_{1,n})$. Thus, $\chi_\delta(L(F_{1,n})) \geq n$.

Consider the following n-coloring of $L(F_{1,n})$:
For $1 \leq i \leq n$, assign the color c_i to e_i. For $1 \leq i \leq n-3$, assign color c_{i+2} to u_i and assign the colors c_1, c_2 to u_{n-2}, u_{n-1} respectively. An easy check shows that this coloring is a 3-dynamic coloring. Hence, $\chi_\delta(L(F_{1,n})) \leq n$. Therefore, $\chi_\delta(L(F_{1,n})) = n$. □

4. Relationship between r-Dynamic Chromatic Number and b-Chromatic Number of Fan Graphs

In [15], the authors show that the b-chromatic number of $M(F_{1,n})$ and of $T(F_{1,n})$ is $n + 1$. Therefore, by Theorems 2.1 and 2.2, we get the following corollary.

Corollary 5. *Let $n \geq 9$. For any fan graph of order n,*

$$\chi_\delta(M(F_{1,n})) = \chi_\delta(T(F_{1,n})) = \chi_\delta(C(F_{1,n})) = \varphi(M(F_{1,n})) = \varphi(T(F_{1,n})) = n+1.$$

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