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THE STABLE B-CHROMATIC GRAPHS

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Abstract: A b-coloring is a proper k-coloring of the vertices of a graph such that each color class has a vertex that is adjacent to a vertex of every other color class. The b-chromatic number of a graph G is the largest integer k such that G admits a b-coloring with k colors. A graph G is said to be *stable b-chromatic* graph if $b(G.uv) = b(G)$ for every $u, v \in V(G)$ with u is adjacent to v. In this paper we obtain some basic properties of b_s-chromatic graphs.

AMS Subject Classification: 05C15
Key Words: b-coloring, b-chromatic number, b-system, b$_s$-chromatic graphs

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand, Lesniak and Zhang [1].

A vertex u is called a *neighbor* of a vertex v in G if uv is an edge of G. The set of all neighbors of v is the *open neighborhood* of v and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v in G. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. We observe that $N[v] \neq N(v)$ but we can have $N[S] = N(S)$ such that S is the set of two vertices from K_3. The *degree* of a vertex v in a graph G is defined as the number of edges incident with

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v and is denoted by $deg(v)$. A vertex of degree zero in G is an *isolated vertex* and a vertex of degree one is a *pendant vertex* or a *leaf.* An edge e in a graph G is called a *pendant edge* if it is incident with a pendant vertex. Any vertex which is adjacent to a pendant vertex is called a *support vertex.* The minimum of $\{deg(v) : v \in V(G)\}$ is denoted by $\delta(G)$ or simply δ and the maximum of $\{deg(v) : v \in V(G)\}$ is denoted by $\Delta(G)$ or simply Δ. A set of vertices S is *independent set* if every two vertices of S is non adjacent.

The *subdivision graph* $S(G)$ of a graph G is the graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. If $G = K_n$ is a complete graph, then the new graph obtained from G by inserting m vertices to every edge of G is denoted by $S_m(G)$.

A graph $G.uv$ is obtained from$G$ by identifying the vertices u and v, add a new vertex (uv) which is adjacent to all the vertices of $G - \{u, v\}$ that were originally adjacent to either u or v.

A graph G is *k-colorable* if there exists an s-coloring of G for some $s \leq k$. The minimum integer k for which graph G is k-colorable is called *chromatic number* and is denoted by $\chi(G)$. In given coloring of G, a set consisting of all those vertices assigned the same color is referred to as a *color class* and it is denoted by $V_1, V_2, \ldots, V_k$.

Irvin and Manlove [2] introduced b-chromatic number of G. A vertex $u \in V_i$ is said to be a *b-vertex* if u has at least one neighbor in each color class except V_i. Otherwise u is said to be *non b-vertex.* A *b-coloring* of a graph G is proper k-coloring such that each color class contains at least one b-vertex. The maximum integer k for which G is b-coloring is called *b-chromatic number* and is denoted $b(G)$. A set S is said to be *b-system* if every vertex $u \in S$ is a b-vertex. Let G be a b -coloring graph. Then there exists a b-system S such that $|S| = k$. Several authors studied b-chromatic number of graph G in [3, 4, 5]. Also easily observed a trivial upper bound of b-chromatic number of graph is $b(G) \leq \Delta + 1$.

A set of vertices of G is a *dominating set* if every vertex of $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called *domination number* of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is referred to as a *minimum dominating set* and is denoted by γ-set. Also a set S of vertices in a graph G is called an *independent dominating set* of G if S is both an independent and a dominating set of G.

In this paper we define a stable b-chromatic graph and we prove some basic properties of b_s-chromatic graphs.

2. Main Results

The b-chromatic number of the $G.uv$ is equal to $b(G)$ or less than $b(G)$ or greater than $b(G)$. So that we define as follows. A graph G is said to be *stable b-chromatic* if $b(G) = b(G.uv)$ and is denoted by b_s-chromatic. A graph G is said to be *lower b-chromatic* if $b(G) < b(G.uv)$ and is denoted by b_l-chromatic. A graph G is said to be *greater b-chromatic* if $b(G) > b(G.uv)$ and is denoted by b_h-chromatic.

In this section we investigate the b_s-chromatic graphs. If G is a b_s-chromatic graph then there exists a b-coloring of G with a b-system S contains every two b-vertices having distinct colors or some of the two b-vertices having same colors or both and is denoted by S_0 or S_1 or $S_2 = S_0 \cup S_1$. Therefore naturally we arise the following question.

Question 1. Characterize the b_s-chromatic graphs with b-system S_0 or S_1 or S_2.

Observation 2. If G is a b_s-chromatic graph then there exists a b-system S contains S_0 or S_1 or S_2.

Observation 3. If G is a b_s-chromatic graph and there exists a b-system S_0 such that $|S_0| = k$ then $deg(u) \geq k-1$ for all $u \in S_0$ and $deg(v) < k-1$ for all $v \in V - S_0$.

Observation 4. If $deg(u) = k-1$ for all $u \in S_0$ then there is a no pendant vertex in $V - S_0$.

Theorem 5. *If G is a b-coloring graph and for every $x, y \in S_0$ such that $N(x) \cap N(y) = \phi$ then G is a b_s-chromatic graph.*

Proof. Let G be b-coloring graph. Then the color classes are $V_0, V_1, \ldots, V_k$ and there exists a b-system S_0 with $|S_0| = k$. Let $x, y \in S_0$ and let $v \in V - S_0$. Then by Observation 3, $deg(v) < k-1$ for all $v \in V - S_0$. If G is a b-coloring graph with $N(x) \cap N(y) = \phi$ for every $x, y \in S_0$ then there is two possibilities of cases as follows.

1. x is adjacent to u for all $x \in S_0$ and $u \in V - S_0$
2. u is adjacent to v for all $u, v \in V - S_0$.

Case 1. x is adjacent to u for all $x \in S_0$ and $u \in V - S_0$.

Let $x \in V_i$ and $u \in V_j$. Then the vertex u has no neighbor in V_j and hence u has neighbor in $V_l, j < l$ or u has neighbor in $V_i, i < j, l$.

Subcase 1. $u \in V_j$ has neighbor in $V_l, j \neq l$.

If $|N(x)| = k-1$, for all $x \in S_0$ then $z \in v_l, l \neq i$ is the neighbor of the vertex u and hence the vertex x has two neighbors in V_k say z, w in the new graph $G.xu$. also x has no neighbor in V_j in $G.xu$ with $|N(x) = k-1$. Since the neighbor of z is not in V_j , it follows that a vertex $z \in V_j$m and hence x is a b-vertex. Suppose z is adjacent to a b-vertex $v \in V_m, m \neq l, j$. Then v has two neighbors $z, t \in V_j$. Thus $t \in V_l, l \neq i, j$ and hence $b(G.xu) = b(G)$.Suppose $|N(x)| > k-1$ with more than two neighbors in V_j. Let $u, v \in V_j$. Then $x$is a b-vertex of $G.xu$ and hence $b(G.xu) = b(G)$.

Subcase 2. $u \in V_j$ has neighbor in V_i.

Let $z \in V_i$ and let $y \in V_j$. If z is adjacent to y then $z \in V_j$, $y \in V_i$ and hence V_j has no b-vertex. Let $w \in V_l$. Then w is a neighbor of y. If $y \in V_l$ then V_l is not an independent. Therefore b-vertex $s \in V_l$ which implies $s \in V_j$. If the vertex w is adjacent to neighbors of s then the $deg(w) < k-1$ and $w \in V_m, l \neq m$ and each color classes are independent. Thus a color class V_l has two b-vertices y and s. Since V_j has no b-vertex, it follows that $s \in V_j$ and V_j is not an independent. Then there exists a vertex $t \in V_j$ is also a neighbor of s and hence s is adjacent to t, where $s, t \in V_j$. If the vertex t is a neighbor of $s \in V_j$ then $t \in V_l, l \neq j$. Thus s is a b-vetex of V_j. Hence $b(G.xu) = b(G)$. If w is not adjacent to neighbor of b-vertex s say g such that $g \in V_p, p \neq l$. Suppose $s \in V_j$. Then V_j is not an independent. Let $s \in V_j$ be a vertex incident an edge $e = st$ where $t, s \in V_j$. Then g is adjacent to t . if $g \notin V_j$ then it is not necessary for g is adjacent to t such that $t \in v_l$ and hence s is a b-vertex of V_j. Thus $b(G.xu) = b(G)$.

Case 2. u is adjacent to v for all $u, v \in V - S_0$.

Let $u \in V_j$ and $v \in V_i$. If the vertex v has neighbor is either a b-vertex or non$b$-vertex in V_j then V_j is not an independent in $G.vu$ Therefore a b-vertex $y \in V_i$ and $u \in V_l, l \neq i, j$ in $G.vu$ then by subcase 2, we get $b(G.xu) = b(G)$.

Hence, G is b_s-*chromatic graph.*

□

Theorem 6. *The graph $G \cong P_n$ or C_n is b_s-chromatic graph iff $n \geq 6$.*

Proof. Suppose $n \geq 6$, then $b(P_n) = b(C_n) = 3$. Since the graphs $P_n.uv$ or $C_n.uv$ is respectively a P_{n-1} or C_{n-1}, it follows that, we get $b(G.uv) = b(G)$.

Suppose $G = P_n$ is a b_s-chromatic graph of order ≤ 5. Then there exists a b-system S_0 for $P_n, n = 2, 5$ and hence $b(G.uv) < b(G)$. Also there exists a b-system S_1 for $P_n, n = 3, 4$. Hence $b(G.uv) = b(G)$.

Suppose $G = C_n$ is a b_s-chromatic graph of order ≤ 5. Then there exists a b-system S_0 for $C_n, n = 3, 5$ and hence $b(G.uv) < b(G)$. Also there exists a b-system S_1 for C_4. Hence, $b(G.uv) > b(G)$. □

Remark 1. From the above Theorem 6 we observed that graph P_n is a b_s-chromatic graph with a b-system S_0 and S_1 for $n \geq 6$ and a b-system S_1 for $n = 3, 4$. Hence, $P_n, n > 2$ is a b_s-chromatic graph.

Observation 7. If $G = K_n$ is the complete graph then G is not a b_s-chromatic graph.

Proof. Let $G = K_n$ be a complete graph with $b(G) = n$. Then there exists a b-system S_0 and hence $V - S_0 = \emptyset$ and $b(G) = n$. Since the graph $K_n.uv \cong K_{n-1}$, it follows that, we get a b-system S_0 for K_{n-1}. Thus $b(K_{n-1}) = n - 1$. Hence K_n is not a b_s-chromatic graph. □

Remark 2. From the Remark 1 and Theorem 6, the graph $S_2(K_2) \cong P_4$ and $S_2(K_3) \cong C_9$ are b_s-chromatic graph.

Definition 8. A graph G is said to be *domination dot stable* if $\gamma(G.uv) = \gamma(G)$ for all $u, v \in V(G)$. u is adjacent to v.

Theorem 9. *If G is a b_s-chromatic graph with a b-system S_0 and $|S_0| = k$ such that $|N(u)| = k - 1$ for all $u \in S_0$ then G is a stable domination.*

Proof. Let G be a b_s-chromatic graph. Then there exists a b-system S_0 with $|S_0| = k$ such that $|N(u)| = k - 1$ for every $u \in S_0$ and hence clearly S_0 is independent and a minimum dominating set of G. Thus S_0 is a independent dominating set of G. Hence G is a stable domination. . □

The converse of the Theorem 9 need not be true.

Observation 10. If G is a domination dot stable graph then G need not be a b_s-chromatic graph. For example, consider the graph $P_6 = (v_1v_2v_3v_4v_5v_6)$ then $\gamma(P_n) = 2$, $n = 5, 6$ and $b(P_6.uv \cong P_5) = 3$. From the Theorem 6, P_5 is not a b_s-chromatic graph.

3. Conclusion and Scope

In this paper we proved some benchmark results of b_s-chromatic graphs with a b-system S_0. This provides scope for further investigation on b_s-chromatic graph with b-system S_1 or S_2.

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