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CIRCULAR AND CUT CIRCULAR INDICES OF CERTAIN GRAPHS

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Abstract: In this paper the Wiener polynomial, Detour polynomial, Circular polynomial and Cut Circular polynomial are studied for certain graphs. The Wiener Index(WI), Detour Index(DI), Circular Index(CI) and Cut Circular Index(CCI) for the graphs are also derived from the respective polynomials. This type of topological indices play an important role in theoretical chemistry.

AMS Subject Classification: 05C12, 05C31

Key Words: Wiener polynomial, Detour polynomial, circular polynomial and cut circular polynomial

1. Introduction

Graph theory has a wide range of applications such as coding theory, x-ray crystallography, radar tracking, remote control, radio-astronomy, communica-

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tion networks, networks flows, chemical graph theory, logistics management, etc., [2, 9, 10, 17]. In Particular, the distance based topological indices in graph theory are very much useful in finding the boiling point of molecular graphs.

Let $G = (V(G), E(G))$ be a simple connected undirected graph, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. Let $u, v \in V(G)$. Let the shortest distance between u and v be denoted by $d(u, v)$ and the longest distance between be denoted by $D(u, v)$. The sum $D(u, v) + d(u, v)$ is known as circular distance and is denoted by $d^0(u, v)$ and the difference $D(u, v) - d(u, v)$ is known as Cut Circular distance and is denoted by $d^{-0}(u, v)$.

The concept of "topological index" was first proposed by Hosoya [11] for characterizing the topological nature of a graph. Such graph in-variants are usually related to the distance function $d(u, v) : V(G) \times V(G) \to R$, where $u, v \in V(G)$. The distance matrix of G is defined as $DM(G) = [d_{ij}], d_{ij} = d(u_i, v_j), i \neq j, d_{ii} = 0$. The Detour Distance Matrix $DDM(G) = [D_{ij}], D_{ij} = D(u_i, v_j), i \neq j, D_{ii} = 0$. The Detour Distance Matrix was introduced in Graph Theory by Fran Harary [9] for describing the connectivity in directed graphs. The Detour Distance Matrix is contrast to the distance matrix that records the length of the longest distance between each pair of vertices [3, 13, 14, 15].

2. Necessary Definitions

In this section the following definitions are used to find out DI,WI, CI and CCI for the graphs under consideration.

Definition 2.1. A Wiener-Detour Matrix (WDM) of the graph G with n vertices and e edges is a square matrix of order $n \times n$ with entries zero along the principal diagonal, above the principal diagonal the entries are $D(u_i, v_j)$ and below the principal diagonal the entries are $d(u_i, v_j)$.

Definition 2.2. The Wiener index is the first order derivative of the Wiener polynomial with respect to x with $x = 1$, i.e. $WI = \frac{d}{dx}[WP(G : x)]_{x=1}$ and the Detour index is the first order derivative of the Detour polynomial with respect to y with $y = 1$, i.e.$DI = \frac{d}{dy}[DP(G : y)]_{y=1}$.

Definition 2.3. A Circular-Cut Circular Matrix (CCCM) of G with n vertices and e edges is a square matrix of order $n \times n$ with entries zero along the principal diagonal, above the principal diagonal the entries are $d^{-0}(u_i, v_j)$

and below the principal diagonal the entries are $d^0(u_i, v_j)$.

Definition 2.4. The Circular index is the first order derivative of the Circular polynomial with respect to z with $z = 1$, i.e. $CI = \frac{d}{dz}[CP(G : z)]_{z=1}$ and the Cut Circular index is the first order derivative of the Detour polynomial with respect to t with $t = 1$, i.e. $CCI = \frac{d}{dt}[CCP(G : t)]_{t=1}$.

Definition 2.5. Circulant graph denoted by $C_n(s)$, where

$$S \subset \{1, 2, \cdots, \lfloor n/2 \rfloor\}, n \geq 3$$

is a graph consisting of the vertex set $V = \{1, 2, \cdots, n\}$, and the edge set $E = \{(i, j) : \text{there is } s \in S \text{ such that } |j - i| \equiv s(modn)\}$.

Definition 2.6. Let $n \geq 1$ be an integer. The Mangoldt function $\wedge(n)$ is defined as follows:

$$\wedge(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.7. Let $n \geq 1$ be an integer. The Mangoldt graph M is defined as the graph whose vertices are the elements of the set $\{1, 2, \cdots, n\}$ and two distinct vertices u, v are adjacent (or(u, v) is an edge) if and only if $\vee(u.v) = 0$ or $u.v$ is not a power of a prime.

3. Wiener Index, Detour Index, Circular Index and Cut Circular Index

Theorem 3.1. *Let $G = C(n, \pm\{1, 2\}), n \geq 6$ be the Circulant graph. Then:*

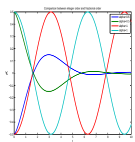
$$\begin{aligned} WP(G : x) =& 2nx + \frac{(n^2 - 5n)}{2}x^2, \\ DP(G : y) =& \frac{n(n-1)}{2}y^{n-1}, \\ CP(G : z) =& 2nz^2 + \frac{(n^2 - 5n)}{2}z^{n+1}, \text{ and} \end{aligned}$$

$$CCP(G:t) = 2nt^{n-2} + \frac{(n^2-5n)}{2}t^{n-3}.$$

Proof. Let $G = C(n, \pm\{1,2\}), n \geq 6$ be the Circulant graph. The Circulant graphs $C(4, \pm\{1,2\})$ and $C(5, \pm\{1,2\})$ are same as the K_4 and K_5 respectively and shown in the figures 1 and 2

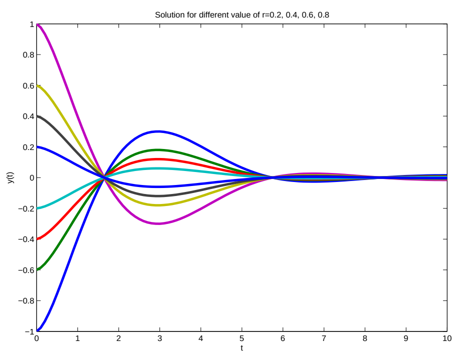
Figure 1: $C(4, \pm\{1,2\})$

Figure 2: $C(5, \pm\{1,2\})$



The general form of WDM and CCCM of $C(n, \pm\{1,2\})$ are shown in the figures 4 and 5 respectively.

Figure 3: $WDMC(n, \pm\{1,2\})$



From the above matrix we can easily derive Weiner polynomial and Detour polynomial as

$$WP(G:x) = \sum_{u,v \in V(G)} x^{d(u,v)} = 2nx + \frac{(n^2 - 5n)}{2} x^2$$

and

$$DP(G:y) = \sum_{u,v \in V(G)} y^{D(u,v)} = \frac{n(n-1)}{2} y^{n-1}.$$

Figure 4: $CCCMC(n, \pm\{1,2\})$

From the above matrix we can easily derive Circular polynomial and Cut Circular polynomial as

$$CP(G:z) = \sum_{u,v \in V(G)} z^{d^0(u,v)} = 2nz^n + \frac{(n^2 - 5n)}{2} z^{n+1}$$

and

$$CCP(G:t) = \sum_{u,v \in V(G)} t^{d^{-0}(u,v)} = 2nt^{n-2} + \frac{(n^2 - 5n)}{2} t^{n-3}. \qquad \square$$

Theorem 3.2. *Let* $G = C(n, \pm\{1,2,3\}), n \geq 8$ *be the Circulant graph. Then:*

$$WP(G:x) = 3nx + \frac{(n^2 - 7n)}{2} x^2,$$

$$DP(G:y) = \frac{n(n-1)}{2} y^{n-1},$$
$$CP(G:z) = 3nz^2 + \frac{(n^2-7n)}{2} z^{n+1}, \text{ and}$$
$$CCP(G:t) = 3nt^{n-2} + \frac{(n^2-7n)}{2} t^{n-3}.$$

Proof. Let $G = C(n, \pm\{1,2,3\}), n \geq 8$ be the Circulant graph. The Circulant graphs $C(6, \pm\{1,2,3\})$ and $C(5, \pm\{1,2,3\})$ are same as the K_6 and K_7 respectively and shown in the figures 5 and 6

Figure 5: $C(6, \pm\{1,2,3\})$

Figure 6: $C(7, \pm\{1,2,3\})$

The general form of WDM and CCCM of $C(n, \pm\{1,2,3\})$ are shown in the figures 7 and 8 respectively.

Figure 7: $WDMC(n, \pm\{1,2,3\})$

From the above matrix we can easily derive Weiner polynomial and Detour polynomial as

$$WP(G:x) = 3nx + \frac{(n^2-7n)}{2} x^2$$

and

$$DP(G:y) = \frac{n(n-1)}{2} y^{n-1}$$

.

From the above matrix we can easily derive Circular polynomial and Cut Circular polynomial as

$$CP(G:z) = 3nz^n + \frac{(n^2-7n)}{2} z^{n+1}$$

Figure 8: $CCCMC(n, \pm\{1,2,3\})$

and

$$CCP(G:t) = 3nt^{n-2} + \frac{(n^2-7n)}{2}t^{n-3}. \qquad \square$$

Theorem 3.3. *Let $G = C(n, \pm\{1,2,3,\cdots,m\}), n \geq 2(m+1)$ be the Circulant graph. Then:*

$$\begin{aligned} WP(G:x) =& mnx + \frac{(n^2-(2m+1)n)}{2}x^2, \\ DP(G:y) =& \frac{n(n-1)}{2}y^{n-1}, \\ CP(G:z) =& mnz^2 + \frac{(n^2-(2m+1)n)}{2}z^{n+1}, \text{ and} \\ CCP(G:t) =& mnt^{n-2} + \frac{(n^2-(2m+1)n)}{2}t^{n-3}. \end{aligned}$$

Also

$$\begin{aligned} WI =& \frac{d}{dx}[WP(G:x)]_{x=1} = n^2-(2m+1)n, \\ DI =& \frac{d}{dy}[DP(G:y)]_{y=1} = \frac{n(n^2-2n+1)}{2}, \\ CI =& \frac{d}{dz}[CP(G:z)]_{z=1} = \frac{(n^2-(2m+1)n)}{2}, \text{ and} \\ CCI =& \frac{d}{dt}[CCP(G:t)]_{t=1} = \frac{n(n^2-4n+(2m+3))}{2}. \end{aligned}$$

Proof. The required polynomials are immediately follows from theorems 3.1 and 3.2 and differentiate the polynomials with respect to x, y, z and t respectively and put $x = y = z = t = 1$, we get the required indices. $\square$

Theorem 3.4. *If $n \geq 2$ is an integer $p_1, p_2, p_3, \cdots, p_i$ are prime numbers $\leq n$ and α_i is the largest positive integer such that $p_i^{\alpha_i} \leq n$, then the Wiener Polynomial of the graph M_n is*

$$WP(M_n:x) = [nC_2 - \sum_{i=1}^{t}(\alpha_i+1)C_2]x + [\sum_{i=1}^{t}(\alpha_i+1)C_2]x^2,$$

$$DP(M_n : y) = \frac{n(n-1)}{2} y^{n-1},$$
$$CP(M_n : z) = [nC_2 - \sum_{i=1}^{t}(\alpha_i + 1)C_2] z^n + [\sum_{i=1}^{t}(\alpha_i + 1)C_2] z^{n+1}, \text{ and}$$
$$CCP(M_n : t) = [nC_2 - \sum_{i=1}^{t}(\alpha_i + 1)C_2] t^{n-2} + [\sum_{i=1}^{t}(\alpha_i + 1)C_2] t^{n-3}.$$

Also

$$WI(M_n) = [nC_2 + \sum_{i=1}^{t}(\alpha_i + 1)C_2],$$
$$DI(Mn) = \frac{n(n^2 - 2n + 1)}{2},$$
$$CI(M_n) = [n(nC_2) + \sum_{i=1}^{t}(\alpha_i + 1)C_2], \text{ and}$$
$$CCI(M_n) = [(n-2)(nC_2) - \sum_{i=1}^{t}(\alpha_i + 1)C_2].$$

Proof. Let $V = \{1, 2, \cdots, n\}, n \geq 2$ and $E = \{(u, v)/\wedge(u, v) = 0, u, v \in V\}$ be the vertex set and edge set of the Mangoldt graph M_n respectively. Let $p_1, p_2, p_3, \cdots, p_i$ be the distinct prime numbers $\leq n$ and α_i is the largest positive integer such that $p_i^{\alpha_i} \leq n$. If p_i is any prime $\leq n$, then there is no edge between any pair of vertices among $1, p_i, p_i^2, \ldots, p_i^{\alpha_i}$ in of M_n. So the $\sum_{i=1}^{t}(\alpha_i + 1)C_2$ number of distinct pairs of vertices is having distance two and the remaining $nC_2 - \sum_{i=1}^{t}(\alpha_i + 1)C_2$ number of distinct pairs of vertices is having distance one. Also the detour distance between any pair of vertices in M_n is $n-1$. Then we get the required polynomials directly.

$$WP(M_n : x) = [nC_2 - \sum_{i=1}^{t}(\alpha_i + 1)C_2] x + [\sum_{i=1}^{t}(\alpha_i + 1)C_2] x^2,$$
$$DP(M_n : y) = \frac{n(n-1)}{2} y^{n-1},$$
$$CP(M_n : z) = [nC_2 - \sum_{i=1}^{t}(\alpha_i + 1)C_2] z^n + [\sum_{i=1}^{t}(\alpha_i + 1)C_2] z^{n+1}, \text{ and}$$
$$CCP(M_n : t) = [nC_2 - \sum_{i=1}^{t}(\alpha_i + 1)C_2] t^{n-2} + [\sum_{i=1}^{t}(\alpha_i + 1)C_2] t^{n-3}.$$

Differentiating the above polynomials with respect to x, y, z and t respectively and put $x = y = z = t = 1$, we get

$$\begin{aligned} WI(M_n) =&[nC_2 + \sum_{i=1}^{t}(\alpha_i + 1)C_2], \\ DI(Mn) =&\frac{n(n^2 - 2n + 1)}{2}, \\ CI(M_n) =&[n(nC_2) + \sum_{i=1}^{t}(\alpha_i + 1)C_2], \text{ and} \\ CCI(M_n) =&[(n-2)(nC_2) - \sum_{i=1}^{t}(\alpha_i + 1)C_2]. \end{aligned}$$

The Mangoldt graph M_{10}, WDM and CCCM of M_{10} are shown in the figures 9, 10 and 11 respectively. □

Figure 9: M_{10}

Figure 10: $WDM(M_{10})$

Figure 11: $CCCM(M_{10})$

Theorem 3.5. *Let G be a $n - 2$ regular graph, where $n \geq 6$ and n even is the number of vertices of G. Then:*

$$\begin{aligned} WP(G:x) =&\frac{(n^2 - 2n)}{2}x + \frac{n}{2}x^2, \\ DP(G:y) =&\frac{n(n-1)}{2}y^{n-1}, \\ CP(G:z) =&\frac{(n^2 - 2n)}{2}z^n + \frac{n}{2}z^{n+1}, \text{ and} \end{aligned}$$

$$CCP(G:t) = \frac{(n^2-2n)}{2}t^{n-2} + \frac{n}{2}t^{n-3}.$$

Also

$$\begin{aligned} WI(G) &= \frac{n^2}{2}, \\ DI(G) &= \frac{n(n^2-2n+1)}{2}, \\ CI(G) &= \frac{n(n^2-n+1)}{2}, \text{ and} \\ CCI(G) &= \frac{n(n^2-3n+1)}{2}. \end{aligned}$$

Proof. Since G is a $n-2$ regular graph, every vertex $u \in G$ is adjacent to all the remaining vertices in G except the vertex $v \in G$ for which $d(u,v) = 2$ and hence $d(u,w) = 1$, where $w(\neq v) \in G$. Also $D(u,v) = n-1, \forall u, v \in G$.

Hence we get

$$\begin{aligned} WP(G:x) &= \frac{(n^2-2n)}{2}x + \frac{n}{2}x^2, \\ DP(G:y) &= \frac{n(n-1)}{2}y^{n-1}, \\ CP(G:z) &= \frac{(n^2-2n)}{2}z^n + \frac{n}{2}z^{n+1} \text{ and} \\ CCP(G:t) &= \frac{(n^2-2n)}{2}t^{n-2} + \frac{n}{2}t^{n-3}. \end{aligned}$$

Differentiating the above polynomials with respect to x, y, z and t respectively and put $x = y = z = t = 1$, we get

$$\begin{aligned} WI(G) &= \frac{n^2}{2}, \\ DI(G) &= \frac{n(n^2-2n+1)}{2}, \\ CI(G) &= \frac{n(n^2-n+1)}{2}, \text{ and} \\ CCI(G) &= \frac{n(n^2-3n+1)}{2}. \end{aligned}$$

□

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