TOTAL VERTEX IRREGULARITY STRENGTH OF CIRCULAR LADDER WINDMILL AND BOW GRAPHS

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Abstract: Let $G(V, E)$ be a simple graph. For a labeling $\partial : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ the weight of a vertex $x$ is defined as $wt(x) = \partial(x) + \sum_{xy \in E} \partial(xy)$. $\partial$ is called a vertex irregular total $k$-labeling if for every pair of distinct vertices $x$ and $y$ $wt(x) \neq wt(y)$. The minimum $k$ for which the graph $G$ has a vertex irregular total $k$-labeling is called the total vertex irregularity strength of $G$ and is denoted by $tvs(G)$. In this paper we determine the total vertex irregularity strength of circular ladder, windmill graph and uniform bow graph.

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1. Introduction

Graph labelings is a mathematical discipline of graph theory closely related
to the field of computer science. It concerns the assignment of values, usually represented by integers, to the edges and/or vertices of a graph. Many of the graph labeling methods were motivated by applications to technology and sports tournament scheduling.

Baca et al. [1] introduced the total vertex irregularity strength of a graph as follows: Let $G(V,E)$ be a simple graph. For a labeling $\partial : V \cup E \rightarrow \{1,2,3,\ldots,k\}$ the weight of a vertex $x$ is defined as $wt(x) = \partial(x) + \sum_{xy \in E} \partial(xy)$. $\partial$ is called a vertex irregular total $k$-labeling if for every pair of distinct vertices $x$ and $y$ $wt(x) \neq wt(y)$. The minimum $k$ for which the graph $G$ has a vertex irregular total $k$-labeling is called the total vertex irregularity strength of $G$ and is denoted by $tvs(G)$. See Figure ?? (a) & (b). They also proved that if $G$ is a $(p,q)$ graph with minimum degree $\delta$ and maximum degree $\Delta$, then \[ \left \lceil \frac{p + \delta}{\Delta + 1} \right \rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1. \]

The following is the key result used for finding the total vertex irregularity strength of circular ladder.

**Theorem 1.** (see [5]) Let $G$ be an $r$-regular graph on $n$ vertices. Then \[ \left \lceil \frac{n+r}{r+1} \right \rceil \leq tvs(G). \]

### 2. Circular Ladder

In this section we determine the total vertex irregularity strength of circular ladder $CL(n)$.

**Definition 1.** (see [4]) A circular ladder $CL(n)$ is the union of an outer cycle $\Gamma_o : u_1u_2\ldots u_n u_1$ and an inner cycle $\Gamma_1 : v_1v_2\ldots v_nv_1$ with additional edges $(u_i, v_i), i = 1, 2, ..., n$ called spokes.
Theorem 2. \( \text{tvs}(CL(n)) = \lceil \frac{2n+3}{4} \rceil, \ n \geq 12. \)

Proof. Let \( V(CL(n)) = \{u_i, v_i, 1 \leq i \leq n\} \) be taken in the anticlockwise order and \( E(CL(n)) = \{e_i, 1 \leq i \leq n\} \cup \{g_i, 1 \leq i \leq n\} \cup \{h_i, 1 \leq i \leq n\} \) where \( e_i = (u_i, u_{i+1}), 1 \leq i \leq n - 1, e_n = (u_n, u_1), g_i = (v_i, v_{i+1}), 1 \leq i \leq n - 1, g_n = (v_n, v_1) \) and \( h_i = (u_i, v_i), 1 \leq i \leq n. \) Let \( k \) denote the number of spokes. Then clearly \( k = \frac{n}{2}. \) The lower bound follows from Theorem 1. To show that \( \lceil \frac{2n+3}{4} \rceil \) is an upper bound for \( \text{tvs}(CL(n)) \) we describe a total \( \lceil \frac{2n+3}{4} \rceil \) labeling for \( CL(n). \) For \( n \geq 12 \) we construct the function \( \varphi \) as follows:

Case 1: Let \( k \equiv 0 \mod 4. \) Then:

\[
\begin{align*}
\varphi(u_i) &= \begin{cases} 
i, & \text{for } 1 \leq i \leq \frac{n}{2}, \\
n + 1 - i, & \text{for } \frac{n}{2} + 1 \leq i \leq n,
\end{cases} \\
\varphi(v_i) &= \begin{cases} 
\begin{cases} 
2 \left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n}{2} - 1, \ i \text{ odd}
\end{cases}, & \text{for } 1 \leq i \leq \frac{n}{2} + 2 \leq i \leq n,
\end{cases} \\
\varphi(e_i) &= \begin{cases} 
\begin{cases} 
i, & \text{for } 1 \leq i \leq \frac{n}{2}, \\
n + 1 - i, & \text{for } \frac{n}{2} + 1 \leq i \leq n,
\end{cases}
\end{cases} \\
\varphi(g_i) &= \begin{cases} 
\begin{cases} 
2 \left\lceil \frac{i}{2} \right\rceil & \text{for } 1 \leq i \leq \frac{n}{2} - 1, \ i \text{ odd}
\end{cases}, & \text{for } 1 \leq i \leq \frac{n}{2} - 1, \ i \text{ even}
\end{cases} \\
\varphi(h_i) &= \begin{cases} 
\begin{cases} 
\left\lceil \frac{2n+3}{4} \right\rceil & \text{for } i = \frac{n}{2}, \\
n + 1 - i & \text{for } \frac{n}{2} + 1 \leq i \leq n,
\end{cases}
\end{cases}
\end{align*}
\]
\[\varphi(h_i) = \begin{cases} 
1 & \text{for } i = 1, \\
3 & \text{for } i = 2, \\
i + 1 & \text{for } 3 \leq i \leq \frac{n}{2}, \\
n + 2 - i & \text{for } \frac{n}{2} + 1 \leq i \leq n, 
\end{cases}\]

See Figure 2.

**Case 2:** Let \( k \equiv 1 \pmod{4} \). Then:

\[\begin{align*}
\varphi(u_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
n + 1 - i & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n, \end{cases} \\
\varphi(v_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
\left\lfloor \frac{2n+3}{4} \right\rfloor & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor, \\
n + 2 - i & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n, \end{cases} \\
\varphi(e_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
n + 1 - i & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n, \end{cases} \\
\varphi(g_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, i \text{ odd} \\
2 \left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, i \text{ even} \\
n + 1 - i & \text{for } \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n, \end{cases} \\
\varphi(h_i) &= \begin{cases} 
i & \text{for } i = 1, \\
3 & \text{for } i = 2, \\
i + 1 & \text{for } 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
n + 2 - i & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n, \end{cases} 
\end{align*}\]

**Case 3:** Let \( k \equiv 2 \pmod{4} \). Then:

\[\begin{align*}
\varphi(u_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \frac{n}{2} + 1, \\
n + 1 - i & \text{for } \frac{n}{2} + 2 \leq i \leq n, \end{cases} \\
\varphi(v_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \frac{n}{2}, \\
\left\lfloor \frac{2n+3}{4} \right\rfloor & \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \\
n + 2 - i & \text{for } \frac{n}{2} + 2 \leq i \leq n, \end{cases} \\
\varphi(e_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \frac{n}{2}, \\
n + 1 - i & \text{for } \frac{n}{2} + 1 \leq i \leq n, \end{cases} \\
\varphi(g_i) &= \begin{cases} 
i & \text{for } 1 \leq i \leq \frac{n}{2}, i \text{ even} \\
2 \left\lfloor \frac{i}{2} \right\rfloor & \text{for } 1 \leq i \leq \frac{n}{2}, i \text{ odd} \\
n + 1 - i & \text{for } \frac{n}{2} + 1 \leq i \leq n, \end{cases} \\
\varphi(h_i) &= \begin{cases} 
i & \text{for } i = 1, \\
3 & \text{for } i = 2, \\
i + 1 & \text{for } 3 \leq i \leq \frac{n}{2}, \\
n + 2 - i & \text{for } \frac{n}{2} + 1 \leq i \leq n, \end{cases} 
\end{align*}\]
Case 4: Let $k \equiv 3(\text{mod } 4)$. Then:

$$
\varphi(u_i) = \begin{cases} 
i, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
n + 1 - i, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n,
\end{cases}
$$
$$
\varphi(v_i) = \begin{cases} 
i, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
n + 2 - i, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n,
\end{cases}
$$
$$
\varphi(e_i) = \begin{cases} 
i, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
n + 1 - i, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n,
\end{cases}
$$
$$
\varphi(g_i) = \begin{cases} 
2 \left\lfloor \frac{i}{2} \right\rfloor, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, i \text{ odd} \\
i, & \text{for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, i \text{ even} \\
n + 1 - i, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n,
\end{cases}
$$
$$
\varphi(h_i) = \begin{cases} 
i, & \text{for } 3 \leq i \leq \left\lceil \frac{n}{3} \right\rceil - 1, \\
\left\lceil \frac{2n+3}{4} \right\rceil, & \text{for } i = \left\lceil \frac{n}{3} \right\rceil, \\
n + 1 - i, & \text{for } \left\lceil \frac{n}{3} \right\rceil + 1 \leq i \leq n.
\end{cases}
$$

So the weights of the vertices of $CL(n)$ under the lableing $\varphi$ constitute the set $\{4, 5, 6, ..., 2n + 3\}$ and the function $\varphi$ is a mapping from $V(CL(n)) \cup E(CL(n))$ into $\{1, 2, 3, ..., \left\lceil \frac{2n+3}{4} \right\rceil\}$. This concludes the proof.

3. Windmill Graph

In this section we determine the total vertex irregularity strength of windmill graph $C_3^{(m)}$.

Definition 2. (see [2]) The windmill graph $C_3^{(m)}$, $m > 1$ is a family of graphs consisting of $m$ copies of $C_3$ with a vertex in common.

Lemma 1. $\left\lceil \frac{2m+2}{3} \right\rceil \leq C_3^{(m)}$, $m > 1$.

Proof. The windmill graph $C_3^{(m)}$ has $2m + 1$ vertices $\{v_1, v_2, v_3, ..., v_{2m}\}$ of degree 2 and 1 vertex $u$ of degree $2m$. The smallest weight of the vertices of degree 2 is 3 and the largest weight of these vertices is $2m + 2$. Since the weight of any vertex of degree 2 is the sum of three positive integers, there exists a vertex of label at least $\left\lceil \frac{2m+2}{3} \right\rceil$. The largest value among the weights of vertices of degree 2 and $2m$ is at least $2m + 3$ and this weight is the sum of at most $2m$ integers. Hence the largest label contributing to this weight
must be at least \( \lceil \frac{2m+3}{2m} \rceil \) and \( \max\{\lceil \frac{2m+2}{3} \rceil, \lceil \frac{2m+3}{2m} \rceil \} \leq \text{tvs}(C^3_m) \). Therefore \( \lceil \frac{2m+2}{3} \rceil \leq \text{tvs}(C^3_m) \).

**Theorem 3.** \( \text{tvs}(C^3_m) = \lceil \frac{2m+2}{3} \rceil \), \( m > 1 \).

**Proof.** By Lemma 1, \( \lceil \frac{2m+2}{3} \rceil \leq \text{tvs}(C^3_m) \). To show that \( \lceil \frac{2m+2}{3} \rceil \) is an upper bound for \( \text{tvs}(C^3_m) \) we describe a total \( \lceil \frac{2m+2}{3} \rceil \) labeling for \( C^3_m \) similar to Theorem 2. See Figure ??.

### 3.1. Bow Graph

In this section we determine the total vertex irregularity strength of uniform bow graph.

**Definition 3.** (see [3]) A shell graph is a cycle \( C_n \) with \( (n-3) \) chords sharing a common end point called the apex. Shell graphs are denoted as \( C(n, n - 3) \). A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence a double shell consists of two edge disjoint shells with a common apex.

**Definition 4.** (see [3]) A bow graph is a a double shell in which each shell has any order. A bow graph in which each shell has the same order \( 'l' \) is a uniform bow graph. A uniform bow graph is denoted as \( B(n) \).

**Theorem 4.** \( \text{tvs}(B(n)) = \lceil \frac{n+1}{4} \rceil \), \( n \geq 5 \).
Proof. The proof is similar to that of Theorem 3.

4. Conclusion

In this paper, we have determined the total vertex irregularity strength of circular ladder, windmill graph and uniform bow graph. Total vertex irregular $k$-labeling for networks like hexagonal network, butterfly network and benes network is under investigation.

References


