Abstract: The $k$-power domination problem is to determine a minimum size vertex set $S \subseteq V(G)$ such that after setting $X = N[S]$ and iteratively adding to $X$ vertices $x$ that have a neighbour $v$ in $X$ such that at most $k$ neighbours of $v$ are not yet in $X$ till we get $X = V(G)$. The least cardinality of such set is called the $k$-power domination number of $G$ and is denoted by $\gamma_{p,k}(G)$. If $k = 0$ and 1 then the problem is called domination and power domination problem respectively. In this paper, we discuss the 2-power domination problem and we compute 2-power domination number to be 2 for certain interconnection networks such as torus, twisted torus, and certain bipartite graphs.

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1. Introduction

Electric power companies need to continually monitor their systems state as defined by a set of state variables, for example the voltage magnitude at loads and machine phase angle at generators. One method of monitoring these variables is to place Phase Measurement Units, called PMUs, at selected locations in the system. The power domination in graphs is the problem of monitoring an electric power system by placing as few measurement units in the system as possible. Let $G(V,E)$ be a graph representing an electric power system, where a vertex represents an electric node and an edge represents a transmission line joining two electrical nodes. The problem is to locate a smallest set of PMUs to
monitor the entire system. A PMU measures the state variable for the vertex at which it is placed and its incident edges and their end vertices. The power domination problem is considered as a variation of the dominating problem. We define a set \( S \) to be a power dominating set if every vertex in \( G \) is observed by \( S \) [1]. The \( k \)-power domination is a generalization of domination and power domination problems. The \( k \)-power domination number of \( G \), denoted by \( \gamma_{p,k}(G) \), is the minimum cardinality of a \( k \)-power dominating set of \( G \). For any graph \( G \), \( 1 \leq \gamma_{p,k}(G) \leq \gamma_p(G) \leq \gamma(G) \) [2].

The \( k \)-power domination number of any connected graph \( G \) of order \( n \), satisfies \( \gamma_{p,k}(G) \leq \frac{n}{k+2} \). Also for any claw-free \((k+2)\)-regular graph of order \( n \), \( \gamma_{p,k}(G) \leq \frac{n}{k+3} \) [2]. Generalized power domination has been well studied for regular graphs [3], Sierpinski graphs [4].

2. Basic Concepts

In this section, we give the basic definitions and preliminaries that are required for the remaining study.

**Definition 1.** For \( v \in V(G) \), the open neighbourhood of \( v \), denoted as \( N_G(v) \), is the set of vertices adjacent to \( v \); and the closed neighbourhood of \( v \), denoted by \( N_G[v] \), is \( N_G(v) \cup \{v\} \). For a set \( S \subseteq V(G) \), the open neighbourhood of \( S \) is defined as \( N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S \) and the closed neighbourhood of \( S \) is defined as \( N_G[S] = N_G(S) \cup S \). For brevity, we denotes \( N_G(v) \) by \( N(v) \), \( N_G[v] \) by \( N[v] \) and \( N_G[S] \) by \( N[S] \).

**Definition 2.** [1] For a graph \( G(V,E) \), \( S \subseteq V \) is a dominating set of \( G \) if every vertex in \( V \setminus S \) has at least one neighbour in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \).

**Definition 3.** [2] Let \( G(V,E) \) be a graph and let \( S \subseteq V(G) \). For \( k \geq 0 \), we define the sets \( M^i(S) \) of vertices monitored by \( S \) at level \( i \), \( i \geq 0 \) is defined inductively as follows:

1. \( M^0(S) = N[S] \)
2. \( M^{i+1}(S) = \bigcup \{N[v] : v \in M^i(S) \text{ such that } |N[v] \setminus M^i(S)| \leq k\} \)

Note that \( M^i(S) \subseteq M^{i+1}(S) \subseteq V(G) \) for any \( i \). Moreover, every time a vertex of the set \( M^i(S) \) has at most \( k \) neighbours outside the set, we add its
neighbours to the next generation $M^{i+1}(S)$. If $M^{i_0}(S) = M^{i_0+1}(S)$ for some $i_0$, then $M^j(S) = M^{i_0}(S)$ for any $j \geq i_0$. We thus define $M^\infty(S) = M^{i_0}(S)$.

**Definition 4.** [2] Let $G = (V, E)$ be a graph, let $S \subseteq V(G)$, and let $k \geq 0$ be an integer. If $M^\infty(S) = V(G)$, then the set $S$ is called a $k$-power dominating set of $G$. The minimum cardinality of a $k$-power dominating set in $G$ is called the $k$-power domination number of $G$ written $\gamma_{p,k}(G)$.

In this paper, we restrict our discussion to $k = 2$. In general, the $k$-power domination problem is NP-complete [2]. In fact, the problem has been shown to be NP-complete even when restricted to bipartite graphs and chordal graphs [1].

**Definition 5.** [9] A bipartite graph is one whose vertex set can be partitioned into two subsets $V_1$ and $V_2$, so that each edge has one end-vertex in $V_1$ and another in $V_2$ such a partition $\{V_1, V_2\}$ is called a bipartition of the graph.

**Definition 6.** [9] A complete bipartite graph is a bipartite graph with bipartition $(V_1, V_2)$ in which each vertex of $V_1$ is joined to each vertex of $V_2$ and each vertex of $V_2$ is joined to each vertex of $V_1$; if $|V_1| = m$ and $|V_2| = n$, such a graph is denoted by $K_{m,n}$.

3. Main Results

In this section, we prove that 2-power domination number is 2 for certain interconnection networks.

3.1. Bipartite Graph

**Theorem 7.** Let $G$ be a bipartite graph with $\delta > 3$. Then $\gamma_{p,2}(G) \neq 1$.

**Proof.** Let $G$ be a bipartite graph with bipartition $(V_1, V_2)$. Suppose $\gamma_{p,2}(G) = 1$. Let $S = \{v\}$ be the 2-power dominating set. Without loss of generality, let $v \in V_1$. Then $N(v) \subseteq V_2$. For every vertex $u \in N(v)$, $\text{deg}(u) > 3$, a contradiction. \qed

**Theorem 8.** Let $G$ be a complete bipartite graph $K_{m,n}$, $m, n \geq 4$. Then
\[ \gamma_{p,2}(G) = 2. \]

**Proof.** By Theorem 3.1, \( \gamma_{p,2}(G) \geq 2 \). Let \( S = \{v_1, v_2\} \), where \( v_1 \in V_1 \) and \( v_2 \in V_2 \). Then \( M^0(S) = N[S] = V(K_{m,n}) \). Hence \( \gamma_{p,2}(K_{m,n}) = 2 \). See Figure 1(a).

![Figure 1(a). 2-power domination of complete bipartite graph \( K_{4,5} \) (b).]

**3.2. Torus**

**Definition 9.** [5] Let \( C_m \) and \( C_n \) be a cycles of length \( m \) and \( n \) respectively, \( m, n \geq 3 \). The torus \( C_m \times C_n \) is a \( P_m \times P_n \) grid with a wraparound edge in each row and column.

**Remark:** The torus \( C_m \times C_n \) has \( mn \) vertices and \( 2mn \) edges, is four-regular and nonplanar. See Figure 1(b).

**Theorem 10.** Let \( G \) be a torus \( C_m \times C_n \) where \( m, n \geq 3 \). Then \( \gamma_{p,2}(G) \geq 2 \).

**Proof.** Suppose \( \gamma_{p,2}(G) = 1 \). Since \( G \) is vertex transitive, let \( S = \{v\} \), where \( v \) is an arbitrarily chosen vertex in \( G \). Now every vertex in \( M^0(S) = N[v] \) has more than two adjacent vertices, a contradiction. Hence \( \gamma_{p,2}(G) \geq 2 \). □
**Theorem 11.** Let $G$ be a torus $C_m \times C_n$ where $m, n \geq 3$. Then $\gamma_{p,2}(G) = 2$.

**Proof.** By Theorem 3.4, $\gamma_{p,2}(G) \geq 2$. Consider the left most top vertex $v_1$ and label it 1 as shown in Figure 1(b). Label the vertex set $N_i(v_1)$, $1 \leq i \leq m + n - 2$, consecutively from 2 to $mn$ beginning with vertex in the highest row to the vertex with the lowest row. Let $S = \{1\}$. Then $M^0(S) = N[S] = \{1, 2, 5, 6, 21\}$. Now every vertex in $M^0(S) = N[S]$ has more than two adjacent vertices. Then let a vertex $v_2$ labelled as 2, where $v_2 \subseteq N(v_1)$. Now $M^0(S) = N[S] = \{1, 2, 3, 7, 22\}$. For every $i \in M^0(S)$, $|N[i] \setminus M^0(S)| \leq 2$. Hence $M^1(S) = \{11, 10, 8, 12, 4, 23, 25, 16, 17\}$. Proceeding inductively, for every $i \in M^j(S)$, $|N[i] \setminus M^j(S)| \leq 2$ and hence $M^{j+1} = N[M^j(S)] = V(C_m \times C_n)$, when $i = m - 1$, $m < n$. Thus $S = \{1, 2\}$ is a 2-power dominating set. □

### 3.3. Twisted torus

**Definition 12.** [6] A $a \times b$, $a, b \geq 3$ is a twisted torus, denoted by $TT(a, b)$ consists of $ab$ nodes arranged with labels $(x, y)$ such that $0 \leq x \leq b - 1$ and $0 \leq y \leq a - 1$. All the inner links form an orthogonal 2D mesh, that is, any node $(x, y)$ such that $0 < x < a - 1$ and $0 < y < b - 1$ is adjacent to the four nodes $(x + 1, y)$, $(x - 1, y)$, $(x, y + 1)$, $(x, y - 1)$. The wraparound links are defined as:

1. $(x, 0)$ is adjacent to $(x + b, b - 1)$ $\forall x : 0 \leq x \leq b - 1$
2. $(x, 0)$ is adjacent to $(x - b, b - 1)$ $\forall x : 0 \leq x \leq b - 1, \ b \leq x \leq a - 1$
3. $(0, y)$ is adjacent to $(a - 1, y)$ $\forall x : 0 \leq x \leq b - 1, \ 0 \leq y \leq b - 1$
4. $(b, y)$ is adjacent to $(a - 1, y - b)$ $\forall x : 0 \leq x \leq b - 1, \ b \leq x \leq b - 1$. See Figure 3.

In other words, twisted torus is only differ from torus in the twist of a columns on the vertical wraparound links [7].

**Theorem 13.** Let $G$ be a twisted torus $TT(a, b)$, $a, b \geq 3$. Then $\gamma_{p,2}(G) \geq 2$.

**Proof.** Similar from Theorem 3.4.
Theorem 14. Let \( G \) be a twisted torus \( TT(a, b) \), \( a, b \geq 3 \). Then \( \gamma_{p,2}(G) = 2 \).

Proof. By Theorem 3.4, \( \gamma_{p,2}(G) \geq 2 \). Consider the left most top vertex \( v_1 \) and label it 1 as shown in Figure 2. Label the vertex set \( N_i(v_1) \), \( 1 \leq i \leq m + n - 2 \), consecutively from 2 to \( mn \) beginning with vertex in the highest row to the vertex with the lowest row. Let \( S = \{1\} \). Then \( M^0(S) = N[S] = \{1, 2, 8, 9, 29\} \). Now every vertex in \( M^0(S) = N[S] \) has more than two adjacent vertices. Then let a vertex \( v_2 \) labelled as 2, where \( v_2 \subseteq N(v_1) \). Now \( M^0(S) = N[S] = \{1, 2, 3, 10, 30\} \). For every \( i \in M^0(S) \), \( |N[i]| \backslash M^0(S) | \leq 2 \). Hence \( M^1(S) = \{16, 17, 18, 11, 4, 31, 7, 28, 21, 22\} \). Proceeding inductively, for every \( i \in M^i(S) \), \( |N[i]| \backslash M^i(S) | \leq 2 \) and hence \( M^{i+1} = N[M^i(S)] = V(TT(a, b)) \), when \( i = a - 1, a < b \). Thus \( S = \{1, 2\} \) is a 2-power dominating set. \( \square \)

4. Conclusion

In this paper, we have obtained the 2-power domination number is 2 for certain classes of graphs.

References


