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SEQUENCE SPACES GENERATED BY SEQUENTIAL MODULUS OF SECOND ORDER

P.D. Srivastava[1], Atanu Manna[2]
[1,2]Department of Mathematics
Indian Institute of Technology Kharagpur
Kharagpur, West Bengal 721302, INDIA

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Abstract: In this article, we have defined the remainder form of the sequential φ-modulus using φ-function by introducing the notion of second order modulus of smoothness for sequences. Structural properties of sequence spaces generated by means of the sequential φ-moduli are investigated.

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1. Introduction

In mathematical analysis, modulus of continuity, modulus of smoothness and its variation are the basic characteristic properties of continuous function. The modulus of continuity and smoothness of functions may be defined on the spaces of continuous function, L^p-spaces etc. The modulus of continuity and smoothness are important tools for approximate a function. It has extensive applications in the theory of approximations and Fourier analysis. Musielak [2]

investigated approximation results by means of translated sequences. Musielak et al.(see [3]) in his consecutive paper obtained approximation results, particularly in Orlicz sequence spaces using sequential modulus defined by the relation $\omega(x, r) = \sup_{m\geq r}\sup_{i\geq m} |t_{i+m} - t_i|$, $r = 0, 1, 2, \ldots$, where $x = \{t_i\}_{i=0}^{\infty}$ and studied a modular space of sequences defined by this modulus (see [2], [3]).
In the present article, we have transferred the second order L^p-modulus defined by

$$\omega^2(f, \delta) = \sup_{0\leq|h|<\delta} \|f(t + 2h) - 2f(t + h) + f(t)\|$$

to the sequential case and studied some structural properties of sequence spaces generated by this modulus.

2. Sequential Modulus of Order Two

Let X denote the space of all real sequences. and let φ be a φ-function (see for definition [1], [8]). For $x \in X$, we write $(\tau_m x)_j = t_j$ for $j < m$ and $= t_{m+j}$ for $j \geq m$, where $m, j = 0, 1, 2, \ldots$. The sequence $\tau_m x = \{(\tau_m x)_j\}_{j=0}^{\infty}$ is called the m-th translation of the sequence $x = \{t_i\}_{i=0}^{\infty}$ and the remainder form of the sequential φ-modulus of the sequence x is defined as

$$\omega_\varphi(x, r) = \sup_{m\geq r} \sum_{i=m}^{\infty} \varphi_i(|t_{i+m} - t_i|),\ r = 0, 1, 2, \ldots \quad \text{(see [1], [2], [4])}.$$

Let I be an identity operator on X. Define sequential modulus of 2nd order of sequence $x = \{t_i\}_{i=0}^{\infty}$ as

$$\omega^2(x, r) = \sup_{m\geq r}\sup_{i\geq 0} |((\tau_m - I)^2 x)_i| = \sup_{m\geq r}\sup_{i\geq m} |t_{i+2m} - 2t_{i+m} + t_i|,\ r = 0, 1, 2, \ldots$$

Define the remainder form of the sequential φ-modulus of 2nd order of x as

$$\omega^2_\varphi(x, r) = \sup_{m\geq r} \sum_{i=1}^{\infty} \varphi_i\Big(|((\tau_m - I)^2 x)_i|\Big) = \sup_{m\geq r} \sum_{i=m}^{\infty} \varphi_i(|\Delta^2_m t_i|), r = 0, 1, 2, \ldots$$

where Δ^2_m is the 2nd order difference operator $\Delta_m : X \longrightarrow X$ defined by $\Delta_m x = (\Delta_m t_i)_{i=0}^{\infty} = (t_{i+m} - t_i)_{i=0}^{\infty}$ for $x = \{t_i\}_{i=0}^{\infty} \in X$ and $\Delta^2_m x = \Delta_m(\Delta_m x) = (t_{i+2m} - 2t_{i+m} + t_i)$.

Let Ψ be a non-negative, non-decreasing function of $u \geq 0$ such that $\Psi(u) \to 0$ as $u \to 0_+$, Ψ is not identically zero and $\{a_r\}$ be the sequence of real numbers

such that $\inf_{r\geq 0} a_r > 0$. (see [4])
We now consider the following classes of sequences

$$X^2_{\varphi_0}(\Psi) = \{x \in X : \ a_r\Psi(\omega^2_\varphi(x,r)) \to 0 \text{ as } r \to \infty\} \text{ and}$$

$$X^2_{\varphi}(\Psi) = \{x \in X : \ a_r\Psi(\omega^2_\varphi(\lambda x,r)) \to 0 \text{ as } r \to \infty \text{ for some } \lambda > 0\}.$$

The following properties of φ-function will be useful to prove our results [1, 4, 5, 6, 7, 9]:
The function Ψ is said to satisfy the condition (Δ_2) for small u (for all u), if there are $u_0 > 0$ and $K > 0$ such that

$$\Psi(2u) \leq K\Psi(u) \text{ for all } 0 < u \leq u_0 \ \ (\text{for all } u \geq 0). \tag{1}$$

The sequence $\varphi = \{\varphi_i\}_{i=0}^\infty$ is said to satisfy the condition **(A)**, if for every $\epsilon > 0$ there exists a $L > 0$ and $\alpha > 0$ such that for all $0 \leq u \leq L$

$$\varphi_i(\alpha u) \leq \epsilon\ \varphi_i(u) \text{ for all } i = 0, 1, 2, \ldots. \tag{2}$$

The sequence $\varphi = \{\varphi_i\}_{i=0}^\infty$ is said to satisfy the condition $(\mathbf{A})'$, if for every $u > 0$ there exists an $\alpha > 0$ such that for all $i = 0, 1, 2, \ldots$.

$$2\varphi_i(\alpha u) \leq \varphi_i(u).$$

The function Ψ satisfy the condition **(B)**, if there exists a $v > 0$ such that for every $\delta > 0$ there is an $\eta > 0$ satisfying the inequality

$$\Psi(\eta u) \leq \delta\Psi(u) \text{ for any } 0 \leq u \leq v.$$

The sequence $\varphi = \{\varphi_i\}_{i=0}^\infty$ of φ-functions is said to satisfy the condition **(C)**, if for every $\eta > 0$ there exists an $\epsilon > 0$ such that for all $u > 0$ and all indices i, the inequality $\varphi_i(u) < \epsilon$ implies $u < \eta$.

Remark 1. *Every s-convex $\varphi-$function* $(0 < s \leq 1)$ *satisfy the condition **(A)** and $(\mathbf{A})'$. There are φ-function satisfying condition **(A)** or $(\mathbf{A})'$, which are not s-convex* $(0 < s \leq 1)$. *For example see [4].*

Remark 2. *Every s-convex φ-function Ψ, $0 < s \leq 1$, satisfies **(B)**. There are φ-function Ψ not satisfying **(B)**. For example see [5].*

3. Main Results

Theorem 3. *Let the functions φ_i satisfy (Δ_2) for all u and the function Ψ satisfies (Δ_2) for small u. Then $x \in X^2_\varphi(\Psi)$ if and only if $a_r\Psi(\omega^2_\varphi(\lambda x, r)) \to 0$ as $r \to \infty$ for all $\lambda > 0$.*

Proof. The proof of this theorem is easy to show. So, we omit the proof. □

Theorem 4. *Let one of the following two conditions hold:*
(P_1) Ψ satisfies condition (Δ_2) for small u,
*(P_2) $\varphi = \{\varphi_i\}$ satisfies **(A)**$'$.*
Then $X^2_\varphi(\Psi)$ is a linear space. Moreover if φ is convex then $X^2_{\varphi 0}(\Psi)$ is a convex set.

Proof. Suppose (P_1) is satisfied and $x_1,\ x_2 \in X^2_\varphi(\Psi)$. Let ξ be a real number. Then there is a number $\lambda > 0$ such that $a_r\Psi(\omega^2_\varphi(\lambda x_i, r)) \to 0$ as $r \to \infty$ for $i = 1, 2$. We have

$$\begin{aligned} a_r\Psi(\omega^2_\varphi(\frac{\lambda}{2}(x_1 + x_2), r)) &\leq a_r\Psi(\omega^2_\varphi(\lambda x_1, r) + \omega^2_\varphi(\lambda x_2, r)) \\ &\leq a_r\Psi(2\omega^2_\varphi(\lambda x_1, r)) + a_r\Psi(2\omega^2_\varphi(\lambda x_2, r)) \quad (3) \end{aligned}$$

By definition of Ψ, there are $\delta; M > 0$ such that $0 < \Psi(u) \leq \delta$ implies $u \leq M$. Since $a = \inf_{r\geq 0} a_r > 0$, so for some $\lambda > 0$, we have $\Psi(\omega^2_\varphi(\lambda x_1, r)) \to 0$ as $r \to \infty$. Consequently, there is a $r_1 > 0$ such that $\omega^2_\varphi(\lambda x_1, r) \leq M$ for $r \geq r_1$. Similarly, there is a $r_2 > 0$ such that $\omega^2_\varphi(\lambda x_2, r) \leq M$ for $r \geq r_2$. Since Ψ satisfies (Δ_2) for small u, so there is a constant $K > 0$ such that $\Psi(2\omega^2_\varphi(\lambda x_i, r)) \leq K\Psi(\omega^2_\varphi(\lambda x_i, r))$ for $r \geq \max\{r_1, r_2\}$ and $i = 1, 2$. Now equation (3) reduces to $a_r\Psi(\omega^2_\varphi(\frac{\lambda}{2}(x_1 + x_2, r))) \leq K\Big[a_r\Psi(\omega^2_\varphi(\lambda x_1, r)) + a_r\Psi(\omega^2_\varphi(\lambda x_2, r))\Big] \to 0$ as $r \to \infty$. Which implies $x_1 + x_2 \in X^2_\varphi(\Psi)$.

Now consider $a_r\Psi(\omega^2_\varphi(\frac{\lambda}{\xi}\xi x_1, r)) = a_r\Psi(\omega^2_\varphi(\lambda x_1, r)) \to 0$ as $r \to \infty$. This implies $\xi x_1 \in X^2_\varphi(\Psi)$. Hence $X^2_\varphi(\Psi)$ is a vector space.

Suppose (P_2) is satisfied. i.e. φ satisfies **(A)**$'$. So, there exist an $\alpha > 0$, we have for $n \geq 1$,

$$\omega^2_\varphi(\alpha\lambda x_1, r) = \sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(\alpha\lambda|\Delta^2_m t_i|) \leq \frac{1}{2}\sup_{m\geq r}\sum_{i=m}^{\infty}\varphi_i(\lambda|\Delta^2_m t_i|) = \frac{1}{2}\omega^2_\varphi(\lambda x_1, r).$$

Similarly, $\omega^2_\varphi(\alpha\lambda x_2, r) \leq \frac{1}{2}\omega^2_\varphi(\lambda x_2, r)$. Now

$$a_r\Psi(\omega^2_\varphi(\frac{\alpha\lambda}{2}(x_1 + x_2), r)) \leq a_r\Psi(\omega^2_\varphi(\alpha\lambda x_1, r) + \omega^2_\varphi(\alpha\lambda x_2, r))$$

$$\begin{aligned} &\le a_r\Psi(\omega_\varphi^2(2\alpha\lambda x_1,r)) + a_r\Psi(\omega_\varphi^2(2\alpha\lambda x_2,r)) \\ &\le a_r\Psi(\omega_\varphi^2(\lambda x_1,r)) + a_r\Psi(\omega_\varphi^2(\lambda x_2,r)) \\ &\quad \to 0 \text{ as } r\to\infty. \end{aligned}$$

So, $x_1 + x_2 \in X_\varphi^2(\Psi)$. Further, it is easy to see that $\xi x_1 \in X_\varphi^2(\Psi)$ for any real number ξ. Therefore $X_\varphi^2(\Psi)$ is a linear space.

The second part of this theorem is easy to proof. So, we omit the details. □

4. Modular Structure on $X_\varphi^2(\Psi)$

For every $x \in X$ we define the functional

$$\varrho(x) = \begin{cases} \sup\limits_{r\ge 0} a_r\Psi\Big(\omega_\varphi^2(x,r)\Big) & \text{if } x \in X_{\varphi_0}^2(\Psi) \\ \infty & \text{if } x \notin X_{\varphi_0}^2(\Psi) \end{cases}$$

Theorem 5. *Let Ψ be a continuous increasing function of $u \ge 0$, $\Psi(0) = 0$ and one of the following two conditions hold:*

(i) Ψ is concave,

(ii) φ_i's are $s-$convex $\varphi-$function for each i.

Then $X_\varphi^2(\Psi)$ is a linear space and ϱ is a pseudomodular in $X_\varphi^2(\Psi)$.

Proof. If Ψ is concave and $\Psi(0) = 0$ then Ψ satisfies the condition (Δ_2) for all $u > 0$ because $\Psi(2u) \le 2\Psi(u)$. Hence by Theorem 4 $X_\varphi^2(\Psi)$ is a linear space. Now, if $x = \{t_i\}_{i=0}^\infty, y = \{s_i\}_{i=0}^\infty \in X, \alpha, \beta \ge 0, \alpha^s + \beta^s = 1$, then

$$\varrho(\alpha x + \beta y) \le \sup_{r\ge 0} a_r\Psi\Big(\sup_{m\ge r}\sum_{i=m}^\infty \varphi_i\big(\alpha|\Delta_m^2 t_i| + \beta|\Delta_m^2 s_i|\big)\Big) \le \varrho(x) + \varrho(y)$$

Therefore ϱ is a pseudomodular.

Now let us suppose the condition (b) i.e. φ_i's are s-convex for each $i = 0,1,2,\ldots$ holds. Then $\varphi = \{\varphi_i\}_{i=0}^\infty$ satisfies $\mathbf{(A)}'$ and Theorem 4 implies $X_\varphi^2(\Psi)$ is a linear space. Now for $\alpha, \beta \ge 0, \alpha^s + \beta^s = 1$,

$$\varrho(\alpha x + \beta y) \;\le\; \sup_{r\ge 0} a_r\Psi\Big(\alpha^s \sup_{m\ge r}\sum_{i=m}^\infty \varphi_i\big(|\Delta_m^2 t_i|\big) + \beta^s \sup_{m\ge r}\sum_{i=m}^\infty \varphi_i\big(|\Delta_m^2 s_i|\big)\Big)$$

$$\begin{aligned} &\le \sup_{r\ge 0} a_r \Psi\Big(\sup_{m\ge r}\sum_{i=m}^{\infty}\varphi_i\big(|\Delta_m^2 t_i|\big)\Big) + \sup_{r\ge 0} a_r \Psi\Big(\sup_{m\ge r}\sum_{i=m}^{\infty}\varphi_i\big(|\Delta_m^2 s_i|\big)\Big) \\ &\le \varrho(x)+\varrho(y). \end{aligned}$$

Therefore $\varrho(\alpha x+\beta y)\le \varrho(x)+\varrho(y)$ for $\alpha,\beta\ge 0, \alpha^s+\beta^s=1$. Other properties of pseudomodular are obvious. Hence ϱ is a pseudomodular. □

It is note that the above definition of pseudomodular generalizes the definition of pseudomodular defined in [6].

Now corresponding to this pseudomodular ϱ we denote

$$X_\varrho = \{x\in X : \varrho(\lambda x)\to 0 \text{ as } \lambda\to 0_+\}.$$

This is called modular space. Now for this pseudomodular ϱ we define a functional

$$|x|_\varrho = \inf\Big\{u>0 : \varrho\Big(\frac{x}{u^{1/s}}\Big)\le u\Big\} \tag{4}$$

for $x\in X_\varrho$. The functional in (4) defines an F-pseudonorm on X_ϱ [see [1]].

5. Completeness

This section deals with the completeness of the spaces X_ϱ and $X_\varphi^2(\Psi)\bigcap X_\varrho$ with respect to both the F-norm $|.|_\varrho$ and modular structure. We begin with the following theorem:

Theorem 6. *Let Ψ be increasing, continuous function of $u\ge 0$, $\Psi(0)=0$ and satisfying the condition **(B)**. Let $\varphi=\{\varphi_i\}_{i=0}^\infty$ satisfy the conditions **(A)** and **(C)**. Moreover, let at least one of the following two conditions hold:*

(1) Ψ is concave,

(2) φ_i's are $s-$convex for each i.

Then X_ϱ is a Fréchet space with respect to the F-norm $|.|_\varrho$.

Proof. Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in X_ϱ, $x_n=\{t_i^n\}_{i=0}^\infty$ and $a=\inf_{r\ge 0} a_r>0$. Without loss of generality, we choose $t_1^n=0, t_2^n=0$ for $n=1,2,3,\ldots$. Then for every $\epsilon>0$, there exists a natural number $N\in\mathbb{N}$ such that $\|x_p-x_q\|_\varrho<\epsilon$ for $p,q>N$ $u_\epsilon>0$ such that $0<u_\epsilon<a\Psi(\epsilon)$, and

$$\varrho\Big(\tfrac{x_p-x_q}{{u_\epsilon}^{1/s}}\Big)\le u_\epsilon \text{ for } p,q>N.$$

The last inequality implies that for $p,q>N$ and $r\ge 0$,

$$a_r \Psi\Big(\omega_\varphi^2\Big(\frac{x_p - x_q}{u_\epsilon^{1/s}}, r\Big)\Big) \le u_\epsilon$$

and consequently we obtain

$$\omega_\varphi^2\Big(\frac{x_p - x_q}{u_\epsilon^{1/s}}, r\Big) \le \Psi_{-1}(\frac{u_\epsilon}{a_r}) \le \Psi_{-1}(\frac{u_\epsilon}{a}) < \epsilon \text{ for } p, q > N \text{and } r \ge 0,$$

where Ψ_{-1} denotes the inverse function to Ψ.

The definition of $\omega_\varphi^2(x, r)$ implies

$$\sum_{i=m}^{d} \varphi_i\Big(\frac{1}{u_\epsilon^{1/s}}|\Delta_m^2(t_i^p - t_i^q)|\Big) \le \Psi_{-1}(\frac{u_\epsilon}{a_r}) < \epsilon$$

$$\text{for } p, q > N, d \ge m \text{ and } i \ge m \ge r \ge 0. \quad (5)$$

Applying the condition **(C)** of φ_i's in (5), we have for every $\eta > 0$

$$u_\epsilon^{-1/s}|\Delta_m^2(t_i^p - t_i^q)| < \eta \;\text{ for } p, q > N \text{ and each } i \ge m \ge r \ge 0. \quad (6)$$

Now (6) implies $|\Delta_m^2(t_i^p - t_i^q)| < \eta u_\epsilon^{1/s} < \eta\{a\Psi(\epsilon)\}^{1/s}$ for $p, q > N$, $i \ge m \ge r \ge 0$ i.e.

$$|(t_{i+2m}^p - t_{i+2m}^q) - 2(t_{i+m}^p - t_{i+m}^q) + (t_i^p - t_i^q)| < \eta\{a\Psi(\epsilon)\}^{1/s}$$

for $p, q > N$, $i \ge m \ge r \ge 0$. Indeed, for $r = 1$ and $m = 1$ we have

$$|(t_{i+2}^p - t_{i+2}^q) - 2(t_{i+1}^p - t_{i+1}^q) + (t_i^p - t_i^q)| < \eta\{a\Psi(\epsilon)\}^{1/s}$$

for $p, q > N$, $i \ge 1$ and so for $i = 1$, $|(t_3^p - t_3^q)| \le 2|(t_2^p - t_2^q)| + |(t_1^p - t_1^q)| + \eta\{a\Psi(\epsilon)\}^{1/s}$, which implies that $\{t_3^p\}_{p=1}^\infty$ is a Cauchy sequence in X_ϱ and hence convergence. In a similar way, we can obtain the convergence of $\{t_i^p\}_{p=1}^\infty$ for $i \ge 4$. Hence $\{t_i^p\}_{p=1}^\infty$ converges for each $i = 1, 2, 3, 4, \dots$. Let $t_0 = 0$, $\lim\limits_{p\to\infty} t_i^p = t_i$ for each $i = 1, 2, 3, 4, \dots$. Taking $q \to \infty$ in (5), we obtain

$$\sum_{i=m}^{d} \varphi_i\Big(\frac{1}{u_\epsilon^{1/s}}|\Delta_m^2(t_i^p - t_i)|\Big) \le \Psi_{-1}(\frac{u_\epsilon}{a_r}) \text{ for } p > N,\ d \ge m \ge r \ge 0,$$

which implies

$$\sum_{i=m}^{\infty} \varphi_i\Big(\frac{1}{u_\epsilon^{1/s}}|\Delta_m^2(t_i^p - t_i)|\Big) \le \Psi_{-1}(\frac{u_\epsilon}{a_r}), \text{ for } p > N,\ m \ge r \ge 0 \text{ and as } d \to \infty. \quad (7)$$

Taking the supremum over $m \ge r$ in both sides of (7), we get

$$\omega_\varphi^2\Big(\frac{x_p - x}{u_\epsilon^{1/s}}, r\Big) \le \Psi_{-1}\Big(\frac{u_\epsilon}{a_r}\Big) \text{ for } p > N \text{ and } r \ge 0.$$

Therefore for $p > N$ and $r \geq 0$, we get

$$a_r \Psi\Big(\omega_\varphi^2\Big(\frac{x_p - x}{u_\epsilon^{1/s}}, r\Big)\Big) \leq u_\epsilon. \tag{8}$$

Now, we prove that $x_p - x \in X_\varrho$ for large p, i.e. $\varrho(\lambda(x_p - x)) \to 0$ as $\lambda \to 0_+$. For this, we choose a natural number N and a fixed $\epsilon > 0$. Then for $p > N$ and $\lambda > 0$, we write

$$\omega_\varphi^2\big(\lambda(x_p - x), r\big) = \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i\Big(\lambda u_\epsilon^{1/s} \frac{|\Delta_m^2(t_i^p - t_i)|}{u_\epsilon^{1/s}}\Big). \tag{9}$$

As $q \to \infty$, (6) becomes

$$\frac{1}{u_\epsilon^{1/s}}|\Delta_m^2(t_i^p - t_i)| \leq \eta \text{ for } i \geq m \geq 0.$$

Now, the condition **(A)** of $\{\varphi_i\}$ with $\eta = L$, $\bar{\epsilon}$ in place of ϵ and $0 < \lambda \leq \frac{\alpha}{u_\epsilon^{1/s}}$. Then for $u = \frac{|\Delta_m^2(t_i^p - t_i)|}{u_\epsilon^{1/s}}$ we get

$$\varphi_i\Big(\lambda u_\epsilon^{1/s} \frac{|\Delta_m^2(t_i^p - t_i)|}{u_\epsilon^{1/s}}\Big) \leq \bar{\epsilon}\varphi_i\Big(\frac{|\Delta_m^2(t_i^p - t_i)|}{u_\epsilon^{1/s}}\Big) \text{ for } i \geq m \geq 0, \quad p \geq N. \tag{10}$$

Then using (10), equation (9) becomes

$$\begin{aligned} \omega_\varphi^2\Big(\lambda(x_p - x), r\Big) &\leq \bar{\epsilon} \sup_{m \geq r} \sum_{i=m}^{\infty} \varphi_i\Big(\frac{|\Delta_m^2(t_i^p - t_i)|}{u_\epsilon^{1/s}}\Big) \\ &= \bar{\epsilon}\omega_\varphi^2\Big(\frac{x_p - x}{u_\epsilon^{1/s}}, r\Big) \leq \bar{\epsilon}\Psi_{-1}(\frac{u_\epsilon}{a_r}) \leq \bar{\epsilon}\epsilon. \end{aligned}$$

Hence for $0 < \lambda \leq \frac{\alpha}{u_\epsilon^{1/s}}$, we have

$$\varrho(\lambda(x_p - x)) = \sup_{r \geq 0} a_r \Psi(\omega_\varphi^2(\lambda(x_p - x), r)) \leq \sup_{r \geq 0} a_r \Psi(\bar{\epsilon}\Psi_{-1}(\frac{u_\epsilon}{a_r})).$$

Now, the condition **(B)** with $v = \Psi_{-1}(\frac{u_\epsilon}{a}))$, $u = \Psi_{-1}(\frac{u_\epsilon}{a_r}))$ and $\bar{\epsilon} = \eta$ implies that for arbitrary $\delta > 0$,

$$\Psi(\bar{\epsilon}\Psi_{-1}(\tfrac{u_\epsilon}{a_r})) \leq \delta\Psi(\Psi_{-1}(\tfrac{u_\epsilon}{a_r})) = \delta\tfrac{u_\epsilon}{a_r}.$$

So, we obtain

$$\varrho(\lambda(x_p - x)) \leq \sup_{r \geq 0} a_r \delta \frac{u_\epsilon}{a_r} = \delta u_\epsilon \text{ for } 0 < \lambda \leq \tfrac{\alpha}{u_\epsilon^{1/s}}$$

Since u_ϵ is fixed, this implies $\varrho\Big(\lambda(x_p - x)\Big) \to 0$ as $\lambda \to 0_+$. Hence $x_p - x \in X_\varrho$ for $p > N$. But linearity of the space X_ϱ implies $x \in X_\varrho$.

Therefore from (8), we have for arbitrary $\epsilon > 0$,

$$\varrho\Big(\frac{x_p - x}{u_\epsilon^{1/s}}\Big) \leq u_\epsilon \text{ for } p > N$$

Thus, $|x_p - x|_\varrho < u_\epsilon < a\Psi(\epsilon)$ for $p > N$, and we get $|x_p - x|_\varrho \to 0$ as $p \to \infty$. Hence X_ϱ is complete. □

Theorem 7. *Suppose that the functions Ψ and φ satisfies the assumptions of Theorem 6 and Theorem 3. Then $X^2_\varphi(\Psi) \bigcap X_\varrho$ is a Fréchet space with respect to the F-norm $|.|_\varrho$.*

Proof. Since $X^2_\varphi(\Psi) \bigcap X_\varrho$ is a subspace of $X\varrho$, so to complete the proof it is sufficient to prove that $X^2_\varphi(\Psi) \bigcap X_\varrho$ is closed in X_ϱ with respect to the Fréchet norm $|.|_\varrho$. Let $x_n \in X^2_\varphi(\Psi)$ for each $n \in \mathbb{N}$ and $x_n \to x$ in X_ϱ. By definition of F-norm convergence, we have for all $\lambda > 0$ $a_r \Psi(\omega^2_\varphi(\lambda(x - x_n), r)) \to 0$ as $n \to \infty$, uniformly with respect to r. By assumption, there exists $M, \delta > 0$ such that $0 < \Psi(u) \leq \delta$ implies $u \leq M$. Taking $\lambda > 0$ fixed, we may find $n_1 > 0$ such that $\Psi(\omega^2_\varphi(\lambda(x - x_n), r)) \leq \delta$ for $n \geq n_1$ and consequently we obtain $\omega^2_\varphi(\lambda(x - x_n), r) \leq M$ for $n \geq n_1$. Let $l > 0$ be a number such that $K \leq 2^l$. Since Ψ satisfies the condition (Δ_2) for small u with a constant $K_1 > 0$, we obtain that

$$\begin{aligned} \Psi(\omega^2_\varphi(\lambda x, r)) &\leq \Psi[2K\omega^2_\varphi(\lambda(x - x_n), r)] + \Psi[2K\omega^2_\varphi(2\lambda x_n, r)] \\ &\leq K_1^{l+1}\Big(\Psi(\omega^2_\varphi(\lambda(x - x_n), r)) + \Psi(\omega^2_\varphi(\lambda x_n, r))\Big) \end{aligned} \qquad (11)$$

for $n \geq n_1$. Now for an arbitrary $\epsilon > 0$ there exists a $n_0 \geq n_1$ such that

$$a_r \Psi(\omega^2_\varphi(\lambda(x - x_{n_o}), r)) < \tfrac{\epsilon}{2K_1^{l+1}}.$$

Also the existence of x_{n_o} in $X^2_\varphi(\Psi)$ implies that $a_r \Psi(\omega^2_\varphi(\lambda x_{n_o}, r)) \to 0$ as $r \to \infty$. Hence there exists an r_0 such that

$$a_r \Psi(\omega^2_\varphi(\lambda x_{n_o}, r)) < \tfrac{\epsilon}{2K_1^{l+1}} \text{ for } r \geq r_0.$$

Therefore for some $\lambda > 0$, from (11) we get

$$a_r \Psi(\omega^2_\varphi(\lambda x, r)) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \quad \text{for } r \geq r_0.$$

This shows that $x \in X^2_\varphi(\Psi)$. Also Theorem 6 implies that $x \in X_\varrho$ and it finishes the proof. □

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References

[1] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture notes in Math., Volume 1034, Springer-Verlag, Berlin (1983).

[2] J. Musielak, Modular approximation by a filtered family of linear operators, functional analysis and approximation, In: *Proceed. Conf. Oberwolfach, August 9-16, 1980*, Birkhauser Verlag (1981), 99-110.

[3] J. Musielak, A. Waszak, Generalized variation and translation operator in some sequence spaces, *Hokkaido Math. Journal*, **17** (1988), 345-353.

[4] J. Musielak, A. Waszak, Sequence spaces generated by moduli of smoothness, *Revista Mathematica de la Universidad Complutense de Madrid*, **8**, No. 1 (1995), 91-105.

[5] J. Musielak, A. Waszak, Remarks on some modular spaces of sequences, *Functiones et Approximatio*, **18** (1989), 143-147.

[6] J. Musielak, W. Orlicz, On modular spaces, *Studia Math.*, **18** (1959), 49-65.

[7] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, Volume 146, Marcel Dekker Inc., New York (1991).

[8] W. Matuszewska, W. Orlicz, On certain properties of φ-functions, *Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr. et Phys.*, **8** (1960), 439-443.

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