International Journal of Pure and Applied Mathematics
Volume 83 No. 5 2013, 643-650
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v83i5.6
ijpam.eu

PROJECTION METHOD FOR NEWTON-TIKHONOV REGULARIZATION FOR NON-LINEAR ILL-POSED HAMMERSTEIN TYPE OPERATOR EQUATIONS

Monnanda Erappa Shobha, Santhosh George
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka
575 025, INDIA

Proceedings of
NCRTMSA – 2012

Abstract: An iteratively regularized projection scheme for the ill-posed Hammerstein type operator equation $KF(x) = f$ has been considered. Here $F : D(F) \subseteq X \to X$ is a non-linear operator, $K : X \to Y$ is a bounded linear operator, X and Y are Hilbert spaces. The method is a combination of discretized Tikhonov regularization and modified Newton's method. It is assumed that the Fréchet derivative of F at x_0 is invertible i.e., the ill-posedness of the operator KF is due to the ill-posedness of the linear operator K. Here x_0 is an initial approximation to the solution $\hat{x}$ of $KF(x) = f$. Adaptive choice of the parameter suggested by Perverzev and Schock(2005) is employed in selecting the regularization parameter α. A numerical example is given to test the reliability of the method.

AMS Subject Classification: 47J06, 47A52, 65N20, 65J20
Key Words: discretized Newton Tikhonov method, ill-posed Hammerstein operator, balancing principle, regularization

1. Introduction

In this paper we study the finite dimensional approximation of an ill-posed operator equation of the form

$$KF(x) = f. \tag{1}$$

Here $F : D(F) \subseteq X \to X$ is a non-linear operator, $K : X \to Y$ is a bounded linear operator, $D(F)$ is the domain of F, X and Y are Hilbert spaces with the corresponding inner product $\langle ., . \rangle$ and norm $\|.\|$ respectively. Let $B(x, r)$ stand for the open ball in X with center $x \in X$ and of radius $r > 0$. Equation (1) is called Hammerstein type operator equation (see [2], [3], [4], [5] and [6]) and is ill-posed in the sense that the solution does not depend continuously on the data f.

Throughout the paper we assume that $f^\delta \in Y$ are the available data with

$$\|f - f^\delta\| \leq \delta. \tag{2}$$

The numerical treatment of ill-posed problems (1) and (2) requires the application of special iterative regularization methods. In this paper we consider the following iterative regularization method for approximately solving (1) and (2):

$$y^{h,\delta}_{n,\alpha} = x^{h,\delta}_{n,\alpha} - P_h F'(x^{h,\delta}_{0,\alpha})^{-1} P_h(F(x^{h,\delta}_{n,\alpha}) - z^{h,\delta}_\alpha), \tag{3}$$

$$x^{h,\delta}_{n+1,\alpha} = y^{h,\delta}_{n,\alpha} - P_h F'(x^{h,\delta}_{0,\alpha})^{-1} P_h(F(y^{h,\delta}_{n,\alpha}) - z^{h,\delta}_\alpha) \tag{4}$$

where $x^{h,\delta}_{0,\alpha} := P_h x_0$ and $z^{h,\delta}_\alpha$ is the discretized Tikhonov regularized solution of $Kz = f^\delta$ which is given by

$$(P_h K^* K P_h + \alpha P_h)(z^{h,\delta}_\alpha - P_h F(x_0)) = P_h K^*[f^\delta - KF(x_0)]. \tag{5}$$

Here $P_h : X \to X_h$ is an orthogonal projection of X into a finite dimensional subspace X_h. Throughout the paper we assume that F possess a uniformly bounded Fréchet derivative for all $x \in D(F)$ i.e., $\|F'(x)\| \leq M$, for some $M > 0$ and $\|F'(x_0)^{-1}\| := \beta_1$.

In Section 2, we state the results of Section 2 in [6] which are needed for proving our main results. Convergence analysis and error estimates are given in Section 3. Numerical example given in Section 4 which confirm the reliability of our method.

2. Preliminaries

Let $\varepsilon_h := \|K(I-P_h)\|$, $\tau_h := \|F'(x)(I-P_h)\|$, $\forall x \in D(F)$. Let $\{b_h : h > 0\}$ is such that $\lim_{h\to 0} \frac{\|(I-P_h)x_0\|}{b_h} = 0$, $\lim_{h\to 0} \frac{\|(I-P_h)F(x_0)\|}{b_h} = 0$ and $\lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ and $\tau_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if K and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \le \tau_0$, $b_h \le b_0$.

In order to guarantee the convergence rates for $\|x_{n,\alpha}^{h,\delta} - \hat{x}\|$, we use the following general source condition on $F(\hat{x}) - F(x_0)$.

Assumption 2.1. (see [6], Assumption 2.1) There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \ge \|K\|^2$ satisfying; (i) $\lim_{\lambda\to 0} \varphi(\lambda) = 0$, (ii) $\sup_{\lambda>0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \le \varphi(\alpha)$, $\forall \lambda \in (0, a]$ and (iii) $F(\hat{x}) - F(x_0) = \varphi(K^*K)w$ for some $w \in X$ such that $\|w\| \le 1$.

We will use the following Theorem from [6] for obtaining our error estimates.

Theorem 2.2. (see [6], Theorem 2.4) *Let $z_\alpha^{h,\delta}$ be as in (5). Further if $b_h \le \frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ and Assumption 2.1 holds. Then*

$$\|F(\hat{x}) - z_{\alpha,h}^\delta\| \le C(\varphi(\alpha) + (\frac{\delta + \varepsilon_h}{\sqrt{\alpha}})).$$

where $C = \max\{M\rho, 1\} + 1$

As in [6], we use the adaptive choice of the parameter suggested by Pereverzev and Schock [7]. Let

$$l := \max\{i : \varphi(\alpha_i) \le \frac{\delta + \varepsilon_h}{\sqrt{\alpha_i}}\} < N, \tag{6}$$

$$k = \max\{i : \alpha_i \in D_N^+\} \tag{7}$$

where $D_N^+ = \{\alpha_i \in D_N : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \le \frac{4C(\delta+\varepsilon_h)}{\sqrt{\alpha_j}}, j = 0, 1, 2, ..., i-1\}$.

Theorem 2.3. (see [6], Theorem 2.5) *Let l be as in (6), k be as in (7) and $z_{\alpha_k}^{h,\delta}$ be as in (5) with $\alpha = \alpha_k$. Then $l \le k$ and*

$$\|F(\hat{x}) - z_{\alpha_k}^{h,\delta}\| \le C(2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta + \varepsilon_h).$$

3. Convergence Analysis

The following assumption is needed to carry out the analysis throughout the paper.

Assumption 3.1. (cf.[1], Assumption 3 (A3)) There exist a constant $k_0 \geq 0$ such that for every $x, u \in B(x_0, r) \cup B(\hat{x}, r) \subseteq D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|.$$

Using Assumption 3.1 one can prove that $F'(P_h x_0)^{-1}$ exists and is bounded. So without loss of generality we assume that

$$\|F'(P_h x_0)^{-1}\| \leq \beta, \tag{8}$$

for some $\beta > 0$.

Lemma 3.2. (cf. [6], equation (3.7)) *Let $b_0 < \frac{1}{k_0}$ and (8) hold. Then*

$$\|P_h F'(P_h x_0)^{-1} P_h F'(P_h x_0)\| \leq 1 + \beta\tau_0.$$

Let

$$e_{n,\alpha_k}^{h,\delta} := \|y_{n,\alpha_k}^{h,\delta} - x_{n,\alpha_k}^{h,\delta}\|, \qquad \forall n \geq 0. \tag{9}$$

For our further analysis, we assume that, $k_0 < \frac{1}{4\beta(1+\beta\tau_0)}$ and

$$\delta_0 + \varepsilon_0 < \frac{1}{4\beta k_0(1 + \beta\tau_0)(M + 1 + C_{M\rho})}\sqrt{\alpha_0}$$

where $C_{M\rho} = \frac{1}{2}\max\{M\rho, 1\}$.

Let $\|\hat{x} - x_0\| \leq \rho$ with

$$\rho < \frac{1}{M}[\frac{1}{4\beta k_0(1 + \beta\tau_0)} - (M + 1 + C_{M\rho})\frac{\delta_0 + \varepsilon_0}{\sqrt{\alpha_0}}],$$

$$\gamma_\rho := \beta[M\rho + (M + 1 + C_{M\rho})(\frac{\varepsilon_0 + \delta_0}{\sqrt{\alpha_0}})] \leq \frac{1}{4k_0(1 + \beta\tau_0)} \tag{10}$$

and let

$$q := (1 + \beta\tau_0)k_0 r, r \in (r_1, r_2) \tag{11}$$

where

$$r_1 = \frac{1 - \sqrt{1 - 4k_0(1 + \beta\tau_0)\gamma_\rho}}{2k_0(1 + \beta\tau_0)}$$

and

$$r_2 = min\{\frac{1}{k_0(1+\beta\tau_0)}, \frac{1+\sqrt{1-4k_0(1+\beta\tau_0)\gamma_\rho}}{2k_0(1+\beta\tau_0)}\}.$$

Note that by (10), r is well defined and $q < 1$.

The proofs of the following results are analogous to the proof of corresponding results in [6], so we ignore the details.

Lemma 3.3. *Let $z^{h,\delta}_{\alpha_k}$ be as in (5) with $\alpha = \alpha_k$ and $e^{h,\delta}_{0,\alpha_k}$ be as in (9). Suppose (8) holds and $b_h \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}$, then $e^{h,\delta}_{0,\alpha_k} \leq \gamma_\rho$.*

Lemma 3.4. *Let $e^{h,\delta}_{n,\alpha_k}$, q be as in (9), (11) respectively. Let the sequences $y^{h,\delta}_{n,\alpha_k}$, $x^{h,\delta}_{n,\alpha_k}$ be as in (3), (4) respectively with $\alpha = \alpha_k$, $\delta \in (0, \delta_0]$, and $\varepsilon_h \in (0, \varepsilon_0]$. Then under the assumptions of Theorem 2.3 and Lemma 3.2, the following hold for all $n \geq 0$.*

(a) $\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\| \leq (1+q)\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|$;

(b) $\|y^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{n,\alpha_k}\| \leq q^2\|y^{h,\delta}_{n-1,\alpha_k} - x^{h,\delta}_{n-1,\alpha_k}\|$;

(c) $e^{h,\delta}_{n,\alpha_k} \leq q^{2n}\gamma_\rho$ *and*

(d) $x^{h,\delta}_{n,\alpha_k}, y^{h,\delta}_{n,\alpha_k} \in B_r(P_h x_0) \quad \forall n \geq 0$.

Theorem 3.5. *Let $y^{h,\delta}_{n,\alpha_k}$ and $x^{h,\delta}_{n,\alpha_k}$ be as in (3) and (4) respectively with $\alpha = \alpha_k$. If Lemma 3.4 holds, then $(x^{h,\delta}_{n,\alpha_k})$ is a Cauchy sequence in $B_r(P_h x_0)$ and converges to $x^{h,\delta}_{\alpha_k} \in \overline{B_r(P_h x_0)}$. Further $P_h F(x^{h,\delta}_{\alpha_k}) = z^{h,\delta}_{\alpha_k}$ and*

$$\|x^{h,\delta}_{n,\alpha_k} - x^{h,\delta}_{\alpha_k}\| \leq C_0 q^{2n}$$

where $C_0 = \frac{\gamma_\rho}{1-q}$.

Remark 3.6. Note that $0 < q < 1$ and hence the sequence $(x^{h,\delta}_{n,\alpha_k})$ converges linearly to $x^{h,\delta}_{\alpha_k}$.

Theorem 3.7. *Suppose $b_h + \rho < r$ and Assumption 2.1 holds. Then*

$$\|\hat{x} - x^{h,\delta}_{\alpha_k}\| \leq \frac{\beta}{(1-q)}\|F(\hat{x}) - z^{h,\delta}_{\alpha_k}\|.$$

The following Theorem is a consequence of Theorem 3.5 and Theorem 3.7.

Theorem 3.8. *Let $x^{h,\delta}_{n,\alpha_k}$ be as in (4), assumptions in Theorem 3.5 and Theorem 3.7 hold. Then*

$$\|\hat{x} - x^{h,\delta}_{n,\alpha_k}\| \leq C_0 q^{2n} + \frac{\beta}{(1-q)}\|F(\hat{x}) - z^{h,\delta}_{\alpha_k}\|$$

where C_0 is as in Theorem 3.5.

Now since $l \leq k$ and $\alpha_\delta \leq \alpha_{l+1} \leq \mu\alpha_l$ we have

$$\frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_l}} \leq \mu\frac{\delta+\varepsilon_h}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha(\delta,h)) = \mu\psi^{-1}(\delta+\varepsilon_h).$$

This leads to the following theorem,

Theorem 3.9. *Let $x^{h,\delta}_{n,\alpha_k}$ be as in (4), assumptions in Theorem 3.8 hold. Let $n_k := \min\{n : q^{2n} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_k}}\}$. Then $\|\hat{x} - x^{h,\delta}_{n_k,\alpha_k}\| = O(\psi^{-1}(\delta+\varepsilon_h))$.*

4. Numerical Example

We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with $dimV_n = n+1$ and let $P_h = P_{\frac{1}{n}}$ denote the orthogonal projection on X with range $R(P_h) = V_n$. We assume that $\|P_h x - x\| \to 0$ *as* $h \to 0$ for all $x \in X$. Precisely we choose V_n as the space of linear splines $\{v_1, v_2, \cdots, v_{n+1}\}$ in a uniform grid of $n+1$ points in $[0,1]$ as a basis of V_n.

Example 4.1. (cf. [1], Section 4.3) In this example we consider the operator $KF : L^2(0,1) \longrightarrow L^2(0,1)$ with $K : L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, 0 \leq s \leq t \leq 1 \\ (1-s)t, 0 \leq t \leq s \leq 1 \end{cases}$$

and $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by $F(u) := u^3$,

Then the Fréchet derivative of F is given by $F'(u)w = 3(u)^2 w$. In our computation, we take

$$y(t) = \frac{837t}{6160} - \frac{t^2}{16} - \frac{t^{11}}{110} - \frac{3t^5}{80} - \frac{3t^8}{112}$$

and

$$y^\delta = y + \delta.$$

Then the exact solution

$$\hat{x}(t) = 0.5 + t^3.$$

We use $x_0(t) = 0.5 + t^3 - \frac{3}{56}(t - t^8)$ as our initial guess.

We choose $\alpha_0 = (1.3)^2(\delta + \varepsilon_h)^2$, $\mu = 1.3$, $\delta = 0.1$ the Lipschitz constant k_0 equals approximately 0.2134 as in [1] and $r = 1$, $\tau_0 = \frac{1}{64}$, so that $q = (1 + \beta\tau_0)k_0 r = 0.2133$. The iterations and corresponding error estimates are given in Table 1. The plots of the exact solution and the approximate solution obtained for n=1024 is given in Figure 1. The last column of the Table 1 shows that the error $\|x_k - \hat{x}\|$ is of order $(\delta + \varepsilon_h)^{1/2}$.

n	k	α_k	$\|x_k - \hat{x}\|$	$\frac{\|x_k-\hat{x}\|}{(\delta+\varepsilon_h)^{1/2}}$
8	4	0.1820	0.5484	1.7273
16	4	0.1065	0.5376	1.6984
32	4	0.1061	0.5301	1.6759
64	4	0.1061	0.5257	1.6624
128	4	0.1061	0.5234	1.6551
256	4	0.1060	0.5222	1.6513
512	4	0.1060	0.5216	1.6493
1024	4	0.1060	0.5213	1.6484

Table 1

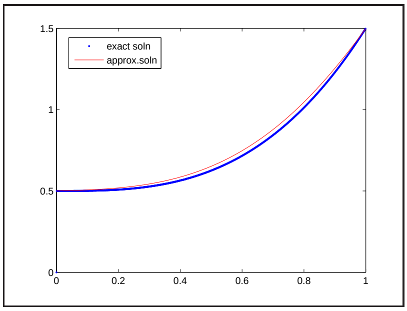

Figure 1

References

[1] E.V. Semenova, Lavrentiev regularization and balancing principle for solving ill-posed problems with monotone operators, *Comput. Methods Appl. Math.*, **4** (2010), 444-454.

[2] S. George, Newton-Tikhonov regularization of ill-posed Hammerstein operator equation, *J.Inverse and Ill-Posed Problems*, **14**, No. 2 (2006), 135-146.

[3] S. George, Newton-Lavrentiev regularization of ill-posed Hammerstein operator equation, *J.Inverse and Ill-Posed Problems*, **14**, No. 6 (2006), 573-582.

[4] S. George, M. Kunhanandan, An iterative regularization method for Ill-posed Hammerstein type operator equation, *J. Inv. Ill-Posed Problems*, **17** (2009), 831-844.

[5] S. George, M.T. Nair, A modified Newton-Lavrentiev regularization for nonlinear ill-posed Hammerstein-type operator equation, *Journal of Complexity*, **24** (2008), 228-240.

[6] Santhosh George, Monnanda Erappa Shobha, Two-Step Newton-Tikhonov Method for Hammerstein-Type Equations: Finite-Dimensional Realization, *ISRN Applied Mathematics*, Volume 2012, Article ID 783579, 22 pages (2012), **doi:** 10.5402/2012/783579.

[7] S. Pereverzev, E. Schock, On the adaptive selection of the parameter in regularization of ill-posed problems, *SIAM. J. Numer. Anal.*, **43**, No. 5 (2005), 2060-2076.

Received: January 10, 2013; **Accepted:** February 13, 2013.