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# SOME CLASS OF OPERATOR IDEALS

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**Abstract:** In 1975, Rhoades [9] generalized the classes of, operators of $l_p$ type and operators of Cesàro type by introducing an arbitrary infinite matrix $A = (a_{nk})$ using approximation numbers of a bounded linear operator. In the same paper Rhoades has proved that for each fixed matrix $A$ satisfying the condition $|a_{n,2k-1}| + |a_{n,2k}| \leq M|a_{nk}|$ on the matrix $A = (a_{nk})$ for each $n, k = 1, 2, \cdots$ and each $p, 0 < p \leq \infty$ the set of $A - p$ type operators is a linear space and raised an open question whether this condition is necessary to be a linear space? In this paper we have answered the question in negation. Further, we have introduced and studied the class $A^{(s)} - p$ of $s$-type $ces_p$ operators using $s$-number sequence and Cesàro sequence spaces. We have also shown that the class $A^{(s)} - p$ forms a quasi-Banach operator ideal. Moreover, the inclusion relations among the operator ideals are established.

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**Key Words:** $s$-numbers, approximation numbers, operator ideals, sequence space

## 1. Introduction

Due to the immense applications in spectral theory, geometry of Banach spaces, theory of eigenvalue distributions etc., the theory of operator ideals occupies a special importance in functional analysis. Many useful operator ideals have been defined by using sequence of $s$-numbers. For the unifications of different $s$-numbers, Pietsch ([4], 1974) defined an axiomatic theory of $s$-numbers in Banach spaces.

For each fixed infinite matrix $A = (a_{nk})$, Rhoades [9] defined $A - p$ space, denoted by $|A, p|$ as

$$|A,p| = \begin{cases} x \in w & : \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}x_k|\Big)^p\Big)^{\frac{1}{p}} < \infty \quad \text{for } 0 < p < \infty \\ x \in w & : \sup_{n\geq 1}\Big(\sum_{k=1}^{\infty}|a_{nk}x_k|\Big) < \infty \quad \text{for } p = \infty, \end{cases}$$

where $w$ is a sequence space of real or complex numbers. $A - p$ spaces contain many known sequence spaces by specifying suitable matrix. In particular, if we choose the matrix $A$ as Cesàro matrix of order 1, then $A - p$ spaces reduce to the Cesàro sequence spaces [10] denoted as $ces_p$ for $1 < p < \infty$ and defined by

$$ces_p = \Big\{x = (x_k) \in w : \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|x_k|\Big)^p < \infty\Big\},$$

The space $ces_p$ is complete with respect to the norm $\|x\| = \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}|x_k|\Big)^p\Big)^{\frac{1}{p}}$. It is easy to verify that if $1 < p \leq q < \infty$, then $ces_p \subseteq ces_q$.

Pietsch [3] defined an operator $T \in \mathscr{L}(E,F)$ to be $l^p$ type operator if $\sum_{n=1}^{\infty}\Big(a_n(T)\Big)^p$ is finite for $0 < p < \infty$, where $(a_n(T))$ is the sequence of approximation numbers of the bounded linear operator $T$. Later on Constantin [2] generalized the class of $l_p$ type operators to the class of $ces - p$ type operators by using the Cesàro sequence spaces. Rhoades [9] further generalized the class of $ces - p$ type operators to the class of $A - p$ type operators. An operator $T \in \mathscr{L}(E,F)$ is said to be $A - p$ type operator if $(a_n(T))$ is an element of the corresponding $A - p$ space, $0 < p \leq \infty$. Let $A = (a_{nk})$ be a fixed matrix satisfying the condition:

$$|a_{n,2k-1}| + |a_{n,2k}| \leq M|a_{nk}| \qquad \text{for each } k \text{ and } n, \tag{1}$$

where $M$ is a constant independent of $n$ and $k$. Rhoades has shown that for each fixed matrix $A$ satisfying the condition (1) and each $p$, the set of $A-p$ type operators forms a linear space and raised an open question whether the condition (1) is necessary.

In this paper we show that the condition (1) on the matrix $A=(a_{nk})$ is sufficient only and it is not a necessary condition. We study a generalized class of operators using the sequence of $s$-numbers. We also prove that the class $A^{(s)}-p$ of $s$-type $ces_p$ operators is a quasi-Banach operator ideal. Moreover, the inclusion relations among the operator ideals are established.

## 2. Preliminaries

Throughout this paper we denote $E$, $F$ as the real or complex Banach spaces and $\mathscr{L}(E,F)$ as the space of all bounded linear operators from $E$ to $F$ and $\mathscr{L}$ be the space of all bounded linear operator from any two arbitrary Banach spaces. We denote $E^{'}$ as the dual of $E$ and $x^{'}$ is the continuous linear functional on $E$. $\mathbb{N}$ and $\mathbb{R}^+$ stand for the set of all natural numbers and the set of all nonnegative real numbers respectively. Let $x^{'}\in E^{'}$ and $y\in F$, then the map $x^{'}\otimes y: E\to F$ is defined by $(x^{'}\otimes y)(x)=x^{'}(x)y,\ \ x\in E$.

**Definition 1.** ([1]) A map $s=(s_n):\mathscr{L}\to\mathbb{R}^+$ assigning to every operator $T\in\mathscr{L}$ a non-negative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an $s$-number sequence if the following conditions are satisfied:

($S1$) monotonicity: $\|T\|=s_1(T)\geq s_2(T)\geq\cdots\geq 0,\quad$ for $T\in\mathscr{L}(E,F)$

($S2$) additivity: $s_{m+n-1}(S+T)\leq s_m(S)+s_n(T),\quad$ for $S,T\in\mathscr{L}(E,F)$, $m,n\in\mathbb{N}$

($S3$) ideal property: $s_n(RST)\leq\|R\|s_n(S)\|T\|,\quad$ for some $R\in\mathscr{L}(F,F_0)$, $S\in\mathscr{L}(E,F)$ and $T\in\mathscr{L}(E_0,E)$ where $E_0,F_0$ are arbitrary Banach spaces

($S4$) rank property: If $rank(T)\leq n$ then $s_n(T)=0$

($S5$) norming property: $s_n(I:l_2^n\to l_2^n)=1$, where $I$ denotes the identity operator on the $n$-dimensional Hilbert space $l_2^n$.

We call $s_n(T)$ the $n$-th $s$-number of the operator $T$. The approximation number is a particular example of $s$-number sequence. The definition is as follows: Let $T\in\mathscr{L}(E,F)$ and $n\in\mathbb{N}$. The $n$-th approximation number, denoted by $a_n(T)$, is defined as

$$a_n(T) = \inf\Big\{\|T-L\|: \quad L \in \mathscr{L}(E,F),\ \mathrm{rank(L) < n}\Big\}.$$

There are several examples of $s$-number sequences namely Gel'fand number $(c_n(T))$, Kolmogorov number $(d_n(T))$, Weyl number $(x_n(T))$, Chang number $(y_n(T))$, Hilbert number $(h_n(T))$ etc. For details on $s$-number sequences, refer ([1], [4], [6], [7]).

**Proposition 2.** ([7], p.115) *Let* $T \in \mathscr{L}(E,F)$. *Then* $h_n(T) \le x_n(T) \le c_n(T) \le a_n(T)$ *and* $h_n(T) \le y_n(T) \le d_n(T) \le a_n(T)$.

The following lemma is required to prove our theorems.

**Lemma 3.** [4] *Let* $S, T \in \mathscr{L}(E,F)$, *then* $|s_n(T) - s_n(S)| \le \|T - S\|$ *for* $n = 1, 2, \cdots$.

**Definition 4.** ([5], [8]) Let $\mathscr{L}$ be the class of all bounded linear operators between any two arbitrary Banach spaces. A sub collection $\mathscr{M}$ of $\mathscr{L}$ is said to be an operator ideal if each component $\mathscr{M}(E,F) = \mathscr{M} \bigcap \mathscr{L}(E,F)$ satisfies the following conditions:

($OI1$) if $x' \in E'$, $y \in F$ then $x' \otimes y \in \mathscr{M}(E,F)$

($OI2$) if $S, T \in \mathscr{M}(E,F)$ then $S + T \in \mathscr{M}(E,F)$

($OI3$) if $S \in \mathscr{M}(E,F)$, $T \in \mathscr{L}(E_0,E)$ and $R \in \mathscr{L}(F,F_0)$ then $RST \in \mathscr{M}(E_0,F_0)$.

**Definition 5.** ([5], [8]) A function $\alpha : \mathscr{M} \to \mathbb{R}^+$ is said to be a quasi-norm on the ideal $\mathscr{M}$ if the following conditions hold:

($QON1$) if $x' \in E'$, $y \in F$ then $\alpha(x' \otimes y) = \|x'\|\|y\|$

($QON2$) if $S, T \in \mathscr{M}(E,F)$ then there exists a constant $C \ge 1$ such that $\alpha(S+T) \le C[\alpha(S) + \alpha(T)]$

($QON3$) if $S \in \mathscr{M}(E,F)$, $T \in \mathscr{L}(E_0,E)$ and $R \in \mathscr{L}(F,F_0)$ then $\alpha(RST) \le \|R\|\alpha(S)\|T\|$.

In particular if $C = 1$ then $\alpha$ becomes a norm on the operator ideal $\mathscr{M}$.

An ideal $\mathscr{M}$ with a quasi-norm $\alpha$, denoted by $[\mathscr{M}, \alpha]$ is said to be a quasi-Banach operator ideal if each component $\mathscr{M}(E,F)$ is complete under the quasi-norm $\alpha$.

### 3. Main Results

In this section we first answer the question raised by Rhoades in negation.

Let $0 < p < \infty$ and $A = (a_{nk})$ be a nonzero diagonal matrix (In particular Identity matrix) such that

$$|a_{2n-1,2n-1}|^p + |a_{2n,2n}|^p \leq M_1 |a_{nn}|^p, \tag{2}$$

where $M_1$ is a constant independent of $n$.

Let $S, T$ be any two $A - p$ type operators. Then

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}a_k(T+S)|\Big)^p\Big)^{\frac{1}{p}} =& \Big(\sum_{n=1}^{\infty}\Big(|a_{nn}|a_n(T+S)\Big)^p\Big)^{\frac{1}{p}}\\ \leq& C_1.M_1^{\frac{1}{p}}\Big[\Big(\sum_{n=1}^{\infty}\Big(|a_{nn}|a_n(T)\Big)^p\Big)^{\frac{1}{p}}\\ &+\Big(\sum_{n=1}^{\infty}\Big(|a_{nn}|a_n(S)\Big)^p\Big)^{\frac{1}{p}}\Big]\\ =& C_1.M_1^{\frac{1}{p}}\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}a_k(T)|\Big)^p\Big)^{\frac{1}{p}}\\ &+\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}a_k(S)|\Big)^p\Big)^{\frac{1}{p}}\Big]\\ <&\infty.\end{aligned}$$

Thus $S + T$ is an $A - p$ type operators. Clearly $\lambda T$ is an $A - p$ type operators for any scalar $\lambda$. Thus the set of $A - p$ type operators is a linear space where $A = (a_{nk})$ be a nonzero diagonal matrix satisfying (2). But the nonzero diagonal matrix $A$ (In particular Identity matrix) does not satisfy the condition (1). Hence the condition (1) on the matrix $A$ is sufficient only and it is not a necessary to be a linear space.

Let $s = (s_n)$ be a sequence of $s$-numbers. We call an operator $T \in \mathscr{L}(E, F)$ is of $s$-type $ces_p$ if

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}s_k(T)\Big)^p < \infty, \qquad 1 < p < \infty.$$

We denote $A^{(s)} - p$ as the class of all $s$-type $ces_p$ operators between arbitrary Banach spaces and $A^{(s)}(E \to F) - p$ as the set of all $s$-type $ces_p$ operators from $E$ to $F$ which is a component of the class $A^{(s)} - p$.

**Theorem 6.** *For $1 < p < \infty$, the class $A^{(s)} - p$ is an operator ideal.*

*Proof.* It is required to prove the conditions $(OI1)$ to $(OI3)$ in order to show that $A^{(s)} - p$ be an operator ideal. Let $x^{'} \in E^{'}$, $y \in F$ then $x^{'} \otimes y$ is a rank one operator. So

$$s_n(x^{'} \otimes y) = 0, \text{ for all } n \geq 2.$$

We have

$$\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(x^{'} \otimes y)\Big)^p\Big)^{\frac{1}{p}} = \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}s_1(x^{'} \otimes y)\Big)^p\Big)^{\frac{1}{p}}$$

$$= \|x^{'} \otimes y\|\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\Big)^p\Big)^{\frac{1}{p}} < \infty.$$

Thus $x^{'} \otimes y \in A^{(s)}(E \to F) - p$; hence $(OI1)$ is proved.

Let $S, T \in A^{(s)}(E \to F) - p$. Since $s$-number is nonnegative and nonincreasing

$$\begin{aligned}\sum_{k=1}^{n} s_k(T+S) &\leq \sum_{k=1}^{n} s_{2k-1}(T+S) + \sum_{k=1}^{n} s_{2k}(T+S)\\ &\leq 2\sum_{k=1}^{n} s_{2k-1}(T+S)\\ &\leq 2\Big(\sum_{k=1}^{n} s_k(T) + \sum_{k=1}^{n} s_k(S)\Big). \qquad (3)\end{aligned}$$

Using Minkowski inequality for $1 < p < \infty$, we have from (3)

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T+S)\Big)^p\Big)^{\frac{1}{p}} &\leq 2\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T) + \frac{1}{n}\sum_{k=1}^{n} s_k(S)\Big)^p\Big)^{\frac{1}{p}}\\ &\leq 2\Big[\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T)\Big)^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(S)\Big)^p\Big)^{\frac{1}{p}}\Big] < \infty.\end{aligned}$$

Thus $S + T \in A^{(s)}(E \to F) - p$; hence $(OI2)$ is proved.

Let $T \in \mathscr{L}(E_0, E)$, $R \in \mathscr{L}(F, F_0)$ and $S \in A^{(s)}(E \to F) - p$.

Using the property $(S3)$ in the Definition 1., we have

$$s_n(RST) \le \|R\| s_n(S) \|T\| \quad \text{for all } n \in \mathbb{N}.$$

So

$$\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(RST)\Big)^p\Big)^{\frac{1}{p}} \le \|R\|\|T\|\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(S)\Big)^p\Big)^{\frac{1}{p}} < \infty.$$

Thus $RST \in A^{(s)}(E_0 \to F_0) - p$ and therefore $(OI3)$ is proved.

Hence the class $A^{(s)} - p$ is an operator ideal. □

**Remark 7.** It is observed that the set $A^{(s)}(E \to F) - p$ of $s$-type $ces_p$ operators from $E$ to $F$ is a linear space. In particular if we take $s$-number sequence as the sequence of approximation numbers then the set $A^{(a)}(E \to F) - p$ coincides with the set of $ces - p$ type operators introduced by Constantin.

**Proposition 8.** *For $1 < p \le q < \infty$, we have $A^{(s)} - p \subseteq A^{(s)} - q$.*

*Proof.* It is trivial as $ces_p \subseteq ces_q$ for $1 < p \le q < \infty$; so we omit the proof. □

Let $A^{(s)} - p$ be an operator ideal. Define $\beta_p^{(s)} : A^{(s)} - p \to \mathbb{R}^+$ for $1 < p < \infty$ by

$$\beta_p^{(s)}(T) = \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T)\Big)^p\Big)^{\frac{1}{p}},$$

where $T \in A^{(s)} - p$.

**Theorem 9.** *The operator ideal $A^{(s)} - p$ is complete under the quasi-norm $\hat{\beta}_p^{(s)}$ i.e. $[A^{(s)} - p, \hat{\beta}_p^{(s)}]$ is a quasi-Banach operator ideal for $1 < p < \infty$, where*

$$\hat{\beta}_p^{(s)}(T) = \frac{\beta_p^{(s)}(T)}{\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\Big)^p\Big)^{\frac{1}{p}}}.$$

*Proof.* Let $1 < p < \infty$. To prove $A^{(s)} - p$ is a quasi-Banach operator ideal, it is enough to prove that each component $A^{(s)}(E \to F) - p$ of $A^{(s)} - p$ is complete under the quasi norm $\hat{\beta}_p^{(s)}$.

We have

$$\beta_p^{(s)}(T) = \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T)\Big)^p\Big)^{\frac{1}{p}}$$

$$\geq \|T\|\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\Big)^p\Big)^{\frac{1}{p}}.$$

$$\Rightarrow \|T\| \leq \hat{\beta}_p^{(s)}(T) \quad \text{for } T \in A^{(s)}(E \to F) - p. \tag{4}$$

Let $(T_m)$ be a Cauchy sequence in $A^{(s)}(E \to F) - p$. Then $\forall\ \epsilon > 0$, there exist $N \in \mathbb{N}$ such that

$$\hat{\beta}_p^{(s)}(T_m - T_l) < \epsilon, \quad \forall\ m, l \geq N. \tag{5}$$

Now from (4),

$$\|T_m - T_l\| \leq \hat{\beta}_p^{(s)}(T_m - T_l).$$

Using (5), we have

$$\|T_m - T_l\| \leq \hat{\beta}_p^{(s)}(T_m - T_l) < \epsilon \quad \forall\ m, l \geq N.$$

Hence $(T_m)$ is a Cauchy sequence in $\mathscr{L}(E, F)$. As $F$ is a Banach space, $\mathscr{L}(E, F)$ is also a Banach space. Therefore $T_m \to T$ as $m \to \infty$ in $\mathscr{L}(E, F)$. We shall now show that $T_m \to T$ as $m \to \infty$ in $A^{(s)}(E \to F) - p$.

Using Lemma 3., we have

$$|s_n(T_l - T_m) - s_n(T - T_m)| \leq \|T_l - T\|.$$

Letting $l \to \infty$, we have

$$s_n(T_l - T_m) \to s_n(T - T_m). \tag{6}$$

From (5), we get

$$\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T_l - T_m)\Big)^p\Big)^{\frac{1}{p}} < \epsilon\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\Big)^p\Big)^{\frac{1}{p}}, \quad \forall\ m, l \geq N.$$

Using (6), as $l \to \infty$ (keeping $m \geq N$ fixed) we have

$$\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T - T_m)\Big)^p\Big)^{\frac{1}{p}} \leq \epsilon\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\Big)^p\Big)^{\frac{1}{p}}$$

$$\Rightarrow \hat{\beta}_p^{(s)}(T - T_m) \leq \epsilon \quad \forall\ m \geq N.$$

This means that $T_m \to T$ under the quasi-norm $\hat{\beta}_p^{(s)}$.

Next we show that $T \in A^{(s)}(E \to F) - p$. Now

$$\sum_{k=1}^{n} s_k(T) \leq \sum_{k=1}^{n} s_{2k-1}(T) + \sum_{k=1}^{n} s_{2k}(T)$$
$$\leq 2\sum_{k=1}^{n} s_{2k-1}(T)$$
$$\leq 2\Big(\sum_{k=1}^{n} s_k(T - T_m) + \sum_{k=1}^{n} s_k(T_m)\Big).$$

Therefore

$$\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T)\Big)^p\Big)^{\frac{1}{p}}$$
$$\leq 2\Big[\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T - T_m)\Big)^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} s_k(T_m)\Big)^p\Big)^{\frac{1}{p}}\Big] < \infty,$$

since $\hat{\beta}_p^{(s)}(T - T_m) \to 0$ as $m \to \infty$ and $(T_m) \in A^{(s)}(E \to F) - p$.
Hence $T \in A^{(s)}(E \to F) - p$. □

Now we state the inclusion relations among the operator ideals generated by different $s$-number sequences.

**Theorem 10.** *Let $1 < p < \infty$. Then*

*(I)* $A^{(a)} - p \subseteq A^{(c)} - p \subseteq A^{(x)} - p \subseteq A^{(d)} - p$

and

*(II)* $A^{(a)} - p \subseteq A^{(d)} - p \subseteq A^{(y)} - p \subseteq A^{(h)} - p$.

*Proof.* Let $1 < p < \infty$. Suppose that $T \in A^{(a)} - p$. Then

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} a_k(T)\Big)^p < \infty.$$

From Proposition 2., we have

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} h_k(T)\Big)^p \leq \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} x_k(T)\Big)^p \leq \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} c_k(T)\Big)^p$$
$$\leq \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n} a_k(T)\Big)^p.$$

This proves $(I)$.

We omit the proof of $(II)$ as it is similar to the previous one. □

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## References

[1] B. Carl, A. Hinrichs, On $s$-numbers and Weyl inequalities of operators in Banach spaces, *Bull. Lond. Math. Soc.*, **41**, No. 2 (2009), 332-340.

[2] Gh. Constantin, Operators of $ces - p$ type, *Rend. Acc. Naz. Lincei.*, **52**, No. 8 (1972), 875-878.

[3] A. Pietsch, Einige neue klassen von kompakten linearen Abbildungen, *Rev. Math. Pures Appl.*, Bucarest, **8** (1963), 427-447.

[4] A. Pietsch, $s$-numbers of operators in Banach spaces, *Studia Math.*, **51** (1974), 201-223.

[5] A. Pietsch, *Operator Ideals*, VEB Deutscher Verlag der Wissenschaften, Berlin (1978).

[6] A. Pietsch, Weyl numbers and eigenvalues of operators in Banach spaces, *Math. Ann.*, **247**, No. 2 (1980), 149-168.

[7] A. Pietsch, *Eigenvalues and s-Numbers*, Cambridge University Press, New York, NY, USA (1986).

[8] J.R. Retherford, Applications of Banach ideals of operators, *Bull. Amer. Math. Soc.*, **81**, No. 6 (1975), 978-1012.

[9] B.E. Rhoades, Operators of $A - p$ type, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, **59**, No-s: 3-4 (1975), 238-241.

[10] J.S. Shiue, On the Cesàro sequence spaces, *Tamkang J. Math.*, **1**, No. 1 (1970), 19-25.

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