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# APPROXIMATION OF SIGNALS (FUNCTIONS) BY (E,q)(C,1) PRODUCT OPERATORS

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**Abstract:** In this paper, we establish a quite new theorem on degree of approximation of a function $\tilde{f}$, conjugate to a $2\pi$ periodic function $f$ belonging to class $Lip\alpha$ by $(E,q)(C,1)$ product operators on a conjugate Fourier series.

**AMS Subject Classification:** 42B05, 42B08
**Key Words:** degree of approximation, $Lip\alpha$ class of function, $(E,q)$ operators, $(C,1)$ operators, $(E,q)(C,1)$ product operators, Fourier series, conjugate Fourier series, Lebesgue integral

## 1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with $s_n$ for its $n^{th}$ partial sum.

Let $\{t_n^E\}$ denote the sequence of $(E, q_n)$ mean of the sequence $\{s_n\}$. If the $(E, q_n)$ transform of $s_n$ is defined as

$$t_n^E(f;x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k(f;x) \to s \text{ as } n \to \infty \tag{1}$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number $s$ by the $(E, q_n)$ method (Hardy [8]).

The $(E, q_n)$ transform reduces to the $(E, 1)$ transform if for all $q_n = 1$.

Let $\{t_n^C\}$ denote the sequence of $(C, 1)$ mean of the sequence $\{s_n\}$. If the $(C, 1)$ transform of $s_n$ is defined as

$$t_n^C(f;x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(f;x) \to s \text{ as } n \to \infty \tag{2}$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number $s$ by $(C, 1)$ method (Cesàro method).

Thus if

$$t_n^{EC}(f;x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{k+1} \sum_{\nu=0}^{k} s_\nu(f;x) \to s \text{ as } n \to \infty, \tag{3}$$

where $\{t_n^{EC}\}$ denote the sequence of $(E, q_n)(C, 1)$ product mean of the sequence $s_n$, the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the number $s$ by $(E, q_n)(C, 1)$ method.

Let $f$ be a $2\pi$-periodic function and Lebesgue integrable. The Fourier series associated with $f$ at a point $x$ is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{4}$$

with partial sums $s_n(f;x)$.

The conjugate series of Fourier series (4) of $f$ is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{5}$$

with partial sums $\tilde{s}_n(f;x)$.

Throughout this paper, we will call (5) as conjugate Fourier series of function $f$.

$L_\infty$ – norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = sup\{|f(x)| : x \in R\} \tag{6}$$

$L_r$- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r \, dx\right)^{\frac{1}{r}} \text{ for some } r \geq 1. \tag{7}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $T_n(x)$ of degree n under sup norm $\| \ \|_\infty$ is defined by

$$\|f(x) - T_n(x)\|_\infty = \sup\{|f(x) - T_n(x)| : x \in R\} \text{ (Zygmund [2])} \tag{8}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{T_n} \|f(x) - T_n(x)\|_r \tag{9}$$

This method of approximation is called Trigonometric Fourier Approximation (TFA).

A function $f \in Lip\alpha$ if

$$| f(x+t) - f(x) |= O(|t|^\alpha) \text{ for } 0 < \alpha \leq 1. \tag{10}$$

We shall use the following notations:

$$\psi(t) = \psi(x,t) = f(x+t) - f(x-t)$$

$$\tilde{K}_n(t) = \frac{1}{2\pi\ (1+q)^n} \sum_{k=0}^{n} \left\{ \binom{n}{k} q^{n-k} \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\cos\left(\nu + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}$$

## 2. The Main Result

Several researchers ([3], [4], [5], [6], [7], [9], [10], [11], [12]) studied error estimates $E_n(f)$ using different linear operators. In the present paper, we establish a theorem on the degree of approximation of a function $\tilde{f}$ conjugate to a periodic function $f$ belonging to the class $Lip\alpha$ by $(E, q_n)(C, 1)$ product mean on its conjugate Fourier series in the following form:

**Theorem 1.** *If* $\{t_n^{EC}\}$ *denote the sequence of* $(E, q_n)(C, 1)$ *product mean of the sequence* $\{s_n\}$ *and the conjugate Fourier series given by* $\tilde{f}$ *converges at the point* $x$ *to the value which is denoted by*

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} \psi(t)\ cot\left(\frac{t}{2}\right)\ dt \quad \text{(Zygmund [1])} \tag{11}$$

*then the degree of approximation of a function* $\tilde{f}$ *conjugate to a* $2\pi$*-periodic function* $f$ *belonging to the class* $Lip\alpha$ *by* $(E, q_n)(C, 1)$ *mean on its conjugate Fourier series (5) is given by*

$$\left\|\tilde{f}(x) - t_n^{EC}(x)\right\|_\infty = O\left\{\frac{1}{(n+1)^\alpha}\right\} \quad \text{for } 0 < \alpha < 1. \tag{12}$$

## 3. Lemmas

For the proof of our theorems, following lemmas are required:

**Lemma 1.** *For* $0 \le t \le \frac{1}{n+1}$,

$$\left|\tilde{K}_n(t)\right| = O\left(\frac{1}{t}\right)$$

*Proof.* For $0 \le t \le \frac{1}{n+1}$, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$ and $|\cos nt| \le 1$

$$\begin{aligned}
\left|\tilde{K}_n(t)\right| &= \frac{1}{2\pi\ (1+q)^n}\left|\sum_{k=0}^{n}\left\{\binom{n}{k} q^{n-k}\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\le \frac{1}{2\pi\ (1+q)^n}\sum_{k=0}^{n}\left\{\binom{n}{k} q^{n-k}\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{\left|\sin\frac{t}{2}\right|}\right\} \\
&\le \frac{1}{2t\ (1+q)^n}\sum_{k=0}^{n}\left\{\binom{n}{k} q^{n-k}\left(\frac{1}{1+k}\right)\sum_{\nu=0}^{k}1\right\} \\
&= \frac{1}{2t\ (1+q)^n}\sum_{k=0}^{n}\left\{\binom{n}{k} q^{n-k}\right\} \\
&= \frac{1}{2t\ (1+q)^n}(1+q)^n
\end{aligned}$$

$$= O\left(\frac{1}{t}\right) \qquad \text{since} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k} = (1+q)^n. \square$$

**Lemma 2.** *For $0 \le a \le b \le \infty$, $0 \le t \le \pi$ and any $n$, we have*

$$\left|\tilde{K}_n(t)\right| = O\left(\frac{1}{t}\right).$$

*Proof.* For $0 \le \frac{1}{n+1} \le t \le \pi$, by applying Jordan's lemma $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$.

$$\begin{aligned}
\left|\tilde{K}_n(t)\right| &= \frac{1}{2\pi\,(1+q)^n}\left|\sum_{k=0}^{n}\left\{\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\sum_{\nu=0}^{k}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right| \\
&\le \frac{1}{2t\,(1+q)^n}\left|\sum_{k=0}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k} e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right| \\
&\le \frac{1}{2t\,(1+q)^n}\left|\sum_{k=0}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k} e^{i\nu t}\right\}\right]\right|\left|e^{i\frac{t}{2}}\right| \\
&\le \frac{1}{2t\,(1+q)^n}\left|\sum_{k=0}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k} e^{i\nu t}\right\}\right]\right| \\
&\le \frac{1}{2t\,(1+q)^n}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k} e^{i\nu t}\right\}\right]\right| \\
&\qquad + \frac{1}{2t\,(1+q)^n}\left|\sum_{k=\tau}^{n}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k} e^{i\nu t}\right\}\right]\right|,
\end{aligned} \tag{13}$$

where $\tau$ denotes the integral part of $\frac{1}{t}$.

Now considering first term of (13),

$$\begin{aligned}
&\frac{1}{2t\,(1+q)^n}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\,Re\left\{\sum_{\nu=0}^{k} e^{i\nu t}\right\}\right]\right| \\
&\le \frac{1}{2t\,(1+q)^n}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\frac{1}{(1+k)}\left\{\sum_{\nu=0}^{k} 1\right\}\right]\right|\left|e^{i\nu t}\right| \\
&\le \frac{1}{2t\,(1+q)^n}\left|\sum_{k=0}^{\tau-1}\left[\left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k}\right]\right|
\end{aligned} \tag{14}$$

Now considering second term of (13) and using Abel's lemma

$$\frac{1}{2t\,(1+q)^n}\left|\sum_{k=\tau}^{n}\left[\binom{n}{k}q^{n-k}\frac{1}{(1+k)}\;Re\left\{\sum_{\nu=0}^{k}e^{i\nu t}\right\}\right]\right|$$
$$\leq \frac{1}{2t\,(1+q)^n}\sum_{k=\tau}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+k)}\max_{0\leq m\leq k}\left|\sum_{\nu=0}^{k}e^{i\nu t}\right|$$
$$\leq \frac{1}{2t\,(1+q)^n}\sum_{k=\tau}^{n}\binom{n}{k}q^{n-k}\frac{1}{(1+k)}(1+k)$$
$$= \frac{1}{2t\,(1+q)^n}\sum_{k=\tau}^{n}\binom{n}{k}q^{n-k} \tag{15}$$

Combining (13), (14) and (15), we get

$$\left|\tilde{K}_n(t)\right| \leq \frac{1}{2t\,(1+q)^n}\sum_{k=0}^{\tau-1}\binom{n}{k}q^{n-k} + \frac{1}{2t\,(1+q)^n}\sum_{k=\tau}^{n}\binom{n}{k}q^{n-k}$$
$$= O\left(\frac{1}{t}\right). \qquad \square$$

## 4. Proof of the Theorem

It is well known that

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\;\frac{\cos\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\;dt$$

Using (5) the $(C,1)$ transform of $\tilde{s}_n(f;x)$ is given by

$$\tilde{f}(x) - t_n^C(x) = \frac{1}{2\pi\;(n+1)}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\;dt$$

The $(E,q_n)(C,1)$ transform of $\tilde{s}_n(f;x)$ is given by

$$\tilde{f}(x) - t_n^{EC}(x)$$
$$= \frac{1}{2\pi\;(1+q)^n}\sum_{k=0}^{n}\left[\binom{n}{k}q^{n-k}\int_0^{\pi}\frac{\psi(t)}{\sin\frac{t}{2}}\left(\frac{1}{k+1}\right)\left\{\sum_{\nu=0}^{k}\cos\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$
$$= \int_0^{\pi}\psi(t)\;\tilde{K}_n(t)\,dt$$

$$= \left[ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right] \psi\,(t)\ \ \tilde{K}_n\,(t)\ \ dt$$

$$\tilde{f}\,(x) - t_n^{EC}(x) = I_{1.1} + I_{1.2}\ \ \text{(say)} \tag{16}$$

Using Leema 1, we have

$$\begin{aligned} \mid I_{1.1} \mid &\leq \int_0^{\frac{1}{n+1}} \mid \psi\,(t) \mid\mid \tilde{K}_n(t) \mid dt \\ &= O \int_0^{\frac{1}{n+1}} \frac{\mid t^{\alpha} \mid}{\mid t \mid} dt \\ &= O \int_0^{\frac{1}{n+1}} t^{\alpha-1} dt \\ &= O \left\{ \frac{t^{\alpha}}{\alpha} \right\}_0^{\frac{1}{n+1}} dt \\ I_{1.1} &= O \left\{ \frac{1}{(n+1)^{\alpha}} \right\} \end{aligned} \tag{17}$$

Using Leema 2, we have

$$\begin{aligned} \mid I_{1.2} \mid &\leq \int_{\frac{1}{n+1}}^{\pi} \mid \psi\,(t) \mid\mid \tilde{K}_n(t) \mid dt \\ &= O \int_{\frac{1}{n+1}}^{\pi} \frac{\mid t^{\alpha} \mid}{\mid t \mid} dt \\ &= O \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1} dt \\ &= O \left\{ \frac{t^{\alpha}}{\alpha} \right\}_{\frac{1}{n+1}}^{\pi} dt \\ I_{1.2} &= O \left\{ \frac{1}{(n+1)^{\alpha}} \right\} \end{aligned} \tag{18}$$

This completes the proof of the theorem.

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