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SOLUTION OF GENERALIZED SPACE-TIME FRACTIONAL TELEGRAPH EQUATION WITH COMPOSITE AND RIESZ-FELLER FRACTIONAL DERIVATIVES

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Abstract: We consider space-time fractional telegraph equation with composite fractional derivative in respect of time and Riesz-Feller fractional derivative in respect of space. We obtain Fourier transform of the solution in closed form in terms of Mittag-Leffler functions.

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1. Introduction

The fractional calculus is an extension of the ordinary calculus [5], which has been widely used in various fields of science and engineering [4]. Physicists have discovered that a number of systems, exhibiting anomalous behavior are usefully described by fractional differential equations.

The telegraph equation is a partial differential equation with constant coefficients given by

$$u_{tt} - c^2 u_{xx} + a u_t + b u = 0. \tag{1}$$

Orsingher and Beghin [6] have shown that the law of the iterated Brownian motion and the telegraph processes with Brownian time are governed by time-fractional telegraph equation and Orsingher and Zhao [7] presented that the transition function of a symmetric process with discontinuous trajectories satisfies the space-fractional telegraph equation. In the present paper we shall consider space-time fractional telegraph equation.

2. Definitions

Definition 1. The *Riemann-Liouville fractional integral* of order $\alpha > 0$ [4] is defined as
$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, 0 < \alpha \leq 1$ with $I_t^0 f(t) = f(t)$.

Definition 2. The *Riemann-Liouville fractional derivative* of order α, $m-1 < \alpha < m$, $m \in \mathbb{N}$ [4] is defined as the left inverse of Riemann-Liouville fractional integral, i.e.

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} D^m \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau. \tag{2}$$

Definition 3. The *Caputo fractional derivative* of order α, $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, is defined as [1]

$${}^C D_t^\alpha f(t) = I_t^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-x)^{\alpha-m+1}} D^m f(x)\, dx. \tag{3}$$

Definition 4. The *Hilfer fractional derivative or composite fractional derivative* of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$, is defined by Hilfer [2] as follows

$$D_t^{\mu,\nu} f(t) = I_t^{\nu(1-\mu)} D I_t^{(1-\nu)(1-\mu)} f(t). \tag{4}$$

Definition 5. Recently this definition is extended for $n-1<\mu<n, 0\leq \nu\leq 1, n\in\mathbb{N}$, and is also termed as generalized Riemann-Liouville fractional derivative by Hilfer *et al.* [3], as follows

$$D_t^{\mu,\nu}f(t)=I_t^{\nu(n-\mu)}D^nI_t^{(1-\nu)(n-\mu)}f(t). \tag{5}$$

In the above definition, type ν allows $D_t^{\mu,\nu}$ to interpolate continuously between the classical Riemann-Liouville fractional derivative and the Caputo fractional derivative, as in the case $\nu=0$, it gives the classical Riemann-Liouville fractional derivative as

$$D_t^{\mu,0}f(t)=D^nI_t^{(n-\mu)}f(t)=D_t^{\mu}f(t),\ \ 0<\mu<1 \tag{6}$$

and in case $\nu=1$, it gives the Caputo fractional derivative

$$D_t^{\mu,1}f(t)=I_t^{(n-\mu)}D^nf(t)={}^CD_t^{\mu}f(t),\ \ 0<\mu<1 \tag{7}$$

The *Laplace transform* of the composite fractional derivative (5) is given by [9]

$$L\left[D_t^{\mu,\nu}f(t);s\right]=s^{\mu}L\left[f(t)\right]-\sum_{k=0}^{n-1}s^{n-k-1-\nu(n-\mu)}D^kI_t^{(1-\mu)(1-\nu)}f(0). \tag{8}$$

Definition 6. The *Riesz-Feller fractional derivative* of order $\gamma,\ 0<\gamma\leq 2$, which is given as a pseudo-differential operator with the Fourier symbol $-|k|^{\gamma}, k\in\mathbb{R}$ is defined as in [9]
$\frac{d^{\gamma}}{d|x|^{\gamma}}g(x)=F^{-1}\left[-|k|^{\gamma}G(k)\right](x)$, where $G(k)=\int_{-\infty}^{\infty}e^{ikx}g(x)\,dx$

3. Main Result

We consider the space-time fractional telegraph equation as

$$D_t^{2\beta,\nu}u(x,t)+2\lambda D_t^{\beta,\nu}u(x,t)=c^2\frac{d^{\alpha}}{d|x|^{\alpha}}u(x,t),\ \ t>0, -\infty<x<\infty,$$

$$1<\alpha\leq 2, \frac{1}{2}<\beta\leq 1, 0\leq\nu\leq 1, \tag{9}$$

with boundary conditions

$$u(\pm\infty,t)=0, \tag{10}$$

and initial conditions

$$\left\{I_t^{(1-\nu)(2-2\beta)}u\left(x,t\right)\right\}_{t\to 0+}=g_1\left(x\right),\left\{I_t^{(1-\nu)(1-\beta)}u\left(x,t\right)\right\}_{t\to 0+}=g_2\left(x\right),$$

$$\left\{\frac{\partial}{\partial t}I_t^{(1-\nu)(2-2\beta)}u\left(x,t\right)\right\}_{t\to 0+}=0, \tag{11}$$

where $D_t^{2\beta,\nu}$ and $D_t^{\beta,\nu}$ are composite fractional derivative operators, defined by (5) and $\frac{d^\alpha}{d|x|^\alpha}$ is the Riesz-Feller fractional derivative operator defined by (6).

We take Fourier transform of (9) and (11) with respect to x and get the problem transformed as follows

$$D_t^{2\beta,\nu}\bar{u}\left(k,t\right)+2\lambda D_t^{\beta,\nu}\bar{u}\left(k,t\right)=-c^2\left|k\right|^\alpha\bar{u}\left(k,t\right),\ t>0, \tag{12}$$

with initial conditions

$$\left\{I_t^{(1-\nu)(2-2\beta)}\bar{u}\left(k,t\right)\right\}_{t\to 0+}=\bar{g}_1\left(k\right),\left\{I_t^{(1-\nu)(1-\beta)}\bar{u}\left(k,t\right)\right\}_{t\to 0+}=\bar{g}_2\left(k\right),$$

$$\left\{\frac{\partial}{\partial t}I_t^{(1-\nu)(2-2\beta)}\bar{u}\left(k,t\right)\right\}_{t\to 0+}=0. \tag{13}$$

where

$$\bar{u}\left(k,t\right)=\int_{-\infty}^{\infty}e^{ikx}u\left(x,t\right)dx \tag{14}$$

and $\bar{g}_1\left(k\right)$and $\bar{g}_2\left(k\right)$are Fourier transforms of $g_1\left(x\right)$ and $g_2\left(x\right)$respectively.

To solve the problem given by (12)-(13), we take Laplace transform of (12) with respect to t, to get

$$\begin{aligned}&s^\mu\bar{U}\left(k,s\right)-s^{1-\nu(2-2\beta)}\left\{I_t^{(1-\nu)(2-2\beta)}\bar{u}\left(k,t\right)\right\}_{t\to 0+}\\&-s^{-\nu(2-2\beta)}\left\{\tfrac{\partial}{\partial t}I_t^{(1-\nu)(2-2\beta)}\bar{u}\left(k,t\right)\right\}_{t\to 0+}\\&+2\lambda\left[s^\beta\bar{U}\left(k,s\right)-s^{-\nu(1-\beta)}\left\{I_t^{(1-\nu)(1-\beta)}\bar{u}\left(k,t\right)\right\}_{t\to 0+}\right]=-c^2\left|k\right|^\alpha\bar{U}\left(k,s\right),\end{aligned} \tag{15}$$

where

$$\bar{U}\left(k,s\right)=\int_0^\infty e^{-st}\bar{u}\left(k,t\right)dt. \tag{16}$$

Using initial conditions (13) and after some simplification we obtain

$$\bar{U}\left(k,s\right)=\frac{s^{1-\nu(2-2\beta)}}{s^{2\beta}+2\lambda s^\beta+c^2\left|k\right|^\alpha}\bar{g}_1\left(k\right)+2\lambda\frac{s^{-\nu(1-\beta)}}{s^{2\beta}+2\lambda s^\beta+c^2\left|k\right|^\alpha}\bar{g}_2\left(k\right). \tag{17}$$

For the purpose of inverse Laplace transform we write (17) as

$$\bar{U}(k,s) = s\frac{s^{\beta-\nu(1-\beta)-\beta}}{s^{\beta}-\eta_1}\frac{s^{\beta-\nu(1-\beta)-\beta}}{s^{\beta}-\eta_2}\bar{g}_1(k) + 2\lambda\frac{s^{\beta-\nu(1-\beta)-\beta}}{s^{\beta}-\eta_1}\frac{1}{s^{\beta}-\eta_2}\bar{g}_2(k), \tag{18}$$

where

$$\eta_1 = -\lambda + \sqrt{\lambda^2 - c^2|k|^{\alpha}}, \eta_2 = -\lambda - \sqrt{\lambda^2 - c^2|k|^{\alpha}}. \tag{19}$$

We now apply the following result [2]

$$\int_0^{\infty} e^{-st}t^{\beta-1}E_{\alpha,\beta}(\rho t^{\alpha})\,dt = \frac{s^{\alpha-\beta}}{s^{\alpha}-\rho},\ \ Re(s) > 0,\ Re(\alpha) > 0,\ Re(\beta) > 0 \tag{20}$$

where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \ \alpha,\beta \in C,\ \ Re(\alpha) > 0, Re(\beta) > 0, \tag{21}$$

to write

$$s\frac{s^{\beta-\nu(1-\beta)-\beta}}{s^{\beta}-\eta_1}\frac{s^{\beta-\nu(1-\beta)-\beta}}{s^{\beta}-\eta_2} = \int_0^{\infty}\int_0^{\infty} se^{-s(u+z)}u^{\nu(1-\beta)+\beta-1}$$
$$E_{\beta,\nu(1-\beta)+\beta}\left(\eta_1 u^{\beta}\right) z^{\nu(1-\beta)+\beta-1}E_{\beta,\nu(1-\beta)+\beta}\left(\eta_2 z^{\beta}\right)dzdu. \tag{22}$$

On substituting $t = u + z$, changing the order of integration, performing integration by parts, and using the results (see [8, p. 26, Eq. (1.108)])

$$\int_0^{t} z^{\gamma-1}E_{\alpha,\gamma}(yz^{a})(t-z)^{\beta-1}E_{\alpha,\beta}(w(t-z)^{\alpha})\,dz$$
$$= \frac{t^{\beta+\gamma-1}}{y-w}\left[yE_{\alpha,\beta+\gamma}(yt^{\alpha}) - wE_{\alpha,\beta+\gamma}(wt^{\alpha})\right], \tag{23}$$

and

$$\frac{d}{dt}\left\{(t-u)^{(\nu-1)(1-\beta)}E_{\beta,\nu(1-\beta)+\beta}\left(\eta_2(t-u)^{\beta}\right)\right\}$$
$$= (t-u)^{\nu(1-\beta)}E_{\beta,(\nu-1)(1-\beta)}\left(\eta_2(t-u)^{\beta}\right), \tag{24}$$

the equation (22) can be expressed as

$$s\frac{s^{\beta-\nu(1-\beta)-\beta}}{s^{\beta}-\eta_1}\frac{s^{\beta-\nu(1-\beta)-\beta}}{s^{\beta}-\eta_2}$$
$$= \int_0^{\infty} e^{-st}\frac{t^{2(\nu-1)(1-\beta)}}{\eta_1-\eta_2}\left[\eta_1 E_{\beta,2\nu(1-\beta)+2\beta-1}\left(\eta_1 t^{\beta}\right) - \eta_2 E_{\beta,2\nu(1-\beta)+2\beta-1}\left(\eta_2 t^{\beta}\right)\right]dt. \tag{25}$$

Similarly we can obtain the result

$$\frac{s^{-\nu(1-\beta)}}{s^{2\beta}+2\lambda s^{\beta}+c^{2}\left|k\right|^{\alpha}}=\int_{0}^{\infty}e^{-st}\frac{t^{\beta+(\nu-1)(1-\beta)}}{\eta_{1}-\eta_{2}}$$
$$\left[\eta_{1}E_{\beta,\nu(1-\beta)+2\beta}\left(\eta_{1}t^{\beta}\right)-\eta_{2}E_{\beta,\nu(1-\beta)+2\beta}\left(\eta_{2}t^{\beta}\right)\right]dt. \quad (26)$$

Taking inverse Laplce transform of (17), using results (25) and (26) and a result $x^{\alpha}E_{\alpha,\alpha+\beta}\left(x^{\alpha}\right)=E_{\alpha,\beta}\left(x^{\alpha}\right)-1$, we get

$$\bar{u}\left(k,t\right)=$$
$$\frac{t^{2(\nu-1)(1-\beta)}}{\eta_{1}-\eta_{2}}\left[\eta_{1}E_{\beta,2\nu(1-\beta)+2\beta-1}\left(\eta_{1}t^{\beta}\right)-\eta_{2}E_{\beta,2\nu(1-\beta)+2\beta-1}\left(\eta_{2}t^{\beta}\right)\right]\bar{g}_{1}\left(k\right)$$
$$+2\lambda\frac{t^{(\nu-1)(1-\beta)}}{\eta_{1}-\eta_{2}}\left[E_{\beta,\nu(1-\beta)+\beta}\left(\eta_{1}t^{\beta}\right)-E_{\beta,\nu(1-\beta)+\beta}\left(\eta_{2}t^{\beta}\right)\right]\bar{g}_{2}\left(k\right). \quad (27)$$

Remark 1. The solution of problem (9)-(11) can be obtained on calculating Fourier inverse of $\bar{u}\left(k,t\right)$.

Remark 2. Setting $\nu=1,\alpha=2$, in (9)-(11) and (27), these equations reduce to the time-fractional telegraph equation with Caputo fractional derivative, which has been solved earlier by Orsingher and Beghin [6] and the solution is same.

Remark 3. Setting (i) $\nu=1$, $g_1(x)=g_2(x)$, (ii) $\nu=0$ in (9)-(11) and (27) successively, we get Fourier transform of the solution of space-time fractional telegraph equation with Caputo and Riemann-Liouville fractional derivative in time respectively.

Remark 4. Setting (i) $\nu=0$, $\alpha=2$, (ii) $\beta=1$, $g_1(x)=g_2(x)$ in (9)-(11) and (27) successively, we get Fourier transform of the solution of time-fractional telegraph equation and space-fractional telegraph equation with Riemann-Liouville fractional derivative, respectively.

Remark 5. Setting$\beta=1$, $\alpha=2$, $g_1(x)=g_2(x)$ in (9)-(11) and (27), we get Fourier transform of the solution of the classical telegraph equation.

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