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STATISTICAL AND LACUNARY CONVERGENCE-ITS LATEST DEVELOPMENT FROM THE POINT OF VIEW OF SEQUENCE SPACES

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The development of the theory of sequence spaces is nowadays effected by the introduction of new convergence methods and theories in the process. Some of them are statistical convergence, lacunary convergence, lacunary statistical convergence etc (see [1], [2], [3], [8], [11]). Here, I have basically concentrated on these concepts.

Statistical convergence while introduced over nearly fifty years ago has only recently become an area of active research in sequence spaces. Fast [5] extended the concept of sequential limit which he called statistical convergence. Schoenberg (1959) gave some basic properties of statistical convergence and studied the concept as summability method.

The basic concept of statistical convergence is based on the notion of natural density of sets A $\subseteq N = \{1, 2, \ldots, n, \ldots\}$, refer to [5], [6]. If A $\subseteq$ N and A(n)=$|A \cap \{1, 2, \ldots, n\}|$, where the vertical bar denotes the cardinality of the set, then the numbers

$$\bar{\delta}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}, \quad \underline{\delta}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$$

are called the upper density and lower density of A. If

$$\bar{\delta}(A) = \underline{\delta}(A) = \delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

then $\delta(A)$ is called the natural density of A.

Definition 1. (see [5], [6]) A sequence $x = (x_k)$ of complex numbers is said to converge statistically to L if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{k \le n : \ |x_k - L| \ge \epsilon\Big\}\Big| = 0.$$

i.e.,

$$\delta\Big(\Big\{k \le n : \ |x_k - L| \ge \epsilon\Big\}\Big) = 0. \tag{0.1}$$

Definition 2. (see [6]) A sequence $x = (x_k)$ is a statistically Cauchy sequence if for each $\epsilon > 0$, there is a positive integer $N = N(\epsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{k \le n : \ |x_N - x_k| \ge \epsilon\Big\}\Big| = 0,$$

or equivalently,

$$\delta\Big(\Big\{k \le n : \ |x_N - x_k| \ge \epsilon\Big\}\Big) = 0.$$

By a.a.k we mean that, if $x = (x_k)$ be a sequence such that x_k satisfies P for all k except a set of natural density zero, then x_k satisfies P for "almost all k" and is written by "a.a.k". So, the conditions (1) and (2) can be written as

$$|x_k - L| < \epsilon \text{ a.a.k} \quad \textit{and} \quad |x_N - x_k| < \epsilon \text{ a.a.k respectively}$$

1. Composite Vector Valued Sequence Space $F(E_k, f)$

Recently, Srivastava and Ghosh [7] introduced new general class F(E_k, f) of vector valued sequence spaces using modulus function under suitable topology.

We generalize the concept of statistical convergence and statistically Cauchy sequence on composite vector valued sequence space F[E,f] using modulus function f (see [7], [9], [10]) as follows:

Definition 3. (see [5], [6]) A sequence $x = (x^{(n)}) = ((x_k^{(n)}))$ in $F(E, f)$ is said to converge statistically to ℓ if for any $\epsilon > 0$,

$$\delta\Big(\Big\{n \in \mathbf{N} : g_F\Big[\Big(f\Big(\|x_k^{(n)} - \ell\|\Big)\Big)\Big] \geq \epsilon\Big\}\Big) = 0$$

for each $k \in \mathbf{N}$, or equivalently,

$$g(x^{(n)} - \ell) < \epsilon \text{ a.a.n}$$

We write it as $x^{(n)} \xrightarrow{stat} \ell$.

Definition 4. (see [5], [6]) A sequence $x = (x^{(n)}) = ((x_k^{(n)}))$ in F(E,f) is said to be statistically Cauchy if for any ϵ>0, $\exists$ a natural number $m = m(\epsilon)$ such that

$$\delta\Big(\Big\{n \in \mathbf{N} : g_F\Big[\Big(f\Big(\|x_k^{(n)} - x_k^{(m)}\|\Big)\Big)\Big] \geq \epsilon\Big\}\Big) = 0$$

for $n \geq m$ and for each k $\in \mathbf{N}$, or equivalently,

$$g(x^{(n)} - x^{(m)}) < \epsilon \text{ a.a.n}$$

2. Main Results

Theorem 2.1. *If $x = ((x_k^{(i)}))$ be a sequence in F[E,f] then the following conditions are equivalent:*

(a) $x^{(i)} \xrightarrow{stat} \ell$, where $\ell \in$ F[E, f],

(b) For any g_F, $\exists$ $y = ((y_k^{(i)}))$ and $z = ((z_k^{(i)}))$ in F[E,f] such that $x_k^{(i)} = y_k^{(i)} + z_k^{(i)}$, $\lim_{i\to\infty} y_k^{(i)} \overset{g}{=} \ell$ and z is statistically null sequence,

(c)For any g_F, there is a subsequence $K = \{i_r\}$ of $\mathbf{N}$ such that $\delta(K) = 1$ and $\lim_{r\to\infty} x_k^{i_r} \overset{g}{=} \ell$, where $\ell \in F[E, f]$.

Theorem 2.2. *The following conditions are equivalent:*

(i) $x^{(n)} \xrightarrow{stot} \ell$.

(ii) $(x^{(n)})$ *is a statistically Cauchy sequence.*

(iii) $(x^{(n)})$ *is a sequence for which there is a convergent sequence* $(y^{(n)})$ *such that for each k,* $x_k^{(n)} = y_k^{(n)}$ *a.a.n.*

Lemma 2.1. *ℓ_∞[F[E, f]] is sequentially complete space with respect to the norm*

$$\bar{g}(x^{(n)}) = \sup_k g_F\Big[\Big(f\Big(\|x^{(n)}_{k,\lambda}\|\Big)\Big)\Big]$$

if F[E, f] is complete.

Theorem 2.3. $\bar{C}[F[E,f]] \bigcap \ell_\infty[F[E,f]]$ *is sequentially complete in ℓ_∞[F[E,f]] if F[E, f] is complete.*

We introduce the following new classes

$$\triangle N_\theta(E,A,M) = \Big\{x = (x_k) : x_k \in E,\ \rho^{(i)} > 0, \tag{2.1}$$

$$\text{there exists } s = (s_i) \in E, \text{ such that} \tag{2.2}$$

$$\lim_{r\to\infty} h_r^{-1} \sum_{i \in I_r} M_i\Big(\frac{\| A_i(\Delta x_k) - s_i e_i \|}{\rho^{(i)}}\Big) = 0\Big\} \tag{2.3}$$

and

$$\triangle N^0_\theta(E,A,M) = \Big\{x = (x_k) : x_k \in E \text{ and } \rho^{(i)} > 0 \tag{2.4}$$

$$\lim_{r\to\infty} h_r^{-1} \sum_{i \in I_r} M_i\Big(\frac{\| A_i(\Delta x_k) \|}{\rho^{(i)}}\Big) = 0\Big\} \tag{2.5}$$

where

$$e_i = \begin{cases} 1 & \text{at the i-th place} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.4. *Let $A = (a_{ik})$ be an infinite matrix of complex numbers and M=(M_i) be a sequence of Orlicz functions satisfying $\triangle_2$ condition. Then $\triangle N_\theta(E,A) \subset \triangle N_\theta(E,A,M)$, where (E,$\|\ .\ \|$) is a normal Banach space*

Theorem 2.5. *Let $A = (a_{ik})$ be an infinite matrix of complex numbers and M=(M_i) be a sequence of Orlicz functions satisfying $\triangle_2$ condition.*

If

$$\lim_{u\to\infty} \inf_i \Big\{\frac{M_i\Big(\frac{u}{\rho^{(i)}}\Big)}{\frac{u}{\rho^{(i)}}} > 0\Big\} \textit{ for some } \rho^{(i)} > 0$$

then $\triangle N_\theta(E,A) = \triangle N_\theta(E,A,M)$.

Lemma 2.2. $| \triangle\sigma_1(A) | \subset \triangle N_0(E, A)$ *if and only if* $\liminf_r q_r > 1$.

Lemma 2.3. $\triangle N_\theta(E, A) \subset | \triangle\sigma_1(A) |$ *if and only if* $\limsup_r q_r < \infty$.

Theorem 2.6. *Let* θ *be a lacunary sequence.Then* $| \triangle\sigma_1(A) | = \triangle N_\theta(E, A)$ *if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty$$

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Theorem 2.7. *Let* $M = (M_i)$ *be a sequence of Orlicz functions and* (M_i) *be point-wise convergent.Then* $\triangle N_\theta(E, A, M) \subset \Delta S_\theta(A)$ *if and only if* $\lim_i M_i(\frac{u}{\rho^{(i)}}) > 0$ *for some u>0,* $\rho^{(i)} > 0$.

Theorem 2.8. *Let* $M = (M_i)$ *be a sequence of Orlicz functions.Then* $\Delta S_\theta(A) \subset \triangle N_\theta(E, A, M)$ *if and only if* $\sup_u \sup_i M_i(\frac{u}{\rho}) < \infty$.

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