Retrial Inventory System with Impatient Customers and Customers search from the Orbit

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Abstract

We present a continuous review \((s, Q)\) inventory system with a service facility consisting of customers searching from the orbit. Arrival of customers are according to a Poisson process, whenever a customer leaves the system after getting service. The service time and the lead time of reorders are assumed to have independent exponential distributions. Any arriving customer, who finds the server is busy, enters into the orbit of finite space. After the completion of each service, the server searches for customers from the orbit with probability \(p > 0\), and remains idle with probability \(1 - p\). Search time is assumed to be negligible. The orbiting customers may renge from the orbit after a random time. The reneging time is exponentially distributed. The joint probability distribution of the number of demands in orbit, the inventory level and the server status are obtained in the steady state case. Various system performance measures are derived and the long-run total expected cost rate are derived in the steady state.

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Key Words: Continuous review inventory system, Positive leadtime, Retrial customers, Impatient Customers, Orbit Search.
1 Introduction

Artalejo et al. [1] were the first to discuss inventory policies with positive lead-time and retrial of customers who could not get service during their earlier attempts to access the service station. Jeganathan et al. [3, 4] studied a retrial inventory system with non-preemptive priority service and second optional service. Krishnamoorthy et al. [6] considered a retrial queue with non-persistent customers and orbital search. Jianan Cui and Jinting Wang [5] developed a queuing inventory system with registration and orbiting searching process. Wchner and Sztrik [7] studied finite source $M/M/S$ retrial queue with search for balking and impatient customer from the orbit.

2 Model Description

We consider a single server continuous review inventory system in which primary customers arrive according to a Poisson stream of rate $\lambda$. Any customer who, upon arrival, finds the server busy immediately leaves the service area and joins the orbit. The interval between two successive repeated attempts is exponentially distributed with parameter $i\theta$ when there are $i$ customers in the orbit. The service time follows an exponentially distributed with rate $\mu$. At the end of a service that customer is provided one item from the inventory. The $(s, Q)$-control policy is adopted. The lead time for replenishment follows an exponential distribution with parameter $\beta$. After the completion of service, the server searches for customers from the orbit with probability $p > 0$, and remains idle with probability $1 - p$. Search time is assumed to be negligible. The orbiting customers may renege from the orbit after a random time. The reneging time is exponentially distributed with rate $\gamma$. The joint probability distribution of the number of demands in orbit, the inventory level and the server status are obtained in the steady state case. Various system performance measures are derived and the long-run total expected cost rate are derived in the steady state.

Notations:

$[A]_{ij}$ : The element/submatrix at $(i, j)$th position of $A$. 
0 : Zero matrix.
I : Identity matrix.
e : A column vector of 1’s of appropriate dimension.

\[ Y(t) = \begin{cases} 0, & \text{if server is idle at time } t. \\ 1, & \text{if server is busy at time } t. \end{cases} \]

\[ E = \{(i, 0, m) : i = 0, 1, \ldots, M, \quad m = 0, 1, \ldots, S\} \cup \{(i, 1, m) : i = 0, 1, \ldots, M, \quad m = 1, 2, \ldots, S\}. \]

### 3 Analysis

Let \( X(t) \) : the number of demands in the orbit, \( Y(t) \) : the server status at time \( t \) and \( L(t) \) : the on hand inventory level at time \( t \).

The stochastic process \( \{(X(t), Y(t), L(t)), \ t \geq 0\} \) is a continuous time Markov chain with the state space given by \( E \).

By ordering states lexicographically, the infinitesimal generator \( A = \{a((i, k, m)), ((j, l, n))\} \) of \( X(t), Y(t), L(t) \) \( t \geq 0 \) can be conveniently expressed in a block partitioned matrix with entries

\[
[A]_{ij} = \begin{cases} A_2 & j = i \quad i = i \\ A_1 & j = i \quad i = 1, 2, \ldots, i - 1 \\ A_0 & j = i \quad i = 0 \\ B & j = i + 1 \quad i = 0, 1, 2, \ldots, i - 1 \\ C & j = i - 1 \quad i = 1, 2, \ldots, i \\ 0 & \text{otherwise} \end{cases}
\]

where

\[
[A]_{01} = \begin{pmatrix} D_{00}(S+1)_{S+S} & D_{01}(S+1)_{S+S} \\ D_{10}(S+1)_{S+S} & D_{11}(S+1)_{S+S} \end{pmatrix}
\]

\[
[D_{00}]_{mn} = \begin{cases} -\lambda, & n = m \quad m = S, S - 1, \ldots, s + 1 \\ -(\lambda + \beta), & n = m \quad m = s, s - 1, \ldots, 0 \\ \beta, & n = m + Q \quad m = s, s - 1, \ldots, 0 \\ 0 & \text{otherwise} \end{cases}
\]
\[
[D_{01}]_{mn} = \begin{cases} 
\lambda & n = m \quad m = 2, 3, \ldots, S \\
0 & \text{otherwise}
\end{cases}
\]

\[
[D_{10}]_{mn} = \begin{cases} 
\mu & n = m - 1 \quad m = S, S - 1, \ldots, 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
[D_{11}]_{mn} = \begin{cases} 
-(\lambda + \mu) & n = m \quad m = S, S - 1, \ldots, s + 1 \\
-(\lambda + \mu + \beta) & n = m \quad m = s, s - 1 \ldots, 1 \\
\beta & n = m + Q \quad m = s, s - 1 \ldots, 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
[A_1]_{01} = \begin{pmatrix} 0 & 1 \\ (E_{00})_{(S+1)\times(S+1)} & (E_{01})_{(S+1)\times S} \\ (E_{10})_{S\times(S+1)} & (E_{11})_{S\times S} \end{pmatrix}
\]

\[
[E_{10}]_{mn} = \begin{cases} 
\mu & n = m - 1 \quad m = S, S - 1, \ldots, 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
[E_{00}]_{mn} = \begin{cases} 
-(\lambda + i\theta) & n = m \quad m = S, S - 1, \ldots, s + 1 \\
(\lambda + i\theta + \beta) & n = m \quad m = s, s - 1 \ldots, 1 \\
-(\lambda + \beta) & n = m \quad m = 0 \\
-\beta & n = m + Q \quad m = s, s - 1 \ldots, 0 \\
0, & \text{otherwise}
\end{cases}
\]

\[
[E_{10}]_{mn} = \begin{cases} 
(1 - p)\mu & n = m - 1 \quad m = S, S - 1, \ldots, 2 \\
\mu & n = 0 \quad m = 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
[E_{11}]_{mn} = \begin{cases} 
-(\lambda + \mu + \gamma) & n = m \quad m = S, S - 1, \ldots, s + 1 \\
-(\lambda + \mu + \gamma + \beta) & n = m \quad m = s, s - 1, \ldots, 1 \\
p\mu & n = m - 1 \quad m = S, S - 1, \ldots, 1 \\
\beta & n = m + Q, \quad m = s, s - 1, \ldots, 1 \\
0 & \text{otherwise}
\end{cases}
\]
\[
[A_{2}]_{01} = \begin{bmatrix}
0 & 1 \\
(F_{00})_{(S+1)\times(S+1)} & (F_{01})_{(S+1)\times S} \\
(F_{10})_{S\times(S+1)} & (F_{11})_{S\times S}
\end{bmatrix}
\]

\[
[F_{00}]_{mn} = \begin{cases}
-i\theta & n = m \\
(i\theta + \beta) & n = m \\
\beta & n = m + Q \\
0 & \text{otherwise}
\end{cases}
\quad m = S, S - 1, \ldots, s + 1
\]

\[
[F_{11}]_{mn} = \begin{cases}
-(\mu + \gamma) & n = m \\
-(\mu + \gamma + \beta) & n = m \\
\beta & n = m + Q \\
\mu & n = m - 1
\end{cases}
\quad m = s, s - 1, \ldots, 1
\]

\[
[B_{1}]_{01} = \begin{bmatrix}
0 & 1 \\
(B_{00})_{(S+1)\times(S+1)} & (B_{01})_{(S+1)\times S} \\
(0)_{S\times(S+1)} & (B_{11})_{S\times S}
\end{bmatrix}
\]

\[
[B_0]_{mn} = \begin{cases}
\lambda & n = m \\
0 & \text{otherwise}
\end{cases}
\quad m = 0
\]

\[
[B_1]_{mn} = \begin{cases}
\lambda & n = m \\
0 & \text{otherwise}
\end{cases}
\quad m = S, S - 1, \ldots, 1
\]

\[
[C_{01}]_{01} = \begin{bmatrix}
0 & 1 \\
(C_{00})_{(S+1)\times(S+1)} & (C_{01})_{(S+1)\times S} \\
(0)_{S\times(S+1)} & (C_{11})_{S\times S}
\end{bmatrix}
\]

\[
[C_{01}]_{mn} = \begin{cases}
i\theta & n = m \\
0 & \text{otherwise}
\end{cases}
\quad m = S, S - 1, \ldots, 1
\]

\[
[C_{11}]_{mn} = \begin{cases}
g & n = m \\
0 & \text{otherwise}
\end{cases}
\quad m = S, S - 1, \ldots, 1
\]

\[
E_{01} = D_{01}, \quad F_{10} = E_{10}
\]
3.1 Steady State Analysis

It can be seen from the structure of $A$ that the homogeneous Markov process $\{(X(t), Y(t), L(t)), t \geq 0\}$ on the finite state space $E$ is irreducible. Hence the limiting distribution $\pi(j,k) = \lim_{t \to \infty} pr \{X(t) = i, Y(t) = j, L(t) = k | X(0), Y(0), L(0)\}$, exists. Let

$$\Pi = (\Pi(0), \Pi(1), \Pi(2), \ldots, \Pi(M))$$

which is partitioned as follows:

$$\Pi(i) = (\pi(i,0,0), \pi(i,0,k), \pi(i,1,k))$$

for $i = 0, 1, 2, \ldots, M$, $k = 1, 2, \ldots, S$.

Then the limiting probability, $\Pi$ satisfies

$$\Pi A = 0, \quad \Pi e = 1 \quad (1)$$

From the structure of $A$, it is a finite QBD matrix, therefore its steady state vector $\Pi$ can be computed by using the following algorithm described by [2].

**Algorithm:**

1. Determine recursively the matrices

$$F_0 = A_0$$

$$F_i = A_1 + B(-F_{i-1}^{-1}) C, \quad i = 1, 2, \ldots, M - 1,$$

$$F_M = A_2 + B(-F_{M-1}^{-1}) C.$$  

2. Compute recursively the vectors $\Pi(i)$ using

$$\Pi(i) = \Pi(i+1) B(-F_i^{-1}), \quad i = 0, 1, 2, \ldots, M - 1$$

3. Solve the system of equations

$$\Pi(M) F_M = 0 \text{ and } \sum_{i=0}^{M} \Pi(i) e = 1.$$  

From the system of equations $\Pi(M) F_M = 0$, vector $\Pi(M)$ could be determined uniquely, up to a multiplicative constant. This constant is decided by

$$\Pi(i) = \Pi(i+1) B(-F_i^{-1}), \quad i = 0, 1, 2, \ldots, M - 1 \text{ and } \sum_{i=0}^{M} \Pi(i) e = 1.$$
4 System Performance Measures

In this section we will list a number of system performance measures. Let $\rho_I$, $\rho_R$, $\rho_0$, $\rho_{OR}$, $\rho_{SR}$, $\rho_g$, and $\rho_L$ denote the Mean inventory level, Expected reorder rate, Expected number of demands in the orbit, Overall rate of retrials, Successful rate of retrials, Expected number of reneging and expected number of lost customers.

\[
\rho_I = \sum_{i=0}^{M} \sum_{k=1}^{S} k \left[ \pi(i,0,k) + \pi(i,1,k) \right]
\]

\[
\rho_R = \mu \sum_{i=0}^{M} \left[ \pi(i,1,s+1) \right]
\]

\[
\rho_0 = \sum_{i=1}^{M-1} \sum_{k=1}^{S} i \left[ \pi(i,0,k) + \pi(i,1,k) \right]
\]

\[
\rho_{OR} = \sum_{i=1}^{M-1} \sum_{k=1}^{S} \theta \left[ \pi(i,0,k) + \pi(i,1,k) \right]
\]

\[
\rho_{SR} = \sum_{i=1}^{M-1} \sum_{k=1}^{S} i \theta \left[ \pi(i,0,k) \right]
\]

\[
\rho_{FSR} = \frac{\rho_{SR}}{\rho_{OR}}
\]

\[
\rho_g = \gamma \sum_{i=1}^{M} \sum_{k=0}^{S} \left[ \pi(i,1,k) \right]
\]

\[
\rho_L = \sum_{k=0}^{S} i \left[ \pi(M,0,k) + \pi(M,1,k) \right]
\]

4.1 Cost Analysis

We consider the costs are $c_h$-the inventory holding cost, $c_s$-the inventory setup cost, $c_b$-cost per blocking customer, $c_o$-Waiting cost of a customer in the orbit, $c_n$-reneging cost per customer per unit time. The long run total expected cost rate is given by

\[
TC(s, S) = c_h \rho_I + c_s \rho_R + c_b \rho_L + c_o \rho_o + c_n \rho_g
\]

5 Conclusion

In this paper, we discussed continuous review retrial inventory system with impatient customers and search customers from the orbit. The joint distribution of the number of customers in the orbit and
the inventory level is obtained in the steady state case. Various system performance measures are derived and the total expected cost rate is derived under a suitable cost structure.

References


