Gracefulness Of The Join Of cycle With Different Graphs

J. Jeba Jesintha¹ and K. Subashini ²
¹PG Department of Mathematics
Women’s Christian College, Chennai
jjjesintha_75@yahoo.com
²Department of Mathematics
Jeppiaar Engineering College, Chennai
k.subashinirajan@gmail.com

Abstract

A graceful labeling of a graph $G$ with $q$ edges is an injection $f : V(G) \rightarrow \{0, 1, 2, ...q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with the vertices $u$ and $v$ is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In this paper, we prove that the cycle of graphs $C(m \circ H)$ where $H$ denotes the caterpillar, $C(m \circ C)$ where $C$ denotes the comb graph, $C(m \circ P)$ where $P$ denotes the path graph and $C(m \circ T)$ where $T$ denotes the coconut tree are graceful when $m \equiv 0, 3$ (mod 4).

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1 Introduction

The most famous and challenging graph labeling method is the graceful labeling of graphs introduced by Rosa [8] in 1967. A graceful labeling of a graph $G$ with $q$ edges is an injection $f :$
V(G) → {0, 1, 2, ..., q} with the property that the resulting edge labels are also distinct, where an edge incident with the vertices u and v is assigned the label |f(u) − f(v)|. A graph which admits a graceful labeling is called a graceful graph. A variety of graphs and families of graphs are known to be graceful for the past five decades. Caterpillars are proved to be graceful by Rosa [8]. Morgan [7] has shown that all lobsters with perfect matchings are graceful. Hrnčiar and Haviar [5] have shown that all trees of diameter five are graceful. Golomb [3] has proved that the complete bipartite graph K_m,n is graceful. Rosa [8] showed that the n-cycle C_n is graceful if and only if n ≡ 0 or 3 (mod 4). Wheels W_n = C_n + K_1 is graceful [4]. Helms are shown to be graceful [1]. Let G_1, G_2, G_3, ..., G_n be n ≥ 2 copies of a graph G. Then the graph G(n) obtained by adding an edge to G_i and G_{i+1}, i = 1, 2, ..., (n − 1) is called the path-union of n copies of the graph G [6]. Kaneria et al. [7] have proved that the path union of complete bipartite graphs is graceful. For an exhaustive survey on graceful graphs refer to the dynamic survey by Gallian [2].

For a cycle C_m, if each vertex of C_m is replaced by connected graphs G_1, G_2, G_3, ..., G_m then the resulting graph is known as cycle of graphs and is denoted as C(G_1, G_2, G_3, ..., G_m). Note that if G_1 = G_2 = ... = G_m = G, then the cycle of graphs is denoted by C(m ◦ G).

In this paper, we prove that the cycle of graphs C(m ◦ H) where H denotes the caterpillar, C(m ◦ C) where C denotes the comb graph, C(m ◦ P) where P denotes the path graph and C(m ◦ T) where T denotes the coconut tree are graceful when m ≡ 0, 3 (mod 4).

2 Main Results

In this section we first recall the definition for caterpillar, coconut tree. Later we prove that the cycle of graphs C(m ◦ H) where H denotes the caterpillar, C(m ◦ C) where C denotes the comb graph, C(m ◦ P) where P denotes the path graph and C(m ◦ T) where T denotes the coconut tree are graceful when m ≡ 0, 3 (mod 4).

Definition 1:
A caterpillar is a tree, the removal of whose pendant vertices produces a path called the spine of the caterpillar.

**Definition 2:**

A coconut tree $CT(m,n)$ is the graph obtained from the path $P_m$ by appending $n$ new pendant edges at an end vertex of $P_m$.

**Theorem 1.** The graph $C(m \circ H)$ where $H$ denotes the caterpillar is graceful.

**Proof.** Let $G$ be a cycle $C_m$ with $m$ vertices that are denoted as $v_1, v_2, ..., v_m$ in the anticlockwise direction. Let $H$ be the caterpillar graph whose path vertices are denoted as $u_1, u_2, ..., u_n$ as shown in Figure 1. The pendant vertices in the caterpillar are denoted as $u_{1,1}, u_{1,2}, ..., u_{1,k_1}, u_{2,1}, u_{2,2}, ..., u_{2,k_2}, ..., u_{n,1}, u_{n,2}, ..., u_{n,k_n}$. Let $k_1, k_2, ..., k_n$ be the number of pendant edges attached to each path vertex of the caterpillar.

**Figure 1: The graphs $G$ and $H$**

Let $C(m \circ H)$ be the cycle of graphs that has been obtained by replacing each vertex of the cycle $C_m$ by the graph $H$ as shown in Figure 2. In other words each vertex $v_i$ for $1 \leq i \leq m$ is identified with the last path vertex $u_n$ of the caterpillar graph $H$. As the $m$ copies of $H$ are attached to the cycle $C_m$ by the identification of $v_i$'s of $C_m$ with $u_n$ of each of the $m$ copies of $H$. We shall denote these identified vertices as $v_1 = u_{1,n}$ for the first copy of $H$, $v_2 = u_{2,n}$ for the second copy of $H$, $v_m = u_{m,n}$ for the $m^{th}$ copy of $H$. In general $v_i = u_{i,n}$ for $1 \leq i \leq m$ for the $i^{th}$ copy of $H$. Now we rename the...
path vertices in the first copy of $H$ in $C(m \circ H)$ as $u_1^1, u_2^1, \ldots, u_n^1$ and $u_{1,1}^1, u_{1,2}^1, \ldots, u_{1,k_1}^1, u_{2,1}^1, u_{2,2}^1, \ldots, u_{2,k_2}^1, \ldots, u_{n,1}^1, u_{n,2}^1, \ldots, u_{n,k_n}^1$ denotes the pendant vertices. The vertices in the second copy of $H$ are renamed as $u_1^2, u_2^2, \ldots, u_n^2$ and the pendant vertices are denoted as $u_{1,1}^2, u_{1,2}^2, \ldots, u_{1,k_1}^2, u_{2,1}^2, u_{2,2}^2, \ldots, u_{2,k_2}^2, \ldots, u_{n,1}^2, u_{n,2}^2, \ldots, u_{n,k_n}^2$. This continues and $u_1^m, u_2^m, \ldots, u_n^m$ denotes the vertices in the last copy of $H$ and $u_{1,1}^m, u_{1,2}^m, \ldots, u_{1,k_1}^m, u_{2,1}^m, u_{2,2}^m, \ldots, u_{2,k_2}^m, \ldots, u_{n,1}^m, u_{n,2}^m, \ldots, u_{n,k_n}^m$ denotes the pendant vertices. Thus $u_i^j$ represent the path vertices in the $j^{th}$ copy of $H$ for $1 \leq i \leq m, 1 \leq j \leq n$ and $u_{j,s}^i$ represents the pendant vertices for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq s \leq k_j$.

**Figure 2: The graph $C(m \circ H)$**

If $p$ denotes number of vertices in $C(m \circ H)$ then $p =$
$m[N + n]$ and if $q$ denotes the number of edges in $C(m \circ H)$ then $q = m[N + n]$ where $N = k_1 + k_2 + ... + k_n$. Also note that the theorem is proved for $m \equiv 0, 3 (mod \, 4)$.

The vertices of $C(m \circ H)$ are labeled as follows depending on the parameter $m$.

Labels for the vertices $u^i_j$ are given below for $1 \leq i \leq m, 1 \leq j \leq n$

$$f(u^i_{2j-1}) = q - (i - 1)(N + n)$$
$$- \sum_{l=2}^{i}(k_{2l-2} + 1) \quad for \quad 1 \leq i \leq \left\lceil \frac{m}{4} \right\rceil, 1 \leq j \leq \left\lfloor \frac{q}{7} \right\rfloor$$

$$f(u^i_{2j-1}) = q - 1 - (i - 1)(N + n)$$
$$- \sum_{l=2}^{i}(k_{2l-2} + 1) \quad for \quad \left\lceil \frac{m}{4} \right\rceil < i \leq \left\lceil \frac{m}{7} \right\rceil, 1 \leq j \leq \left\lfloor \frac{q}{7} \right\rfloor$$

$$f(u^i_{2j}) = (N + n - 1) + (i - 1)(N + n)$$
$$- \sum_{l=2}^{i} k_{2l-2} \quad for \quad 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor, 1 \leq j \leq \left\lfloor \frac{q}{7} \right\rfloor$$

$$f(u^i_{2j}) = k_1 + (i - 1)(N + n)$$
$$+ \sum_{l=2}^{i} k_{2l-2} \quad for \quad 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor, 1 \leq j \leq \left\lfloor \frac{q}{7} \right\rfloor$$

$$f(u^i_{2j}) = q - N + 1 - (i - 1)(N + n)$$
$$+ \sum_{l=2}^{i} (k_{2l-1} + 1) \quad for \quad 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor, 1 \leq j \leq \left\lfloor \frac{q}{7} \right\rfloor$$

$$f(u^i_{2j}) = q - N - (i - 1)(N + n)$$
$$+ \sum_{l=2}^{i} (k_{2l-1} + 1) \quad for \quad \left\lceil \frac{m}{4} \right\rceil < i \leq \left\lceil \frac{m}{7} \right\rceil, 1 \leq j \leq \left\lfloor \frac{q}{7} \right\rfloor$$

Labels for the vertices $u^i_j s$ for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq s \leq k_i$ are given below

$$f(u^i_{j,s}) = f(u^i_{js}) - s \quad for \quad 1 \leq i \leq m, \quad 1 \leq j \leq \left\lfloor \frac{m}{4} \right\rfloor, 1 \leq s \leq k_i$$

$$f(u^i_{j,s}) = f(u^i_{j,s}) - s \quad for \quad 1 \leq i \leq m, \quad 1 \leq j \leq \left\lceil \frac{m}{7} \right\rceil, 1 \leq s \leq k_i$$

From the above definition it is clear that all the vertex labels are distinct. The edge labels can be computed from the above vertex labels and are also found to be distinct from 1 to $q$.

Therefore, The graph $C(m \circ H)$ where $H$ denotes the caterpillar is graceful for $m \equiv 0, 3 (mod \, 4)$. The theorem is illustrated below in Figure 3.

**Illustration:** Here $m = 4, n = 4, k_1 = 4, k_2 = 5, k_3 = 6, k_4 = 3, N = 18, q = 88$
Corollary 1:

The graph $C(m \circ C)$ where $C$ denotes the comb graph is graceful.

**Proof:**

In the above theorem substituting $k_1 = k_2 = \ldots = k_n = 1$ we get the comb graph.

Corollary 2:

The graph $C(m \circ P)$ where $P$ denotes the path graph is graceful.

**Proof:**

In the above theorem substituting $k_1 = k_2 = \ldots = k_n = 0$ we get the path graph.
Corollary 3:
The graph $C(m \circ T)$ where $T$ denotes the coconut tree is graceful.

Proof:
In the above theorem substituting $k_1 = k_2 = ... = k_{n-1} = 0, k_n = k_n$ we get the coconut tree.

\[\square\]

3 Conclusion

In this paper we have proved that the cycle of graphs $C(m \circ H)$ where $H$ denotes the caterpillar, $C(m \circ \mathcal{C})$ where $\mathcal{C}$ denotes the comb graph, $C(m \circ P)$ where $P$ denotes the path graph and $C(m \circ T)$ where $T$ denotes the coconut tree are graceful when $m \equiv 0, 3(\text{mod } 4)$. Further we intend to prove this result for some other connected graphs.

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References


