Embedding BC Graphs into Combs

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Abstract
In this paper, we determine the wirelength of embedding BC graphs $X_n$ into combs $CO(2^{n-1})$.

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1 Introduction
The hypercube network is considered as the most popular interconnection network for its attractive properties like regularity, symmetry and strong connectivity. Many hypercube variants, such as crossed cube, Mobius cube, twisted cube have been proposed individually to achieve diameter less than the diameter of hypercube. Hypercubes and these three variants possess common characteristics like recursive constructability and bijective connection. Based on these characters, Fan and He proposed the definition of $n$-dimensional bijective connection graphs (in brief, BC graphs) [1]. It was proved that the hypercubes and these three hypercube variants namely crossed cubes, Mobius cubes and twisted cubes are all BC graphs. It is convenient that once a property was proved on BC graphs, the individual kind of BC graphs will inherit that property. Later, locally twisted cubes and spined cubes are also considered as BC graphs. In this paper, we compute the wirelength of embedding BC graphs into combs.

2 Basic concepts
In this section we give the basic definitions and preliminaries related to wirelength problems.
**Definition 1.** [2] Let $G$ and $H$ be finite graphs. An embedding $\phi = (f, P_f)$ of $G$ into $H$ is defined as follows:

1. $f$ is a one-to-one map from $V(G) \to V(H)$
2. $P_f$ is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v) \text{ is a path in } H \text{ between } f(u) \text{ and } f(v) \text{ for } (u,v) \in E(G)\}$.

For brevity, we denote the pair $(f, P_f)$ as $f$.

**Definition 2.** Let $f : G \to H$ be an embedding. For $e \in E(H)$, let $EC_f(e)$ denote the number of edges $(u,v)$ of $G$ such that $e$ is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in $H$. In other words,

$$EC_f(e) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}|.$$  

Then the edge congestion of $f : G \to H$ is $EC_f(G,H) = \max EC_f(e)$ where the maximum is taken over all edges $e$ of $H$. The edge congestion of $G$ into $H$ is defined as $EC(G,H) = \min EC_f(G,H)$, where the minimum is taken over all embeddings $f : G \to H$.

If $S$ is any subset of $E(H)$, then $EC_f(S) = \sum_{e \in S} EC_f(e)$.

The congestion problem is to determine that embedding whose congestion is $EC(G,H)$. The congestion problem is NP-complete [3].

**Definition 3.** [4] The wirelength of an embedding $f$ of $G$ into $H$ is given by

$$WL_f(G,H) = \sum_{(u,v) \in E(G)} d_H(f(u), f(v)) = \sum_{e \in E(H)} EC_f(e)$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u,v)$ in $H$.

The wirelength of $G$ into $H$ is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings $f$ of $G$ into $H$.

The wirelength problem [4, 2, 5, 6] of a graph $G$ into $H$ is to find an embedding of $G$ into $H$ that induces the minimum wirelength $WL(G,H)$. The wirelength problem is NP-complete [3].
Definition 4. [7] Let $G$ be a graph and $A \subseteq V(G)$. Denote
$$I_G(A) = \{(u, v) \in E(G) : u, v \in A\}, \quad I_G(m) = \max_{A \subseteq V(G), |A| = m} |I_G(A)|$$
and
$$\theta_G(A) = \{(u, v) \in E(G) : u \in A, v \notin A\}, \quad \theta_G(m) = \min_{A \subseteq V(G), |A| = m} |\theta_G(A)|.$$  

For a given $m$, where $m = 1, 2, \ldots, n$, we consider the problem of finding a subset $A$ of vertices of $G$ such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal [8]. Moreover, for a regular graph $G$, $I_G$ and $\theta_G$ are equivalent in the sense that a solution for one also becomes a solution for the other [8]. The problem of finding $I_G$ is called maximum subgraph problem [3].

Lemma 5. (Modified Congestion Lemma) [9] Let $G$ and $H$ be any arbitrary graphs and let $f$ be an embedding of $G$ into $H$. Let $S$ be an edge cut of $H$ such that the removal of edges of $S$ leaves $H$ into 2 components $H_1$ and $H_2$ and let $G_1$ and $G_2$ be subgraphs of $G$ induced by $f^{-1}(H_1)$ and $f^{-1}(H_2)$ respectively. Furthermore, suppose $S$ satisfies the following conditions:

(i) For every edge $(a, b) \in E(G_i), i = 1, 2, P_f(a, b)$ has no edges in $S$.

(ii) For every edge $(a, b)$ in $E(G)$ with $a \in V(G_1)$ and $b \in V(G_2)$, $P_f(a, b)$ has exactly one edge in $S$.

(iii) $V(G_1)$ and $V(G_2)$ are optimal sets.

Then $EC_f(S)$ is minimum, that is, $EC_f(S) \leq EC_g(S)$ for any other embedding $g$ of $G$ into $H$. Further $EC_f(S) = \sum_{v \in V(G_1)} \deg_G(v) - 2|E(G_1)| = \sum_{v \in V(G_2)} \deg_G(v) - 2|E(G_2)|$.

Lemma 6. (Edge Congestion Lemma)[10] Let $G$ and $H$ be graphs of the same order and let $f : G \rightarrow H$ be an embedding. Let $S$ be an edge cut of $H$ satisfying the conditions of the modified congestion lemma. Then $EC(G, H) \geq \frac{EC_f(S)}{|S|}$.
Lemma 7. (Partition Lemma) \[4\] Let \( f : G \to H \) be an embedding. Let \( \{ S_1, S_2, \ldots, S_p \} \) be a partition of \( E(H) \) such that each \( S_i \) is an edge cut of \( H \) satisfying the conditions of Congestion Lemma. Then
\[
WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i).
\]

3 Wirelength of embedding BC graphs into combs

In this section, we compute the Wirelength of embedding BC graphs into combs.

We begin with the definition of a BC graph.

Definition 8. \[11\] Let \( G_1 = (V_1, E_1), G_2 = (V_2, E_2) \) be two vertex disjoint graphs of the same order. A bijective connection between \( G_1 \) and \( G_2 \) is defined as an edge set \( E = \{(v, \phi(v))\} \), where \( \phi : V_1 \to V_2 \) is a bijection. Let \( G = G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E) \).

An \( n \)-dimensional BC graph is an \( n \)-regular graph with \( 2^n \) nodes and \( n2^n - 1 \) edges. The family of \( n \)-dimensional BC graphs is denoted by \( X_n \).

The graph \( X_n \) can be recursively defined as follows:

Definition 9. \[11\] The one-dimensional BC network is a complete graph \( K_2 \) on two vertices. The family of one-dimensional BC networks is denoted by \( X_1 = \{K_2\} \). When \( n \geq 2 \), \( G \in X_n \) if and only if \( G = G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup E) \) for some \( G_1, G_2 \in X_{n-1} \).

For different bijections, there exist a large number of different BC graphs. Figure 1 demonstrates two four-dimensional BC graphs with Figure 1 (a) representing the hypercube \( Q^4 \) of dimension 4 and Figure 1 (b) representing the spined cube \( SQ^4 \) of dimension 4.

Lemma 10. \[11\] Let \( G \) be an \( n \)-dimensional BC graph. For an integer \( m \) which can be uniquely written as \( m = \sum_{k=0}^{r-1} 2^{l_k} \) for some nonnegative integers \( r \) and \( l_0 > l_1 > \ldots > l_{r-1} \), the maximum
Figure 1: (a) Hypercube $Q^4$ and (b) spined cube $SQ^4$

The number of edges joining vertices from a set of $m$ vertices is $IG(m) = \sum_{k=0}^{r-1} (k/2 + k)2^k$ where $1 \leq m \leq 2^n$, $n \geq 1$.

**Remark.** An ordering $L$ is said to be an optimal order of $G$, if $L_i = \{1, 2, \ldots, i\}$ is an optimal set in the sense that the vertices labelled 1, 2, $\ldots$, $i$ in $G$ induce a subgraph with maximum number of edges among all subgraphs of $G$ on $i$ vertices, $1 \leq i \leq |V(G)|$.

**Definition 11.** A comb is obtained from a path (called the backbone or spine) $v_1, v_2, \ldots, v_n$ by attaching a pendant edge at each $v_i$, $1 \leq i \leq n$ and is denoted by $CO(n)$.

A tree is called a caterpillar if the deletion of vertices of degree one leaves a path. This path is called its spine. If there is exactly one vertex adjacent to each of the spine vertices, then the caterpillar is called a comb.

**Algorithm**

**Input:** A BC graph $X_n$ with optimal order and a comb $CO(2^n-1)$ on $2^n$ vertices.

**Algorithm:** Label the vertices of $X_n$ using optimal order and the vertices of $CO(2^n-1)$ as $0, 1, 2, \ldots, 2^n - 1$ in inorder traversal. Let $f(x) = x$ for all $x \in V(X_n)$ and for $(a, b) \in E(X_n)$, let $P_f(a, b)$ be a path between $f(a)$ and $f(b)$ in $CO(2^n-1)$. 
Output : An embedding $f$ with minimum wirelength.

Proof of correctness : We assume that the labels represent the vertices to which they are assigned.

For $i = 0, 1, 2, \ldots, n - 2$, let $S_i = \{(2i + 1, 2i + 3)\}$ be the cut edge of $CO(2^n - 1)$ such that $S_i$ disconnects $CO(2^n - 1)$ into two components $H_{i1}$ and $H_{i2}$, where $V(H_{i1}) = \{0, 1, 2, \ldots, 2i + 1\}$. Let $G_{i1}$ and $G_{i2}$ be the inverse images of $H_{i1}$ and $H_{i2}$ under $f$ respectively. By Remark, $G_{i1}$ is an optimal set in $G$.

For $j = 0, 1, 2, \ldots, n - 1$, let $S'_j = \{(2j, 2j + 1)\}$ be the cut edge of $CO(2^n - 1)$ such that $S'_j$ disconnects $CO(2^n - 1)$ into two components $H'_{j1}$ and $H'_{j2}$, where $V(H'_{j1})$ is a single vertex. Let $G'_{j1}$ and $G'_{j2}$ be the inverse images of $H'_{j1}$ and $H'_{j2}$ under $f$ respectively. Clearly, $G'_{j1}$ is an optimal set in $G$.

Now, \{S_0, S_1, \ldots, S_{n-2}, S'_0, S'_1, \ldots, S'_{n-1}\} is an edge partition of $E(CO(2^n - 1))$. Moreover, each edge cut $S_i$ as well as $S'_j$ satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ and $EC_f(S'_j)$ is minimum for $i = 0, 1, 2, \ldots, n - 2$ and $j = 0, 1, 2, \ldots, n - 1$. The Partition Lemma implies that $WL_f(X_n, CO(2^n - 1))$ is minimum.

Theorem 12. The wirelength of embedding $n$-dimensional BC graph, $X_n$ into $CO(2^n - 1)$, is

$$WL(X_n, CO(2^n - 1)) = n^2 + n \sum_{i=0}^{n-2} \sum_{k=0}^{r-1} y^k - 2 \sum_{i=0}^{n-2} \sum_{k=0}^{r-1} ((l_k/2 + k)2^k).$$

Proof. Following the notation used in Algorithm, we have
For $i = 0, 1, 2, \ldots, n - 2$, let $2^i$ be uniquely written as $\sum_{k=0}^{r-1} 2^k$. Then

$$EC_f(S_i) = n(2^i) - 2IG(2^i)$$

For $j = 0, 1, 2, \ldots, n - 1$, let $EC_f(S'_j) = n$

Hence,

$$WL(X_n, CO(2^{n-1})) = \sum_{i=0}^{n-2} EC_f(S_i) + \sum_{j=1}^{n-1} EC_f(S'_j)$$

$$= n\sum_{i=0}^{n-2} \sum_{k=0}^{r-1} 2^k - 2\sum_{i=0}^{n-2} \sum_{k=0}^{r-1} ((l_k/2 + k)2^k) + \sum_{j=0}^{n-1} n$$

$$= n^2 + n\sum_{i=0}^{n-2} \sum_{k=0}^{r-1} 2^k - 2\sum_{i=0}^{n-2} \sum_{k=0}^{r-1} ((l_k/2 + k)2^k)$$

$\square$

4 Conclusion

In this paper, we solve the wirelength of embedding BC graphs into combs.

References


