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# Modified Ostrowski Methods For Solving Nonlinear Equation

S. Parimala[1], M. Kalyanasundaram[2] and J. Jayakumar[1]
[1]Pondicherry Engineering College, Pondicherry, India
[2]Saveetha Engineering College, Chennai, India
parimalas@pec.edu, kalyan742@pec.edu, jjayakumar@pec.edu

**Abstract**

In this paper, we have obtained modified Ostrowski's methods for solving nonlinear equation with convergence order eight and six. The efficiency indices of the proposed methods are 1.516 and 1.565, which are better than the efficiency index of Newton's method (1.414). Also it is observed from the numerical illustrations, the proposed methods take less number of iterations than Newton's method. Few other methods are compared with the proposed methods, where the number of iterations for those methods are either same or more than the presented methods. Some examples are given to illustrate the performance of the new methods.

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**Key Words:**Non-linear equation, Higher-order method, Optimal order, Kung-Traub conjecture, Multi-point iteration.

## 1 Introduction

The numerical solution of a nonlinear equation $f(x) = 0$ is a fundamental task in scientific computation. The most famous approach is probably Newton's method (NM):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

It uses two function evaluations $f$ and $f'$ to achieve second-order convergence. However, there are many iterative methods available in the literature, some are specifically developed and analyzed for solving nonlinear equations that improve classical methods, such as Newton's method (NM) and Halley's iteration method etc. Some of the iterative methods are given below: Halley's iteration method is given by

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}. \tag{2}$$

Arithmetic mean Newton's method (AM) with cubic convergence (see [15]) is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}. \tag{3}$$

This method is of order three with three function evaluations per full iteration having efficiency index (EI) $= 1.442$. The following method is known as Harmonic mean Newton's method (HM) with cubic convergence [4]:

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right). \tag{4}$$

The Newton-Steffensen method (SM) with cubic convergence [13] is given by

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f'(x_n)[f(x_n) - f(y_n)]}. \tag{5}$$

Halley's iteration method has third order convergence with three function evaluations. Obviously, $f''$ is difficult to calculate and computationally more costly. Therefore, $f''$ in Equation (2) is approximated using the finite difference; still, the convergence order and total number of function evaluations are maintained [3]. Such a third order method similar to Equation (2) after approximating $f''$ in Halley's iteration method is given below:

$$y_n = x_n - \beta\frac{f(x_n)}{f'(x_n)}, \ x_{n+1} = x_n - \frac{2\beta f(x_n)}{(2\beta - 1)f'(x_n) + f'(y_n)}, \ \beta \neq 0. \tag{6}$$

The efficiency index (EI) of an iterative method is measured using the formula $p^{\frac{1}{d}}$, where $p$ is the local order of convergence and $d$ is the number of function evaluations per full iteration cycle [11]. Kung–Traub [7] conjectured that the order of convergence of any multi-point without memory method with $d$ function evaluations cannot exceed the bound $2^{d-1}$, the "optimal order". Newton's method is an optimal second order method, whereas Jarratt's method [6] is an example of an optimal fourth order method. Recently, some optimal and non-optimal multi-point iterative methods have been developed in [1, 2, 8, 9, 12, 14] and the references therein.

In this paper, we present two new three-step iterative methods for solving nonlinear equations with eighth and sixth order convergence by introducing Newton's step in Ostrowski's method [11] and approximating the derivative to improve the efficiency index by reducing the number of functional evaluations per step. In Section 2 new methods are given. In Section 3, we establish the convergence order of the new methods. Section 4 confirms the theoretical results from computational tests and compare the new variants with classical methods. Finally, Section 5 draws conclusion on our work.

## 2 Construction of the methods

Let us consider the following fourth order method,

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = x_n - \frac{f(x_n)}{f'(x_n)}\left[\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\right], \tag{7}$$

which is the well known Ostrowski's method [11].

**New Eighth order method (M1)**: This method is obtained by introducing one more Newton's step in the method (7), then we get

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{8}$$

Thus, we obtain a new three-step iterative method (8), which has convergence order eight with five function evaluations. The efficiency of the method (8) is found to be $EI = 1.516$, which shows that it is higher than Newton's method (1.414).

**New Sixth order method (M2)**: In order to improve the efficiency index of the method (8), we modify it by using divided difference and approximate $f'(z_n)$ by a combination of already computed function values as follows:

$$f'(z_n) \approx \frac{f(z_n) - f(y_n)}{z_n - y_n} = f[y_n, z_n]. \tag{9}$$

Using (9) in method (8), we obtain the following method:

$$x_{n+1} = z_n - \frac{f(z_n)}{f[y_n, z_n]}. \tag{10}$$

This method (10) has $EI = 1.565$ which is better than Newton's method and the method (8). Thus, we have obtained a new three-step iterative method (10) which has convergence order six with four function evaluations.

# 3 Convergence Analysis

**Theorem 1.** *Let a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a simple root $x^*$ in the open interval $D$. If $x_0$ is chosen in a sufficiently small neighborhood of $x^*$, then the method* (8) *is of local eighth order convergence.*

*Proof.* Let $e_n = x_n - \alpha$. Using the Taylor series, then we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + \ldots], \tag{11}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + \ldots], \tag{12}$$

where $c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}$, $k \geq 2$. Now substituting (11), (12) in (3), then

$$\begin{aligned} y_n = {} & \alpha + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (-8c_2^4 \\ & +20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 + (16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 \\ & +28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)e_n^6 - 2(16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 \\ & +6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8c_6) - 3c_7)e_n^7 + \ldots \end{aligned} \tag{13}$$

Expanding $f(y_n)$ about $\alpha$ and taking into account (13), we have

$$\begin{aligned} f(y_n) = {} & f'(\alpha)[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 - 2(6c_2^4 \\ & - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 + (28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 - 17c_3c_4 \\ & + c_2(37c_3^2 - 13c_5) + 5c_6)e_n^6 - 2(32c_2^6 - 103c_2^4c_3 - 9c_3^3 + 52c_2^3c_4 + 6c_4^2 \\ & +c_2^2(80c_3^2 - 22c_5) + 11c_3c_5 + c_2(-52c_3c_4 + 8c_6) - 3c_7)e_n^7 + \ldots]. \end{aligned} \tag{14}$$

Now, using (11), (12) and (14) in (7), then we have

$$\begin{aligned} z_n = {} & \alpha + (c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 + (10c_2^5 - 30c_2^3c_3 \\ & + 12c_2^2c_4 - 7c_3c_4 + 3c_2(6c_3^2 - c_5))e_n^6 - 2(10c_2^6 - 40c_2^4c_3 - 6c_3^3 + 20c_2^3c_4 \\ & + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) + 5c_3c_5 + c_2(-26c_3c_4 + 2c_6))e_n^7 + \ldots. \end{aligned} \tag{15}$$

Expanding $f(z_n)$ about $\alpha$ and taking into account (15), we have

$$f(z_n) = f'(\alpha)[(c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 \\ +(10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 + 12c_2^2c_4 - 7c_3c_4 - 3c_2c_5)e_n^6 + \ldots], \tag{16}$$

$$f'(z_n) = f'(\alpha)[1 + 2c_2^2(c_2^2 - c_3)e_n^4 - (4c_2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4))e_n^5 \\ + (2c_2(10c_2^5 - 30c_2^3c_3 + 12c_2^2c_4 - 7c_3c_4 + 3c_2(6c_3^2 - c_5)))e_n^6 + \ldots]. \tag{17}$$

Using (15), (16) and (17) in (8), we get $e_{n+1} = \left(c_2^3 - c_3^2\right)e_n^8 + O(e_n^9)$, which shows eighth order convergence. □

**Theorem 2.** *Let a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a simple root $x^*$ in the open interval $D$. If $x_0$ is chosen in a sufficiently small neighborhood of $x^*$, then the method* (10) *is of local sixth order convergence.*

*Proof.* Replacing $f'(z_n)$ in (8) using the formula $f'(z_n) \approx \frac{f(z_n)-f(y_n)}{z_n-y_n}$, we obtain

$$\frac{f(z_n) - f(y_n)}{z_n - y_n} = f'(\alpha)[(1 + c_2^2e_n^2 + (-2c_2^3 + 2c_2c_3))e_n^3 + (c_2(5c_2^3 \\ -7c_2c_3 + 3c_4))e_n^4 - (4c_2)(3c_2^4 - 6c_2^2c_3 + c_3^2 + 3c_2c_4 - c_5)e_n^5 + \ldots]. \tag{18}$$

Using (18) in (10), we get $e_{n+1} = \left(c_2^5 - c_2^3c_3\right)e_n^6 + O(e_n^7)$. □

# 4 Computational examples

The contribution given in Section 3 is supported here through numerical work. The effectiveness of the new methods are compared to the existing methods such as Newton's method **(NM)**, Arithmetic mean Newton method **(AM)** [15], Harmonic mean Newton method **(HM)** [4], Newton-Steffensen method **(SM)** [13], method proposed by Chun et al **(CH)** [2], method proposed by Hu et al **(HF)** [5] and method proposed by Noor et al. **(NK)** [10] . The methods presented in this paper are denoted as **M1** and **M2**. Numerical results on some test functions are given for the proposed methods along with some existing methods. Numerical computations have been carried out in the MATLAB software. The computational order of convergence is given by

$$\rho = \frac{\ln |(x_N - x_{N-1})/(x_{N-1} - x_{N-2})|}{\ln |(x_{N-1} - x_{N-2})/(x_{N-2} - x_{N-3})|}.$$

The test functions and their simple zeros are given below:

$f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}$, $x^* = -0.7848959876612125352...$

$f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, $x^* = -1.2076478271309189270...$

$f_3(x) = sin(x) + cos(x) + x$, $x^* = -0.4566247045676308244...$

$f_4(x) = (x+2)e^x - 1$, $x^* = -0.4428544010023885831...$

$f_5(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}$, $x^* = 0.4099920179891371316...$

Table 1: Comparison of Efficiency Indices (EI) and Optimality

| Methods | $p$ | $d$ | EI | Optimal/Non-optimal |
|---|---|---|---|---|
| **NM** | 2 | 2 | 1.414 | Optimal |
| **AM** | 3 | 3 | 1.442 | Non-optimal |
| **HM** | 3 | 3 | 1.442 | Non-optimal |
| **SM** | 3 | 3 | 1.442 | Non-optimal |
| **CH** | 6 | 4 | 1.565 | Non-optimal |
| **HF** | 7 | 5 | 1.475 | Non-optimal |
| **NK** | 7 | 5 | 1.475 | Non-optimal |
| **M1** | 8 | 5 | 1.516 | Non-optimal |
| **M2** | 6 | 4 | 1.565 | Non-optimal |

Table 1 shows the efficiency indices of the new methods with some known methods. The results of comparisons are given in Table 2. We observe that if the initial points are very close to the root, then all the methods take less number of iterations and produce least error. Also, the present new methods **M1** and **M2** have better efficiency when compared with other methods. For the test function $f_2(x)$, **HF** method diverges, whereas it converges for the proposed new methods.

# 5 Conclusions

We have composed known Ostrowski's method with Newton's method obtaining an eighth order method and then replaced the derivative by a divided difference obtaining a sixth order method with efficiency index 1.565. It is clear that our new methods require five function evaluations and four function evaluations per iterative step respectively to obtain eighth order convergence and sixth order convergence. The proposed methods **M1** and **M2** have better efficiency, whereas **M2** has the highest efficiency among the methods discussed. Moreover, the proposed new method **M2** requires less cpu time for convergence when compared with other methods.

Table 2: Numerical results for test functions

| $f(x)$ | Methods | $x_0$ | $N$ | $\rho$ | $error$ | $cpu(s)$ | $x_0$ | $N$ | $\rho$ | $error$ | $cpu(s)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_1(x)$ | **NM** | $-1.2$ | 7 | 1.99 | 1.57e-060 | 0.653 | $-0.5$ | 8 | 1.99 | 6.42e-071 | 0.518 |
| | **AM** | | 5 | 3.00 | 6.56e-052 | 0.681 | | 6 | 2.99 | 5.53e-147 | 0.780 |
| | **HM** | | 5 | 2.99 | 1.04e-051 | 0.647 | | 5 | 3.00 | 4.40e-057 | 0.799 |
| | **SM** | | 5 | 2.99 | 3.12e-072 | 0.586 | | 6 | 3.00 | 7.27e-130 | 0.710 |
| | **CH** | | 4 | 5.99 | 3.57e-124 | 0.715 | | 4 | 5.99 | 7.65e-67 | 0.744 |
| | **HF** | | 4 | 6.99 | 1.09e-270 | 0.691 | | 4 | 7.00 | 8.95e-064 | 0.792 |
| | **NK** | | 4 | 7.00 | 2.25e-297 | 0.726 | | 4 | 7.00 | 8.36e-201 | 0.732 |
| | **M1** | | 4 | 7.99 | 2.64e-256 | 0.713 | | 4 | 8.00 | 3.85e-254 | 0.743 |
| | **M2** | | 4 | 5.99 | 2.14e-165 | 0.628 | | 4 | 5.99 | 7.76e-111 | 0.671 |
| $f_2(x)$ | **NM** | $-1.7$ | 9 | 2.00 | 4.38e-054 | 0.568 | $-0.8$ | 9 | 2.00 | 2.13e-055 | 0.585 |
| | **AM** | | 7 | 3.00 | 4.32e-124 | 0.703 | | 7 | 3.00 | 1.03e-086 | 0.624 |
| | **HM** | | 6 | 3.00 | 1.29e-072 | 0.577 | | 6 | 3.00 | 9.90e-140 | 0.547 |
| | **SM** | | 6 | 2.99 | 2.79e-051 | 0.584 | | 7 | 3.00 | 3.55e-149 | 0.617 |
| | **CH** | | 5 | 5.99 | 2.30e-245 | 0.715 | | 4 | 5.99 | 1.34e-189 | 0.744 |
| | **HF** | | 4 | 7.05 | 1.51e-062 | 0.520 | | Div | | Div | Div |
| | **NK** | | 4 | 6.99 | 1.62e-101 | 0.584 | | 4 | 7.00 | 2.27e-074 | 0.550 |
| | **M1** | | 4 | 8.00 | 1.28e-164 | 0.676 | | 4 | 8.00 | 1.13e-276 | 0.637 |
| | **M2** | | 4 | 6.00 | 1.55e-63 | 0.670 | | 4 | 6.00 | 1.65e-95 | 0.652 |
| $f_3(x)$ | **NM** | 0.5 | 7 | 1.99 | 1.08e-055 | 0.433 | $-1.5$ | 7 | 1.99 | 1.16e-058 | 0.405 |
| | **AM** | | 5 | 2.99 | 2.75e-066 | 0.494 | | 6 | 2.99 | 9.23e-149 | 0.517 |
| | **HM** | | 6 | 2.99 | 1.62e-137 | 0.512 | | 6 | 2.99 | 5.06e-143 | 0.531 |
| | **SM** | | 5 | 3.00 | 1.30e-059 | 0.559 | | 5 | 2.99 | 1.39e-107 | 0.459 |
| | **CH** | | 4 | 5.99 | 7.81e-78 | 0.599 | | 4 | 5.99 | 2.92e-118 | 0.544 |
| | **HF** | | 4 | 7.00 | 3.06e-114 | 0.588 | | 4 | 7.00 | 1.94e-253 | 0.490 |
| | **NK** | | 4 | 7.00 | 1.53e-208 | 0.673 | | 4 | 6.99 | 9.43e-287 | 0.558 |
| | **M1** | | 4 | 7.99 | 1.95e-283 | 0.514 | | 4 | 7.99 | 1.74e-181 | 0.550 |
| | **M2** | | 4 | 5.99 | 2.03e-137 | 0.472 | | 4 | 5.99 | 9.29e-162 | 0.431 |
| $f_4(x)$ | **NM** | $-0.2$ | 7 | 1.99 | 7.24e-052 | 0.706 | $-0.9$ | 8 | 1.99 | 3.10e-058 | 0.448 |
| | **AM** | | 5 | 2.99 | 2.67e-061 | 0.625 | | 6 | 2.99 | 1.48e-093 | 0.490 |
| | **HM** | | 5 | 3.00 | 2.38e-082 | 0.721 | | 5 | 2.99 | 1.83e-076 | 0.435 |
| | **SM** | | 5 | 2.99 | 1.75e-065 | 0.650 | | 6 | 2.99 | 1.80e-109 | 0.483 |
| | **CH** | | 4 | 5.99 | 3.90e-156 | 0.638 | | 4 | 5.99 | 1.092e-70 | 0.618 |
| | **HF** | | 4 | 6.99 | 1.86e-274 | 0.605 | | 5 | 7.00 | 4.10e-275 | 0.542 |
| | **NK** | | 4 | 6.99 | 3.96e-305 | 0.728 | | 4 | 7.00 | 3.86e-171 | 0.504 |
| | **M1** | | 3 | 7.97 | 1.31e-58 | 0.439 | | 4 | 7.99 | 1.41e-298 | 0.616 |
| | **M2** | | 4 | 5.99 | 6.62e-190 | 0.561 | | 4 | 5.99 | 1.31e-117 | 0.483 |
| $f_5(x)$ | **NM** | 0.8 | 8 | 1.99 | 3.21e-072 | 0.456 | 0.2 | 8 | 1.99 | 8.25e-076 | 0.398 |
| | **AM** | | 6 | 3.00 | 1.70e-136 | 0.505 | | 6 | 2.99 | 2.60e-143 | 0.503 |
| | **HM** | | 5 | 2.99 | 2.35e-094 | 0.464 | | 5 | 2.99 | 1.84e-098 | 0.441 |
| | **SM** | | 5 | 3.00 | 1.84e-136 | 0.502 | | 6 | 3.00 | 2.82e-143 | 0.520 |
| | **CH** | | 4 | 5.99 | 3.46e-100 | 0.665 | | 4 | 5.99 | 3.13e-85 | 0.559 |
| | **HF** | | 4 | 6.99 | 3.55e-159 | 0.476 | | 4 | 7.00 | 2.70e-099 | 0.465 |
| | **NK** | | 4 | 6.99 | 2.50e-209 | 0.558 | | 4 | 7.00 | 1.26e-209 | 0.511 |
| | **M1** | | 4 | 7.99 | 1.22e-286 | 0.502 | | 4 | 7.99 | 5.31e-301 | 0.505 |
| | **M2** | | 4 | 5.99 | 2.37e-121 | 0.469 | | 4 | 5.99 | 2.07e-127 | 0.462 |

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