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# An $M^X$/G(a,b)/1 Queueing System With Two Fluctuating Modes Of Service Under Bernoulli Vacation Schedule For Unreliable Server and Delaying Repair

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**Abstract**

This paper deals with a batch arrival queueing system prepared with a single server providing service to a batch of customers with dissimilar service rate in two fluctuating modes of service. By using the supplementary variable technique, we derive the probability generating function of the number of customers in the queue at a random epoch under the steady state conditions. Performance measures like mean queue size has been obtained explicitly.

**Key Words: General bulk service, Two modes of service, Bernoulli vacation, Delay time, Repair time**.

**AMS Subject Classification (2010): 60K25, 90B22, 68M20**

## 1 Introduction

A comprehensive study on batch service queue was initiated with the work of Bailey [3] . Some general bulk service results have discussed by Holman et al. [6]. Ayyappan and Shymala [1] analyzed a batch arrival single server with two stage heterogeneous service, Bernoulli schedule vacation, feed back, random breakdown and delayed repair time. Recently, Madan [7] has studied the server providing general service in three fluctuating modes and also Baruah et al. [2] investigated the server providing general service in two fluctuating modes. Takagi [9] has given extensive studies on vacation models. Recently, Charan Jeet Singh et al. [4] examined an $M^X$/G/1 unreliable retrial queue with an option of additional service and Bernoulli vacation. Rajadurai et al. [8] and many others have discussed about the queueing system with unreliable server. In reality, it is not possible to start the repair process immediately due to shortage of man power or the sufficient apparatus needed for the repairs. This period of waiting time is termed as delay time. Choudhary and Lotfi [5] have discussed about unreliable server and delaying repair time.

This paper is structured as follows. In section 2 the brief description of the mathematical model. In section 3, we present the definitions whereas in section 4, the probability generating function of the queue size at a random epoch, stability condition and the mean queue size has been derived explicitly.

# 2 Mathematical description of the queueing model

To describe the required queueing model, we consider a $M^X$/G(a,b)/1 queueing system with batch arrival which follows a compound Poisson process and the bulk service is rendered under "GBSR" rule. There is a single server providing service to a batch of customers in two fluctuating modes with probabilities $p_1$ and $p_2$. We presumed that the probability of providing service in the First Mode Service (FMS) is $p_1$ and Second Mode Service (SMS) is $p_2$ (i.e. $p_1 + p_2 = 1$). After completion of any one of these two modes of service, the server will go for vacation with probability $\theta$. If not the server continue to serve the next batch, if exist, with probability $1 - \theta$. Otherwise, the server will stay in the system idle till a new batch of customers arrive. While the server is functioning with any mode of service, it may break down and is assumed to occur according to a Poisson stream with mean breakdown rates $\alpha_1$ for FMS and $\alpha_2$ for SMS. As soon as breakdown occurs, the server will not be sent to the repair process instantly and there is a delay time to start the repair process of respective mode of service.

The two modes of service time, vacation time, delay time to repair and repair time follow general distribution and the notations used for their Cumulative Distribution Function (CDF), the probability density functions (pdf) are given in Table 1.

Table 1: Some notations for distribution function

| Time | CDF | Hazard rate | pdf |
|---|---|---|---|
| $i^{th}$ mode of service | $B_i(x)$ | $\mu_i(x)$ | $b_i(v) = \mu_i(v)e^{-\int_0^v \mu_i(x)dx}$ |
| Bernoulli vacation | $V(x)$ | $\gamma(x)$ | $v(r) = \gamma(r)e^{-\int_0^r \gamma(x)dx}$ |
| Delay time to repair under $i^{th}$ mode of service | $D_i(y)$ | $\xi_i(y)$ | $d_i(u) = \xi_i(u)e^{-\int_0^u \xi_i(y)dy}$ |
| Repair under $i^{th}$ mode of service | $R_i(y)$ | $\beta_i(y)$ | $r_i(w) = \beta_i(w)e^{-\int_0^w \beta_i(y)dx}$ |

# 3 Definitions

We define $P_n^{(i)}(x,t)$ = Probability that at time $t$, the server is active providing $i^{th}$ mode of service $(i = 1, 2)$ and there are $n$ $(n \geq 0)$ customers in the queue excluding the batch being served and the elapsed service time on a batch of customers undergoing service is $x.V_n(x,t)$ = Probability that at time $t$, the server is on vacation with elapsed vacation time is $x$ and there are $n$ $(n \geq 0)$ customers in the queue. $D_n^{(i)}(x,y,t)$ = Probability that at time $t$, the server is under $i^{th}$ mode of delay $(i = 1, 2)$(before entering into the $i^{th}$ repair) with the elapsed service time on a batch of customers undergoing service is $x$, the elapsed delay time of server is $y$ and there are $n$ $(n \geq 0)$ customers in the queue.$R_n^{(i)}(x,y,t)$ = Probability that at time $t$, the server is under $i^{th}$ mode of repair $(i = 1, 2)$

(breakdown during $i^{th}$ mode of service time) with the elapsed service time on a batch of customers undergoing service is $x$, the elapsed repair time of server is $y$ and there are $n$ ($n \geq 0$) customers in the queue. $Q_r(t)$ = Probability that at time $t$, there are $r$ $(0 \leq r \leq a-1)$ customers in the system and the server is idle but available in the system.

We can set limiting probabilities for $x \geq 0$ and $n \geq 0$ and $i = 1, 2$
$\sum_{r=0}^{a-1} Q_r = \lim_{t\to\infty} Q_r(t)$; $P^{(i)}(x) = \lim_{t\to\infty} P_n^{(i)}(x,t)$
$V(x) = \lim_{t\to\infty} V_n(x,t)$; $A^{(i)}(x,y) = \lim_{t\to\infty} A^{(i)}(x,y,t)$.where$A = R, D$.
The Kolmogorov forward equations to govern the system under steady-state conditons, where $i = 1, 2$ denotes the FMS and SMS respectively

$$\frac{\partial}{\partial x}P_n^{(i)}(x) + \frac{\partial}{\partial t}P_n^{(i)}(x) + (\lambda + \mu_i(x) + \alpha_i)P_n^{(i)}(x) = \lambda(1-\delta_{n,0})\sum_{k=1}^{n} c_k P_{n-k}^{(i)}(x) + \int_0^\infty R_n^{(i)}(x,y)\beta_i(y)dy,\ n \geq 0 \tag{1}$$

$$\frac{\partial}{\partial x}V_n(x) + \frac{\partial}{\partial t}V_n(x) + (\lambda + \gamma(x))V_n(x) = \lambda(1-\delta_{n,0})\sum_{k=1}^{n} c_k V_{n-k}(x),\ n \geq 0, \tag{2}$$

$$\frac{\partial}{\partial y}D_n^{(i)}(x,y) + \frac{\partial}{\partial t}D_n^{(i)}(x,y) + (\lambda + \xi_i(y))D_n^{(i)}(x,y) = \lambda(1-\delta_{n,0})\sum_{k=1}^{n} c_k D_{n-k}^{(i)}(x,y),\ n \geq 0, \tag{3}$$

$$\frac{\partial}{\partial y}R_n^{(i)}(x,y) + \frac{\partial}{\partial t}R_n^{(i)}(x,y) + (\lambda + \beta_i(y))R_n^{(i)}(x,y) = \lambda(1-\delta_{n,0})\sum_{k=1}^{n} c_k R_{n-k}^{(i)}(x,y),\ n \geq 0, \tag{4}$$

$$-\lambda Q_r = \lambda(1-\delta_{n,0})\sum_{k=1}^{r} c_k Q_{r-k} + \int_0^\infty V_r(x)\gamma(x)dx + (1-\theta)\Big[\int_0^\infty P_r^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_r^{(2)}(x)\mu_2(x)dx\Big],\ 1 \leq r \leq a-1. \tag{5}$$

To solve the equations (1) to (5), the following boundary conditions at $x = 0$ and $y = 0$ are considered,

$$P_0^{(i)}(0) = p_i\big[\lambda \sum_{r=a}^{b}\sum_{k=0}^{a-1} c_{r-k}Q_k + \sum_{r=a}^{b}\int_0^\infty V_r(x)\gamma(x)dx + (1-\theta)\sum_{r=a}^{b}\big(\int_0^\infty P_r^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_r^{(2)}(x,t)\mu_2(x)dx\big)\big], i = 1,2 \quad (6)$$

$$P_n^{(i)}(0) = p_i\big[\lambda \sum_{k=0}^{a-1} c_{b+n-k}Q_k + \int_0^\infty V_{n+b}(x)\gamma(x)dx + (1-\theta)\big(\int_0^\infty P_{n+b}^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_{n+b}^{(2)}(x)\mu_2(x)dx\big)\big],\ n \geq 1, i = 1,2 \quad (7)$$

$$V_n(0) = \theta[\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_n^{(2)}(x)\mu_2(x)dx], \quad n \geq 0 \quad (8)$$

$$D_n^{(i)}(x,0) = \alpha_i P_n^{(i)}(x),\ n \geq 0, i = 1,2 \quad (9)$$

$$R_n^{(i)}(x,0) = \int_0^\infty D_n^{(i)}(x,y)\xi_i(y)dy,\ n \geq 0, i = 1,2. \quad (10)$$

The normalizing condition is

$$\sum_{r=0}^{a-1} Q_r + \sum_{n=0}^{\infty}\big(\int_0^\infty P_n(x)dx + \int_0^\infty V_n(x)dx + \int_0^\infty\int_0^\infty D_n(x,y)dxdy + \int_0^\infty\int_0^\infty R_n(x,y)dxdy\big) = 1 \quad (11)$$

The probability generating functions for $i = 1,2$ and $|z| \leq 1$

$$P_q^{(i)}(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x);\ V_q(x,z) = \sum_{n=0}^{\infty} z^n V_n(x)$$

$$C(z) = \sum_{n=1}^{\infty} c_n z^n; B_q^{(i)}(x,y,z) = \sum_{n=0}^{\infty} z^n B_n^{(i)}(x,y) \quad (12)$$

$$B_q^{(i)}(x,0,z) = \sum_{n=0}^{\infty} z^n B_n^{(i)}(x,0), \text{where } B = D, R$$

# 4 Queue size distribution at random epoch

By solving the above equations and using eqns (11) and (12), we get

$$P(z) = \frac{\left(\begin{array}{l} \big[\lambda \sum_{r=0}^{a-1} Q_r (C(z)z^r - z^b) + \lambda \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r (z^b - z^{n+r}) \\ + \sum_{r=0}^{b-1} (z^b - z^r) W_r \big] \times \big[\phi(z)\big(p_1 \psi_2(z)(1 - \bar{B}_1(\psi_1(z))) \\ + p_2 \psi_1(z)(1 - \bar{B}_2(\psi_2(z)))\big) + \theta \psi_1(z)\psi_2(z) \\ (1 - \bar{V}(\phi(z))) \big(p_1 \bar{B}_1(\psi_1(z)) + p_2 \bar{B}_2(\psi_2(z))\big) \\ + p_1 \alpha_1 \psi_2(z)(1 - \bar{B}_1(\psi_1(z)))(1 - \bar{R}_1(\phi(z)))\bar{D}_1(\phi(z)) \\ + p_2 \alpha_2 \psi_1(z)(1 - \bar{B}_2(\psi_2(z)))(1 - \bar{R}_2(\phi(z)))\bar{D}_2(\phi(z)) \\ + p_1 \alpha_1 \psi_2(z)(1 - \bar{B}_1(\psi_1(z)))(1 - \bar{D}_1(\phi(z))) \\ + p_2 \alpha_2 \psi_1(z)(1 - \bar{B}_2(\psi_2(z)))(1 - \bar{D}_2(\phi(z)))\big] \\ + \big[z^b - \big((1-\theta) + \theta \bar{V}(\phi(z))\big) \\ \big(p_1 \bar{B}_1(\psi_1(z)) + p_2 \bar{B}_2(\psi_2(z))\big)\big] \\ \times [\phi(z)\psi_1(z)\psi_2(z) Q(z)] \end{array}\right)}{\begin{array}{l} \big[z^b - \big((1-\theta) + \theta \bar{V}(\phi(z))\big) \big(p_1 \bar{B}_1(\psi_1(z)) \\ + p_2 \bar{B}_2(\psi_2(z))\big)\big] \times [\phi(z)\psi_1(z)\psi_2(z)] \end{array}}.$$

The stability condition is

$$\rho = \frac{\begin{array}{l} \big[\lambda E(I)(p_1(1 + \alpha_1(E(R_1) + E(D_1)))E(B_1) \\ + p_2(1 + \alpha_2(E(R_2) + E(D_2)))E(B_2) + \theta E(V))\big] \end{array}}{b} \text{ then } \rho < 1$$

By differenting the above equation, we get the mean queue size of this model is

$$L_q = \lim_{z \to 1} \frac{d}{dz} P(z) = \big[\frac{N^v(1) D'^v(1) - D^v(1) N'^v(1)}{5(D'^v)^2}\big]$$

$$D^{IV} = -24(\lambda E(I))^3 S_1 S_2 (b - \lambda E(I)(\theta E(V) + A_2))$$

$$\begin{aligned} D^V = & -60(\lambda E(I))^2 \big[(\lambda E(I) S_1 S_2) \times \big(b(b-1) - \theta \lambda E(I(I-1)) \\ & E(V) - \theta(\lambda E(I))^2 (E(V^2) + 2E(V)A_2) - (p_1 C_1 E(B_1) + p_1 \\ & (\lambda E(I) S_1)^2 E(B_1^2) + p_2 C_2 E(B_2) + p_2 (\lambda E(I) S_2)^2 E(B_2^2))\big) + (S_2 \\ & C_1 + \lambda E(I(I-1)) S_1 S_2 + S_1 C_2) \times (b - \lambda E(I)(\theta E(V) + A_2))\big] \end{aligned}$$

$$N^{IV} = -24(\lambda E(I))^3 S_1 S_2 \big[X_1 (A_2 + \theta E(v)) + (b - \lambda E(I)(\theta E(V)$$

$$+ A_2)) \sum_{r=0}^{a-1} Q_r \big]$$

$$N^v = -60(\lambda E(I))^2 \big[ X_2 \lambda E(I) S_1 S_2 (A_2 + \theta E(v)) + X_1 C_1 S_2 E(B_2)$$
$$p_2 + X_1 \big( \lambda E(I(I-1)) A_1 + p_1 S_1 E(B_1) C_2 + S_2 p_1 (C_1 E(B_1)$$
$$+ (\lambda E(I) S_1)^2 E(B_1^2)) + p_2 S_1 (C_2 E(B_2) + (\lambda E(I) S_2)^2 E(B_2^2))$$
$$+ \theta E(V) S_2 C_1 + \theta E(V) S_1 C_2 + \theta S_1 S_2 (2(\lambda E(I))^2 E(V) A_2$$
$$+ C_3) + p_1 \alpha_1 (S_1 E(B_1) C_2 + S_2 (C_1 E(B_1) + (\lambda E(I) S_1)^2$$
$$E(B_1^2)))(E(R_1) + E(D_1)) + p_1 \alpha_1 S_1 S_2 E(B_1)(\lambda E(I(I-1))$$
$$(E(R_1) + E(D_1)) + (\lambda E(I))^2 (E(R_1^2) + E(D_1^2)))$$
$$+ 2 S_2 S_1 (\lambda E(I))^2 C_4 + p_2 \alpha_2 (S_2 E(B_2) C_1 + S_1 (C_2 E(B_2)$$
$$+ (\lambda E(I) S_2)^2 E(B_2^2)))(E(R_2) + E(D_2)) + p_2 \alpha_2 S_1 S_2 E(B_2)$$
$$(\lambda E(I(I-1))(E(R_2) + E(D_2)) + (\lambda E(I))^2 (E(R_2^2)$$
$$+ E(D_2^2)))\big) + \big( b(b-1) - \theta \lambda E(I(I-1)) E(V) - \theta (\lambda E(I))^2$$
$$(E(V^2) + 2E(V) A_2) - (p_1 C_1 E(B_1) + p_1 (\lambda E(I) S_1)^2 E(B_1^2)$$
$$+ p_2 C_2 E(B_2) + p_2 (\lambda E(I) S_2)^2 E(B_2^2))\big) \times ((\lambda E(I)) S_1 S_2$$
$$\sum_{r=0}^{a-1} Q_r) + \big( \sum_{r=0}^{a-1} Q_r S_2 (2 S_1 \lambda E(I(I-1)) + \alpha_1 (\lambda E(I))^2 T_1) + S_1$$
$$(2 \lambda E(I) S_2 \sum_{r=0}^{a-1} Q_r r + \sum_{r=0}^{a-1} Q_r C_2)\big) \times (b - \lambda E(I)(\theta E(V) + A_2))\big]$$

$$X_1 = \lambda \sum_{r=0}^{a-1} Q_r (E(I) + r - b) + \lambda \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r (b - n - r) + \sum_{r=0}^{b-1}$$
$$(b - r) W_r ; X_2 = \lambda \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r (b(b-1) - (n+r)(n+r-1))$$
$$+ \lambda \sum_{r=0}^{a-1} Q_r (E(I(I-1)) + 2E(I) r + r(r-1) - b(b-1))$$
$$+ \sum_{r=0}^{b-1} (b(b-1) - r(r-1)) W_r$$
$$S_i = 1 + \alpha_i (E(R_i) + E(D_i)); C_i = \lambda E(I(I-1)) S_i + \alpha_i (\lambda E(I))^2 T_i$$
$$A_2 = p_1 S_1 E(B_1) + p_2 S_2 E(B_2); T_i = E(R_i^2) + 2E(D_i) E(R_i) + E(D_i^2)$$
$$C_3 = \lambda E(I(I-1)) E(V) + (\lambda E(I))^2 E(V^2); A_1 = A_2 S_2; i = 1, 2$$
$$C_4 = p_1 \alpha_1 E(B_1) E(R_1) E(D_1) + p_2 \alpha_2 E(B_2) E(R_2) E(D_2)$$

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