International Journal of Pure and Applied Mathematics
Volume 120 No. 8 2018, 29-37
ISSN: 1314-3395 (on-line version)
url: http://www.acadpubl.eu/hub/
Special Issue
http://www.acadpubl.eu/hub/

Minimum bb-dominating Energy of Graphs

Sayinath Udupa[1] and R. S. Bhat[*1]
[1]Department of Mathematics, MIT,
A constituent unit of Manipal Academy Of Higher Education,
Manipal- 576104, Karnataka, India.
sayinath.udupa@manipal.edu
rs.bhat@manipal.edu

Abstract

Let $B(G)$ denotes the set of all blocks of a graph G. Two blocks in G are adjacent if there is a common cutvertex incident on them. Two blocks $b_1, b_2 \in B(G)$ are said to block-block dominate (bb-dominate) each other if there is a common vertex incident with b_1 and b_2. A set $L \subseteq B(G)$ is said to be a bb-dominating set (BBD-set) if every block in G is bb-dominated by some block in L. The bb-domination number $\gamma_{bb} = \gamma_{bb}(G)$ is the order of a minimum bb-dominating set of G. In this paper we introduce new kind of graph energy, the minimum bb-dominating energy of the graph denoting it as $E_{bb}(G)$. It depends both on underlying graph of G and its particular minimum bb-dominating set (γ_{bb}-set) of G. Upper and lower bounds for $E_{bb}(G)$ are established.

AMS Subject Classification: 05C69 and 05C50.

Key Words and Phrases: Energy of the graph, the minimum bb-dominating energy of the graph.

1 Introduction

By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q. A vertex $v \in V$ is a cutvertex if $G - \{v\}$

is disconnected. A graph which has no cutvertex is called non separable. A maximal non-separable subgraph is a block of G. Let $B(G)$ denote the set of all blocks of G with $|B(G)| = m$. Two blocks in G are adjacent if there is a common cutvertex incident on them. A block-graph $B_G(G)$ is a graph with vertex set $B(G)$ and any two vertices in $B_G(G)$ are adjacent if and only if corresponding blocks are adjacent in G. The number of edges in the block graph $B_G(G)$ is denoted as q_b. P. G. Bhat et.al [1] defined bb-degree and bb-dominating sets as follows. block-block degree (bb-degree) of a block h, $d_{bb}(h)$ is the number of blocks adjacent to h. Two blocks $b_1, b_2 \in B(G)$ are said to block-block dominate (bb-dominate) each other if there is a common vertex incident with b_1 and b_2. A set $L \subseteq B(G)$ is said to be a bb-dominating set (BBD-set) if every block in G is bb-dominated by some block in L. The bb-domination number $\gamma_{bb} = \gamma_{bb}(G)$ is the order of a minimum bb-dominating set of G. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. These eigenvalues arranged in non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G) \ldots, \lambda_p(G)$. Then the energy of the graph G is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i(G)|$. Various properties of energy of the graph may be found in [2, 3]. In this paper we introduce a new matrix, called minimum bb-dominating matrix of a graph and study its energy.

2 Minimum bb-dominating energy of a graph

Let γ_{bb}-set be a minimum bb-dominating set of a graph G. The minimum bb-dominating matrix of G is the $m \times m$ matrix $A_{bb}(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } b_i \text{ and } b_j \text{ are adjacent} \\ 1 & \text{if } i = j \text{ and } \ b_i \in \gamma_{bb}\text{-set} \\ 0 & \text{otherwise} \end{cases}$$

The characteristic polynomial of $A_{bb}(G)$ is denoted by

$$f_m(G, \lambda) = det(\lambda I - A_{bb}(G))$$

The minimum bb-dominating eigenvalues of the graph G are the eigenvalues of $A_{bb}(G)$. Since $A_{bb}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. The minimum bb-dominating energy of G is then defined as

$$E_{bb}(G) = \sum_{i=1}^{m} |\lambda_i(G)|.$$

In this paper we discuss some basic properties of minimum bb-dominating energy of the graph $E_{bb}(G)$ and derive an upper and lower bound for $E_{bb}(G)$.

3 Basic properties of minimum bb-dominating energy of the graph

Theorem 1. *If $\lambda_1(G), \lambda_2(G) \ldots, \lambda_m(G)$ are the eigenvalues of $A_{bb}(G)$, then*

$$\sum_{i=1}^{m} \lambda_i = \gamma_{bb} \qquad (1)$$

$$\sum_{i=1}^{m} \lambda_i^2 = 2q_b + \gamma_{bb}. \qquad (2)$$

where q_b is the number of edges in the block graph $B_G(G)$.

Proof. (1) We know that the sum of the eigenvalues of $A_{bb}(G)$ is equal to trace of $A_{bb}(G)$. Therefore

$$\sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} a_{ii} = \gamma_{bb}.$$

(2) The sum of the squares of the eigenvalues of $A_{bb}(G)$ is the trace of $(A_{bb}(G))^2$. Therefore

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i^2 &= \sum_{i=1}^{m}\sum_{j=1}^{m} (a_{ij}a_{ji}) \\ &= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{m} (a_{ij})^2 \\ &= 2q_b + \gamma_{bb}\end{aligned}$$

□

Theorem 2. *Let G be a graph with m blocks, q_b edges, and the minimum bb-dominating set γ_{bb}-set. Let $f_m(G, \lambda) = c_0\lambda^m + c_1\lambda^{m-1} + c_2\lambda^{m-2} + \cdots + c_m$ be the characteristic polynomial of G, Then*

$$c_0 = 1 \tag{3}$$

$$c_1 = -\gamma_{bb} \tag{4}$$

$$c_2 = \binom{\gamma_{bb}}{2} - q_b \tag{5}$$

$$c_3 = \gamma_{bb}q_b - \sum_{h\in\gamma_{bb}} d_{bb}(h) - \binom{\gamma_{bb}}{3} - 2\triangle \tag{6}$$

where q_b and $\triangle$ respectively denote the number of edges and triangles in the block graph $B_G(G)$.

Proof. (3) directly follows from the definition of $f_m(G, \lambda)$.
(4) Since the sum of diagonal elements of $A_{bb}(G)$ is equal to γ_{bb}, the sum of determinants of all 1×1 principal sub-matrices of $A_{bb}(G)$ is the trace of $A_{bb}(G)$, which is equal to γ_{bb}. Thus $(-1)^1c_1 = \gamma_{bb}$
(5) $(-1)^2c_2$ is equal to the sum of determinants of all 2×2 principal sub-matrices of $A_{bb}(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le m} (a_{ii}a_{jj}) - \sum_{i\le i<j\le m} (a_{ij})^2 \\ &= \binom{\gamma_{bb}}{2} - q_b \end{aligned}$$

(6) We have

$$c_3 = (-1)^3 \sum_{1\le i<j<k\le m} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$$

$$= -\sum_{1\le i<j<k\le m} (a_{ii}a_{jj}a_{kk}) + \sum_{1\le i<j<k\le m} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij})$$

$$- \sum_{1\le i<j<k\le m} (a_{ij}a_{jk}a_{ki}) - \sum_{1\le i<j<k\le m} (a_{ik}a_{kj}a_{ji})$$

$$= -\binom{\gamma_{bb}}{3} + \sum_{1\le i<j<k\le m} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}) - 2\triangle$$

$$= -\binom{\gamma_{bb}}{3} + \left[\sum_{i=1}^{m} a_{ii}\right]\left[\sum_{1\le i<j<k\le m} a_{jk}\right] - \sum_{i=1}^{m} a_{ii} \sum_{k=1,k\ne i} a_{ik} - 2\triangle$$

$$c_3 = \gamma_{bb}q_b - \sum_{h\in\gamma_{bb}} d_{bb}(h) - \binom{\gamma_{bb}}{3} - 2\triangle$$

□

Theorem 3. *If $\lambda_1(G)$ is the largest eigenvalue of the minimum bb-dominating matrix $A_{bb}(G)$, then*

$$\lambda_1(G) \ge \frac{2q_b + \gamma_{bb}}{m}. \tag{7}$$

Proof. Let X be any non zero vector, then we have

$\lambda_1\left(A_{bb}(G)\right) = \begin{matrix}\max \\ X\ne 0\end{matrix}\left[\dfrac{X'AX}{X'X}\right]$ (see [4])

$\lambda_1(A_{bb}(G)) \ge \left[\dfrac{J'AJ}{J'J}\right] = \dfrac{2q_b+\gamma_{bb}}{m}$, where J is the all one's vector.

□

Bounds for $E_{bb}(G)$ similar to McClelland's inequalities [5] for graph energy, are given in the following theorems.

Theorem 4. *Let G be the graph with m blocks, q_b edges, and minimum bb-dominating set γ_{bb}-set of G. Then*

$$E_{bb}(G) \le \sqrt{m\left(2q_b + \gamma_{bb}\right)} \tag{8}$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ be the eigenvalues of $A_{bb}(G)$. Using Cauchy-Schwarz inequality, $\left(\sum_{i=1}^{m} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{m} a_i^2\right)\left(\sum_{i=1}^{m} b_i^2\right)$ choose $a_i = 1$ and $b_i = |\lambda_i|$,
$(E_{bb}(G))^2 = \left(\sum_{i=1}^{m} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{m} 1\right)\left(\sum_{i=1}^{m} |\lambda_i|^2\right) = m\left(2q_b + \gamma_{bb}\right)$ by using theorem 1. Hence result follows. □

Following proposition gives lower bound for $E_{bb}(G)$ in terms of number of blocks and number of edges in the block graph $B_G(G)$.

Theorem 5. *Let G be the graph with m blocks and q_b edges, and let minimum bb-dominating set be γ_{bb}-set. If P is determinant of $A_{bb}(G)$, then*

$$E_{bb}(G) \geq \sqrt{2q_b + \gamma_{bb} + m(m-1)P^{\frac{2}{m}}} \tag{9}$$

Proof.

$$\begin{aligned}(E_{bb}(G))^2 &= \left(\sum_{i=1}^{m} |\lambda_i|\right)^2 \\ &= \left(\sum_{i=1}^{m} |\lambda_i|\right)\left(\sum_{j=1}^{m} |\lambda_j|\right) \\ &= \sum_{i=1}^{m} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|.\end{aligned}$$

Now employing the inequality between the arithmetic and geometric means, we obtain

$$\frac{1}{m(m-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{m(m-1)}}$$

$$\sum_{i\neq j}|\lambda_i||\lambda_j| \geq m(m-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{m(m-1)}}$$ Thus

$$\begin{aligned}(E_{bb}(G))^2 &\geq \sum_{i=1}^{m}|\lambda_i|^2 + m(m-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{m(m-1)}}\\ &\geq \sum_{i=1}^{m}|\lambda_i|^2 + m(m-1)\left(\prod_{i=1}^{m}|\lambda_i|^{2(m-1)}\right)^{\frac{1}{m(m-1)}}\\ &\geq \sum_{i=1}^{m}|\lambda_i|^2 + m(m-1)\left(\prod_{i=1}^{m}|\lambda_i|\right)^{\frac{2}{m}}\end{aligned}$$

$$(E_{bb}(G))^2 \geq 2q_b + \gamma_{bb} + m(m-1)P^{\frac{2}{m}} \text{ (From the theorem 1)}$$

Hence result follows. □

Theorem 6. *If G is a graph with m blocks, q_b edges and $2q_b + \gamma_{bb} \geq m$, then*

$$E_{bb}(G) \leq \frac{2q_b + \gamma_{bb}}{m} + \sqrt{(m-1)\left((2q_b + \gamma_{bb}) - \left(\frac{2q_b + \gamma_{bb}}{m}\right)^2\right)} \tag{10}$$

Proof. Using Cauchy-schwarz inequality, $\left(\sum_{i=2}^{m} a_i b_i\right)^2 \leq \left(\sum_{i=2}^{m} a_i^2\right)\left(\sum_{i=2}^{m} b_i^2\right)$ choose $a_i = 1$ and $b_i = |\lambda_i|$

$$\left(\sum_{i=2}^{m}|\lambda_i|\right)^2 \leq \left(\sum_{i=2}^{m}1\right)\left(\sum_{i=2}^{m}|\lambda_i|^2\right)$$

$$\left(\sum_{i=1}^{m}|\lambda_i| - \lambda_1\right)^2 \leq (m-1)\left(\sum_{i=1}^{m}\lambda_i^2 - \lambda_1^2\right)$$

$$E_{bb}(G) \leq \lambda_1 + \sqrt{(m-1)\left(2q_b + \gamma_{bb} - \lambda_1^2\right)}$$

Let $f(t) = t + \sqrt{(m-1)\left(2q_b + \gamma_{bb} - t^2\right)}$
for decreasing function $f^{'}(t) \leq 0$.
$f^{'}(t) = 1 - \dfrac{t(m-1)}{\sqrt{(m-1)\left(2q_b + \gamma_{bb} - t^2\right)}} \leq 0.$
Therefore $t \geq \sqrt{\dfrac{2q_b + \gamma_{bb}}{m}}$
Since $2q_b + \gamma_{bb} \geq m$, we have

$$\sqrt{\frac{2q_b + \gamma_{bb}}{m}} \leq \frac{2q_b + \gamma_{bb}}{m} \leq \lambda_1$$

$$f(\lambda_1) \leq f\left(\frac{2q_b + \gamma_{bb}}{m}\right). \text{ Therefore}$$

$$E_{bb}(G) \leq f\left(\frac{2q_b + \gamma_{bb}}{m}\right) \text{Hence result follows.}$$

□

References

[1] P.G. Bhat, R.S. Bhat, Surekha R. Bhat, Relationship between Block Domination parameters of a graph,*Discrete Mathematics,Algorithms and Applications*, **5** (2013), 181-190.

[2] I. Gutman, X. Li, J. Zhang, *Graph energy*, in: M. Dehmer, F. Emmert-Streib (Eds.), Analysis of Complex Networks. From Biology to Linguistrics, Wiley-VCH, Weinheim, (2009), 145-174.

[3] Chandrashekar Adiga, Abdelmejid Bayad, I. Gutman, Shrikanth Avant Srinivas, The minimum covering energy of a graph, *Kragujevac J. Sci.*, **34** (2012) 39-56.

[4] R. B. Bapat, *Graphs and Matrices*, Hindustan Book Agency, (2011).

[5] B. J. McClelland, Properties of the latent roots of a matrix : The estimation of π-electron energies, *J. Chem. Phys*, **54** (1971), 640-643.