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Extremally Disconnectedness in Nano Topology

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Abstract

The mathematical intuitive mind is always in search of new concept. Here we find some interesting characterizations of nano extremally disconnectedness in terms of nano semi-open and nano regular open sets. Also we investigate some of the main properties of these sets and study their relations to nano open sets.

AMS Subject Classification: 54A05, 54A10, 54G05

Key Words and Phrases: nano semi-open, nano pre-open, nano extremally disconnected, nano regular open, nano topology.

1 Introduction

Extremally disconnected topological spaces were introduced by Gillman and Jerison [2]. In 1964, Corson and Michael [1] introduced the notion of locally dense sets, also called pre-open sets by Mashhour et al. [8]. The class of pre-open sets properly contain the class of open sets. As the intersection of two pre-open sets may fail to be pre-open, the class of pre-open sets do not always form a topology. In a sub-maximal space, i.e. a topological space X in which every dense subset is open, the collection of all pre-open sets form a topology. Indeed, many notions in Topology can be defined in terms of pre-open sets [3,7]. Lellis Thivagar [4,5] introduced nano topological space with respect to a subset X of an universe which is defined in terms of lower and upper approximations of X. The elements of a nano topological space are called the nano-open sets and their complements are called nano closed sets. Certain weak forms of nano-open sets such as nano α-open sets, nano semi-open sets and nano pre-open sets [5] have also been established. The driving force behind this paper is to derive a few characterizations of nano extremally disconnectedness in terms of nano semi-open and nano regular open sets etc. Further we push this idea to find some new classes nano pre-open sets. Also we investigate some of the main properties of these sets and study their relations to nano open sets.

2 Preliminaries

Now we would like to present some basic notions of nano topology and results of weaker forms of nano open sets which will be necessary to discuss extremally disconnected space.

Definition 1. [5]: Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and R be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, R)$ is said to be the approximation space. Let $X \subseteq \mathcal{U}$.

(i) The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and its is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in\mathcal{U}} \{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by X.

(ii) The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in\mathcal{U}} \{R(x) : R(x) \cap X \neq \emptyset\}$

(iii) The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2. [4, 5]: Let $\mathcal{U}$ be an universe, R be an equivalence relation on $\mathcal{U}$ and $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq \mathcal{U}$. $\tau_R(X)$ satisfies the following axioms:

(i) $\mathcal{U}$ and $\emptyset \in \tau_R(X)$.

(ii) The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

(iii) The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

That is,$\tau_R(X)$ forms a topology on $\mathcal{U}$ called the nano topology on $\mathcal{U}$ with respect to X. We call $(\mathcal{U}, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called nano open sets.

Definition 3. [4, 5]: If $(\mathcal{U}, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq \mathcal{U}$ and if $A \subseteq \mathcal{U}$, then the nano interior of A is defined as the largest nano open set contained in A and is denoted by $Nint(A)$. The nano closure of A is defined as the smallest nano closed set containing A and is denoted by $Ncl(A)$.

Proposition 4. [4, 5] Let $\mathcal{U}$ be a non-empty finite universe and $X \subseteq \mathcal{U}$.

(i) If $L_R(X) = \emptyset$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset\}$, the indiscrete nano topology on $\mathcal{U}$.

(ii) If $L_R(X) = U_R(X) = X$, then the nano topology, $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X)\}$

(iii) If $L_R(X) = \emptyset$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset, U_R(X)\}$.

(iv) If $L_R(X) \neq \emptyset$ and $U_R(X) = \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), B_R(X)\}$.

(v) If $L_R(X) \neq U_R(X)$ where $L_R(X) \neq \emptyset$ and $U_R(X) \neq \mathcal{U}$, then $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), U_R(X), B_R(X)\}$ is the discrete nano topology on $\mathcal{U}$

Definition 5. [5, 6]: Let $(\mathcal{U}, \tau_R(X))$ be a nano topological space and $A \subseteq \mathcal{U}$. Then A is said to be

(i) nano semi-open if $A \subseteq Ncl(Nint(A))$

(ii) nano pre-open if $A \subseteq Nint(Ncl(A))$
(iii) nano regular open if $A = Nint(Ncl(A))$

Definition 6. [9] A topological space (X, τ) is said to be extremally disconnected if the closure of every open set of X is open in X.

3 Various open sets in nano extremally disconnected space

We shall derive nano extremally disconnected space with some suitable examples and prove some related theorems that are useful to expand the nano topology.

Definition 7. A nano topological space $(\mathcal{U}, \tau_R(X))$ is extremally disconnected, if the nano closure of each nano open set is nano open in $\mathcal{U}$

Example 8. Let $\mathcal{U} = \{a, b, c\}, \mathcal{U}/R = \{\{a\}, \{b, c\}\}$ and $X = \{a, b\}$. Then $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a\}, \{b, c\}\}$. Then the nano closed sets in $\mathcal{U}$ are $\mathcal{U}, \emptyset, \{a\}$ and $\{b, c\}$. Therefore $Ncl(\mathcal{U}) = \mathcal{U}, Ncl(\emptyset) = \emptyset$, $Ncl\{a\} = \{a\}$ and $Ncl(\{b, c\}) = \{b, c\}$. That is, nano closure of each nano open set in $\mathcal{U}$ is nano open. Therefore $(\mathcal{U}, \tau_R(X))$ is extremally disconnected.

Theorem 9. A nano topological space $(\mathcal{U}, \tau_R(X))$ is extremally disconnected if and only if $U_R(X) = \mathcal{U}$.

Proof. Let $(\mathcal{U}, \tau_R(X))$ be extremally disconnected. That is, nano closure of each nano open set is nano open. That is, $\mathcal{U}, \emptyset, [B_R(X)]^c$ and $[L_R(X)]^c$ are nano open. That is, $\mathcal{U}, \emptyset, B_R(X)$ and $L_R(X)$ are nano closed. Therefore, $Ncl(B_R(X) = B_R(X)$ and $Ncl(L_R(X) = L_R(X)$. That is, $[L_R(X)]^c = B_R(X)$ and $[B_R(X)]^c = L_R(X)$. That is, $[L_R(X)]^c = U_R(X) - L_R(X) = U_R(X) \cap [L_R(X)]^c$. Therefore, $[L_R(X)]^c \subseteq U_R(X)$. Since $L_R(X) \subseteq U_R(X), [U_R(X)]^c \subseteq [L_R(X)]^c$. Hence $[U_R(X)]^c \subseteq [L_R(X)]^c \subseteq U_R(X$. This is possible when $[U_R(X)]^c = \emptyset$. This implies $U_R(X) = \mathcal{U}$.

Conversely, let $U_R(X) = \mathcal{U}$. If $L_R(X) = \emptyset$ then $\tau_R(X) = \{\mathcal{U}, \emptyset\}$

where nano closure of each nano open set is obviously nano open. If $L_R(X) \neq \emptyset$, then $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), B_R(X)\}$ where $B_R(X) = [L_R(X)]^c$ and hence $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), [L_R(X)]^c\}$. Therefore each set in $\tau_R(X)$ is both nano open and nano closed. Hence nano closure of each nano open is nano open. Therefore, $(\mathcal{U}, \tau_R(X))$ is extremally disconnected. Thus $\mathcal{U}$ is extremally disconnected if and only if $U_R(X) = \mathcal{U}$. □

Theorem 10. If $L_R(X) = U_R(X)$ where $X \subseteq \mathcal{U}$, then $(\mathcal{U}, \tau_R(X))$ is extremally disconnected.

Proof. Since $L_R(X) = U_R(X), B_R(X) = \emptyset$ and hence $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X)\}$ where $Ncl(\mathcal{U}) = \mathcal{U}, Ncl(\emptyset) = \emptyset$ and $Ncl(L_R(X)) = [B_R(X)]^c = \emptyset^c = \mathcal{U}$. That is,the nano closure of each nano open set in $\mathcal{U}$ is nano open. Therefore, $(\mathcal{U}, \tau_R(X))$ is extremally disconnected. The converse of the above theorem is not true by the following example. □

Example 11. Let $\mathcal{U} = \{a, b, c\}$ with $\mathcal{U}/R = \{\{a, b\}, \{c\}\}$. Let $X = \{a, c\} \subseteq \mathcal{U}$. Then $L_R(X) = \{c\}, U_R(X) = \mathcal{U}, B_R(X) = \{a, b\}$. Therefore $L_R(X) \neq U_R(X)$. The nano topology $\tau_R(X) = \{\mathcal{U}, \emptyset, \{c\}, \{a, b\}\}$ where $Ncl(\mathcal{U}) = \mathcal{U}, Ncl(\emptyset) = \emptyset, Ncl(\{c\}) = \{c\}$ and $Ncl(\{a, b\} = \{a, b\}$. That is, the nano closure of each nano open set is nano open. Hence $\mathcal{U}$ is extremally disconnected, but $L_R(X) \neq U_R(X)$.

Theorem 12. If $L_R(X) = \emptyset$, then $(\mathcal{U}, \tau_R(X))$ is extremally disconnected.

Proof. : Since $L_R(X) = \emptyset, B_R(X) = U_R(X)$. We have $\tau_R(X) = \{\mathcal{U}, \emptyset, U_R(X)\}$ where $Ncl(\mathcal{U}) = \mathcal{U}, Ncl(\emptyset) = \emptyset$ and $Ncl(U_R(X)) = \mathcal{U}$. That is, the nano closure of each nano open set is nano open is $\mathcal{U}$. Hence $\mathcal{U}$ is extremally disconnected. □

Theorem 13. If $L_R(X) \neq \emptyset$ and $U_R(X) = \mathcal{U}$, then $(\mathcal{U}, \tau_R(X))$ is extremally disconnected.

Proof. Since $L_R(X) \neq \emptyset$ and $U_R(X) = \mathcal{U}$ we have $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), B_R(X)\}$ where $Ncl(\mathcal{U}) = \mathcal{U}, Ncl(\emptyset) = \emptyset$ and $Ncl(U_R(X)) = \mathcal{U}$ That is, the nano closure of each nano open set is nano open is $\mathcal{U}$. Hence $\mathcal{U}$ is extremally disconnected. □

Remark 14. If $L_R(X) \neq U_R(X)$ where $L_R(X) \neq \emptyset$ and $U_R(X) \neq \mathcal{U}$ then $(\mathcal{U}, \tau_R(X))$ need not be extremally disconnected.

Example 15. Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/R = \{\{a\}, \{b\}, \{c, d\}\}$ and let $X = \{a, c\}$. Then $L_R(X) = \{a\}, U_R(X) = \{a, c, d\}$, $B_R(X) = \{c, d\}$ thus $\tau_R(X) = \{\mathcal{U}, \emptyset, \{a\}, \{a, c, d\}, \{c, d\}\}$. Now $Ncl(\mathcal{U}) = \mathcal{U}, Ncl(\emptyset) = \emptyset, Ncl\{a\} = \{a, b\} \notin \tau_R(X)$, $Ncl\{a, c, d\} = \mathcal{U}, Ncl\{c, d\} = \{b, c, d\} \notin \tau_R(X)$ that is each nano open set is not nano open hence it is not extremally disconnected.

Theorem 16. If $U_R(X) = U$ is a nano topological space, then $U, \emptyset, L_R(X)$ and $B_R(X)$ are the only nano semi-open sets in U.

Proof. : Let $\tau_R(X) = \{\mathcal{U}, \emptyset, L_R(X), B_R(X)\}$. If $L_R(X) \neq \emptyset$ and A be a non empty subset of $\mathcal{U}$. If $A \subset L_R(X)$, then $Ncl(Nint(A)) = \emptyset$ and thus implies A is not nano semi open in $\mathcal{U}$. If $A = L_R(X)$, then $Ncl(Nint(A)) = Ncl(L_R(X)) = L_R(X)$, therefore $A \subseteq Ncl(Nint(A))$. This gives A is nano semi open in $\mathcal{U}$. If $A \supset L_R(X)$, then $Ncl(Nint(A)) = Ncl(L_R(X)) = L_R(X)$. Hence $A \nsubseteq Ncl(Nint(A))$ and A is not nano semi open in $\mathcal{U}$. If $A \subset B_R(X)$, then $Ncl(Nint(A)) = \emptyset$ and this gives that A is not nano semi open in $\mathcal{U}$. If $A = B_R(X)$, then $Ncl(Nint(A)) = Ncl(B_R(X)) = B_R(X)$ and hence $A \subseteq Ncl(Nint(A))$. Therefore A is nano semi open in $\mathcal{U}$. If $A \supset B_R(X)$, then $Ncl(Nint(A)) = Ncl(B_R(X)) = B_R(X) \subset A$. Hence A is not nano semi open in $\mathcal{U}$. If A has atleast one element of $L_R(X)$ and $B_R(X)$, then $ncl(Nint(A)) = Ncl(\emptyset) = \emptyset$ and hence A is not nano semi open in $\mathcal{U}$. Thus $\mathcal{U}, \emptyset, L_R(X)$ and $B_R(X)$ are the only nano semi open sets in $\mathcal{U}$, if $U_R(X) = \mathcal{U}$ and $L_R(X) \neq \emptyset$. If $L_R(X) = \emptyset$, then $\mathcal{U}$ and $\emptyset$ are the only nano semi open sets in $\mathcal{U}$ since $\mathcal{U}$ and $\emptyset$ are the only sets in $\mathcal{U}$ which are nano open and nano closed. □

Theorem 17. $(\mathcal{U}, \tau_R(X)) = NSO(\mathcal{U}, X)$ forms a nano topology, if $\mathcal{U}$ is extremally disconnected.

Proof. Suppose $\mathcal{U}$ is not extremally disconnected then there exists a nano open set A such that $Nint(Ncl(A)) \neq Ncl(A)$. Let $x \in Ncl(A) - Nint(Ncl(A))$. Let $B = \{x\} \cup Nint(Ncl(A))$ and $C = (Nint(Ncl(A)))^c = Ncl(Nint(A^c))$. Now $Ncl(Nint(B)) \supseteq Ncl(Nint(Ncl(A))) = Ncl(A) \supseteq \{x\}$. Also $Ncl(Nint(C)) =$

$Ncl(Nint(Ncl(Nint(A^c)))) = Ncl(Nint(A^c)) = C \supseteq \{x\}$. Thus B and C are nano semi open sets, but $B \cap C = \{x\}$ is not nano semi open, which is contradiction. □

Remark 18. In general, $NSO(\mathcal{U}, \tau_R(X)) \nsubseteq NPO(\mathcal{U}, \tau_R(X))$. However, if $(\mathcal{U}, \tau_R(X))$ is nano extremally disconnected then $NSO(\mathcal{U}, \tau_R(X)) \subseteq NPO(\mathcal{U}, \tau_R(X))$

Theorem 19. The following statements are equivalent for any nano topological space $(\mathcal{U}, \tau_R(X))$:

(i) $(\mathcal{U}, \tau_R(X))$ is nano extremally disconnected.

(ii) $Nscl(A) = Nint(Ncl(A)) = Ncl(Nint(A))$ for every $A \in NSO(\mathcal{U}, \tau_R(X))$.

(iii) $Nscl(A) \in NR(\mathcal{U}, \tau_R(X))$ for every $A \in NSO(\mathcal{U}, \tau_R(X))$.

(iv) $Nscl(A) \in NRO(\mathcal{U}, \tau_R(X))$ for every $A \in NSO(\mathcal{U}, \tau_R(X))$.

Proof. (i) ⇒ (ii). Assume $(\mathcal{U}, \tau_R(X))$ is nano extremally disconnected and $A \in SO(\mathcal{U}, \tau_R(X))$. Implies $A \in NPO(\mathcal{U}, \tau_R(X))$ and hence $Nscl(A) = Nint(Ncl(A))$. To prove $Nscl(A) = Ncl(Nint(A))$, for each $A \in NSO(\mathcal{U}, \tau_R(X))$, $Nscl(A) \in NSR(\mathcal{U}, \tau_R(X))$. Obviously $Nscl(Ncl(Nint(A))) = Ncl(Nint(A))$, since $NC(\mathcal{U}, \tau_R(X)) \subset NSC(\mathcal{U}, \tau_R(X))$. So we have $Nscl(A) \subset Nscl(Ncl(Nint(A))) = Ncl(Nint(A)) \subset Ncl(Nscl(A)) = Nint(Ncl(Nscl(A)))$, because $Ncl(Nscl(A)) \in NPO(\mathcal{U}, \tau_R(X))$. But $Nint(Ncl(Nscl(A))) \subset Nscl(A)$, thus our equality is obvious. Implications (ii) ⇒ (iii) and (iii) ⇒ (iv) are obvious.
(iv) ⇒ (i). Suppose that for an arbitrary $A \in NSO(\mathcal{U}, \tau_R(X))$, $Nscl(A) \in NRO(\mathcal{U}, \tau_R(X))$, that is $Nint(Ncl(Nscl(A))) = Nscl(A)$. Hence we see that $A \in Nint(Ncl(Nscl(A))) \in Nint(Ncl(A))$. Therefore $A \in NPO(\mathcal{U}, \tau_R(X))$. This shows the extremally disconnectedness of $(\mathcal{U}, \tau_R(X))$. □

Conclusion: In this paper, we have discussed some of the properties of nano extremally disconnectedness in terms of nano regular/pre/semi open sets. Our results contribute to the stream of investigations concerning various types of characterizations of nano extremally disconnected spaces. This concept can be extended to other brances like algebraic topology, graph theory, computing algorithm etc.

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