International Journal of Pure and Applied Mathematics
Volume 120 No. 8 2018, 9-17
ISSN: 1314-3395 (on-line version)
url: http://www.acadpubl.eu/hub/
Special Issue
http://www.acadpubl.eu/hub/

Square Difference Labeling of Torus Network

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Abstract

A function f of a graph $G(p,q)$ admits *square difference labeling* if there exist a bijection $f : V(G) \rightarrow \{0,1,2,\ldots,p-1\}$ such that the induced function $f^* : E(G) \rightarrow N$ given by $f^*(uv) = \left|(f(u))^2 - (f(v))^2\right|$ for every $uv \in E(G)$ are all distinct. A graph which admits square difference labeling is called *square difference graph*. In this paper we investigate the square difference labeling of Torus Network $T(m,n)$.

Key Words : Square difference labeling, Square difference graph, Torus Network.

1 Introduction

Graph labeling is a relation between numbers and structure of graphs. A dynamic survey to know about the numerous graph labeling methods is given by Gallian [1] and it is published by Electronic Journal of Combinatorics.

The concept of square difference labeling was introduced by J.Shiama [4]. J. A. Cynthia *et al.* investigated the square difference labeling of the circulant network $G(n; \pm\{1,2\})$[2].

1.1 Definition

Let G be (p,q) graph. Graph G admits *square difference labeling* if there exist a bijection $f : V(G) \rightarrow \{0,1,2,...p-1\}$ such that the induced function $f^* : E(G) \to N$ given by $f^*(uv) = |(f(u))^2 - (f(v))^2|$ for every $uv \in E(G)$ are all distinct.

Any graph which admits *square difference labeling* is called *square difference graph.*

2 Torus Network

The Torus forms a basic class of interconnection networks. They are frequently used in large scale supercomputers as a cost efficient alternative to other topologies. Recently it was demonstrated that torus network for computer cluster can be built from affordable commodity hardware such as infinite band [3].

2.1 Definition

Let C_k be the cycle of length k with the vertex set $(0, 1, \ldots, k-1\}$. Two vertices $u, v \in V(C_k)$ are adjacent in C_k if and only if $u = v \pm 1(mod\ k)$. The Torus Network [6] $T(k_1, \ldots, k_n)$ with $n \geq 2$ and $k_i \geq 2$, for all i, is defined to be $T(k_1, \ldots, k_n) = C_{k_1} \times \cdots \times C_{k_n}$ with vertex set $\{u_1, \ldots, u_n : u_i \in \{0, \ldots, k_i - 1, 1 \leq i \leq n\}\}$. Two vertices $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are adjacent in $T(k_1, \ldots, k_n)$ iff there exist some $j \in \{1, \ldots, n\}$ such that $u_j = v_j \pm 1(mod\ k_j)$ and $u_i = v_i$ for $i \in \{1, \ldots, n\} / \{j\}$. $T(k_1, \ldots, k_n)$ is a connected $2n$-regular graph consisting of $k_1, \ldots, k_n$ vertices. Let $T(k_1, k_2)$ be a two dimension torus where $k_1 \geq 3$ and $k_2 \geq 3$. Then $T(k_1, k_2) = C_{k_1} \times C_{k_2}$ where $C_{k_1} \times C_{k_2}$ consists of k_2 copies of C_{k_1}.

3 Square Difference Labeling of Torus Network

Let us denote the vertices of a $m \times n$ Torus network (i, j) where i denotes the row and j denotes the column. $T(m, n)$ has mn vertices and $2mn$ edges.

we define the edge sets of $T(m, n)$ as folows:

$$E(G) = \begin{cases} u_{ij}u_{ij+1} & , 1 \leq i \leq m\ ; 1 \leq j \leq n-1 \\ u_{ij}u_{i+1j} & , 1 \leq i \leq m-1\ ; 1 \leq j \leq n \\ u_{i1}u_{in} & , 1 \leq i \leq m \\ u_{1j}u_{mj} & , 1 \leq j \leq n \end{cases}$$

Theorem 1. *The torus network $T(m,n)$, m (even) and n (odd); $m \geq 4$, $n > 4$ is a square difference graph.*

Proof. Define the vertex labeling $f : V(G) \to \{0, 1, 2, \ldots, mn-1\}$ as follows :

$$f(u_{1j}) = j - 1,\ 1 \leq j \leq n$$

$$f(u_{ij}) = \begin{cases} mn - j - \left\lfloor \frac{m-i}{2} \right\rfloor n\ ,\ i = 2t,\ 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor,\ 1 \leq j \leq n \\ mn - j - \left\lfloor \frac{2m-i}{2} \right\rfloor n\ , i = 2t+1\ ,\ 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor - 1\ , \\ \qquad 1 \leq j \leq n \end{cases}$$

To find the edge labelings:

Case 1: Consider the edge set $E(G) = u_{ij}u_{ij+1}\ , 1 \leq i \leq m\ , 1 \leq j \leq n-1$

Sub Case 1.1: $i = 1\ , 1 \leq j \leq n-1$

$$f^*(u_{1j}u_{ij+1}) = \left|j^2 - (j-1)^2\right| = 2j - 1.$$

Sub Case 1.2: $i = 2t\ , 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor\ , \leq j \leq n-1$

$$f^*(u_{ij}u_{ij+1}) = 2\left(mn - \left\lfloor \frac{m-i}{2} \right\rfloor n\right) - 2j - 1$$

Sub Case 1.3: $i = 2t+1\ , 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor - 1\ , 1 \leq j \leq n-1$

$$f^*(u_{ij}u_{ij+1}) = 2\left(mn - \left\lfloor \frac{2m-i}{2} \right\rfloor n\right) - 2j - 1$$

Case 2: Consider the edge set $E(G) = u_{ij}u_{i+1j}\ , 1 \leq i \leq m-1\ , 1 \leq j \leq n$

Sub Case 2.1: $i = 1\ , 1 \leq j \leq n$

$$f^*(u_{1j}u_{2j}) = 2j - 1 + (mn - \left\lfloor \frac{m-i}{2} \right\rfloor n)(mn - 2j - \left\lfloor \frac{m-i}{2} \right\rfloor n)$$

Sub Case 2.2: $i = 2t\ , 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor\ , 1 \leq j \leq n, 1 \leq i \leq m-1$

$$f^*(u_{ij}u_{i+1j}) = \left(mn - \left\lfloor \tfrac{m-i}{2} \right\rfloor n\right)\left(mn - \left\lfloor \tfrac{m-i}{2} \right\rfloor n - 2j\right) -$$
$$\left(mn - \left\lfloor \frac{2m-(i+1)}{2} \right\rfloor n\right)(mn - \left\lfloor \frac{2m-(i+1)}{2} \right\rfloor n - 2j)$$

Figure 1: Square Difference Labeling of Torus Network T(4,5)

Sub Case 2.3: $i = 2t + 1, 1 \leq t \leq \lfloor \frac{m}{2} \rfloor - 1, 1 \leq j \leq n$,
$1 \leq i \leq m - 1$

$$f^*(u_{ij}u_{i+1j}) = (mn - \lfloor \tfrac{m-(i+1)}{2} \rfloor n)(mn - \lfloor \tfrac{m-(i+1)}{2} \rfloor n - 2j)$$
$$-(mn - \lfloor \frac{2m-i}{2} \rfloor n)(mn - \lfloor \frac{2m-i}{2} \rfloor n - 2j)$$

Case 3: Consider the edge set $E(G) = u_{i1}u_{in}, 1 \leq i \leq m$
Sub Case 3.1: $i = 1, n = j$
$f^*(u_{11}u_{1n}) = \left|(n-1)^2 - (1-1)^2\right| = (n-1)^2$

Sub Case 3.2: $i = 2t\ , 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq i \leq m\ , n = j$
$f^*(u_{i1}u_{in}) = 2\left(mn - \left\lfloor \frac{m-i}{2} \right\rfloor n\right)(n-1) - n^2 + 1$

Sub Case 3.3: $i = 2t + 1\ , 1 \leq t \leq \left\lfloor \frac{m}{2} \right\rfloor - 1\ , 1 \leq i \leq m\ , n = j$

$$f^*(u_{i1}u_{in}) = 2\left(mn - \left\lfloor \frac{2m-i}{2} \right\rfloor n\right)(n-1) - n^2 + 1$$

Case 4: Consider the edge set $E(G) = u_{1j}u_{mj}\ , 1 \leq j \leq n\ , m = i$
$f^*(u_{1j}u_{mj}) = \left(mn - \left\lfloor \frac{m-i}{2} \right\rfloor n\right)\left(mn - \left\lfloor \frac{m-i}{2} \right\rfloor n - 2j\right) + 2j - 1$

Therefore the induced edge labeling function are all distinct. Hence the Torus Network $T(m,n)$, m (even) and n (odd); $m \geq 4,\ n > 4$ is a square difference graph.

□

Theorem 2. *The torus network $T(m,n)$, m (odd) and n (odd); $m \geq 5,\ n > 5$ is a square difference graph.*

Proof. Define the vertex labeling $f : V(G) \rightarrow \{0, 1, 2, \ldots, mn - 1\}$ as follows :
$f(u_{i1}) = i - 1,\ 1 \leq i \leq m$

$$f(u_{ij}) = \begin{cases} m(n-1) + (i-1) - \left\lfloor \frac{n-j}{2} \right\rfloor m\ ,\ j = 2t,\ 1 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor \\ m(n-1) + (i-1) - \left\lfloor \frac{2n-j}{2} \right\rfloor m\ , j = 2t+1\ , \\ \qquad 1 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor \end{cases}$$

To find the edge labelings:
Case 1: Consider the edge set $E(G) = u_{ij}u_{ij+1}$, $1 \leq i \leq m$, $1 \leq j \leq n-1$
Sub Case 1.1: $j = 1$, $1 \leq i \leq m$

$f^*(u_{i1}u_{i2}) = (m(n-1) - \lfloor \frac{n-2}{2} \rfloor m)(m(n-1) - \lfloor \frac{n-2}{2} \rfloor m + 2(i-1)).$

Sub Case 1.2: $j = 2t$, $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq i \leq m, 1 \leq j \leq n-1$

$$f^*(u_{ij}u_{ij+1}) = 2m(m(n-1) + (i-1))(\lfloor \tfrac{2n-(j+1)}{2} \rfloor - \lfloor \tfrac{n-j}{2} \rfloor)$$
$$-m^2((\lfloor \frac{2n-(j+1)}{2} \rfloor)^2 - (\lfloor \frac{n-j}{2} \rfloor)^2)$$

Sub Case 1.3: $j = 2t+1$, $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq j \leq n-1, 1 \leq i \leq m$

$$f^*(u_{ij}u_{ij+1}) = 2m(m(n-1) + (i-1))(\lfloor \frac{2n-j)}{2} \rfloor - \lfloor \frac{n-(j+1)}{2} \rfloor)$$
$$-m^2((\lfloor \frac{2n-j}{2} \rfloor)^2 - (\lfloor \frac{n-(j+1)}{2} \rfloor)^2)$$

Case 2: Consider the edge set $E(G) = u_{ij}u_{i+1j}$, $1 \leq i \leq m-1$, $1 \leq j \leq n$
Sub Case 2.1: $j = 1$, $1 \leq j \leq n, 1 \leq i \leq m-1$

$$f^*(u_{i1}u_{i+11}) = \left|(f(u_{i+11}))^2 - (f(u_{i1}))^2\right| = 2i-1$$

Sub Case 2.2: $j = 2t$, $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq j \leq n, 1 \leq i \leq m-1$

$$f^*(u_{ij}u_{i+1j}) = 2m(n-1-\lfloor \frac{n-j}{2} \rfloor) + 2i-1$$

Sub Case 2.3: $j = 2t+1, 1 \leq t \leq \lfloor \frac{n}{2} \rfloor, 1 \leq j \leq n, 1 \leq i \leq m-1$

$$f^*(u_{ij}u_{i+1j}) = 2m(n-1-\lfloor \frac{2n-j}{2} \rfloor) + 2i-1$$

Case 3: Consider the edge set $E(G) = u_{1j}u_{mj}$, $1 \leq j \leq n$, $m = i$
Sub Case 3.1: $j = 1, m = i$
$f^*(u_{11}u_{m1}) = (m-1)^2$.
Sub Case 3.2: $j = 2t$, $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq j \leq n$, $m = i$

$f^*(u_{1j}u_{mj}) = 2m(m-1)\left(n-1-\lfloor \frac{n-j}{2} \rfloor\right) + (m-1)^2$

Sub Case 3.3: $j = 2t+1\ , 1 \le t \le \lfloor \frac{n}{2} \rfloor\ , 1 \le j \le n\ , m = i$

$$f^*(u_{1j}u_{mj}) = 2m(m-1)\left(n-1-\left\lfloor \frac{2n-j}{2} \right\rfloor\right) + (m-1)^2$$

Case 4: Consider the edge set $E(G) = u_{i1}u_{in}, 1 \le i \le m, n = j$

$f^*(u_{i1}u_{in}) = \left(m(n-1) - \lfloor \frac{n}{2} \rfloor m\right)\left(m(n-1) - \lfloor \frac{n}{2} \rfloor m + 2(i-1)\right)$

Therefore the induced edge labeling function are all distinct. Hence the Torus Network $T(m,n)$, m (odd) and n (odd); $m \ge 5$, $n > 5$ is a square difference graph.

□

Theorem 3. *The torus network $T(m,n)$, m (odd) and n (even); $m \ge 5$, $n > 5$ is a square difference graph.*

Proof. Define the vertex labeling $f : V(G) \to \{0, 1, 2, \ldots, mn-1\}$ as follows :

$f(u_{i1}) = i-1,\ 1 \le i \le m$

$$f(u_{ij}) = \begin{cases} n(m-1) + (i-1) - \lfloor \frac{n-i}{2} \rfloor m\ , j = 2t,\ 1 \le t \le \lfloor \frac{n}{2} \rfloor, \\ \qquad 1 \le i \le m \\ n(m-1) + (i-1) - \lfloor \frac{2n-j}{2} \rfloor m\ , j = 2t+1\ , \\ \qquad 1 \le t \le \lfloor \frac{n}{2} \rfloor - 1, 1 \le i \le m \end{cases}$$

To find the edge labelings:

Case 1: Consider the edge set $E(G) = u_{ij}u_{ij+1}\ , 1 \le i \le m\ , 1 \le j \le n-1$

Sub Case 1.1: $j = 1\ , 1 \le i \le m$

$f^*(u_{i1}u_{i2}) = (n(m-1) - \lfloor \frac{n-2}{2} \rfloor m)(n(m-1) - \lfloor \frac{n-2}{2} \rfloor m + 2i) + 2i - 1.$

Sub Case 1.2: $j = 2t\ , 1 \le t \le \lfloor \frac{n}{2} \rfloor\ , 1 \le i \le m, 1 \le j \le n-1$

$$f^*(u_{ij}u_{ij+1}) = 2m(n(m-1)+i)(\lfloor \frac{2n-(j+1)}{2} \rfloor - \lfloor \frac{n-j}{2} \rfloor)$$
$$-m^2((\lfloor \frac{2n-(j+1)}{2} \rfloor)^2 - (\lfloor \frac{n-j}{2} \rfloor)^2$$

Sub Case 1.3: $j = 2t{+}1\ , 1 \leq t \leq \lfloor \frac{n}{2} \rfloor {-}1\ , 1 \leq j \leq n{-}1, 1 \leq i \leq m$
$f^*\left(u_{ij}u_{ij+1}\right) = 2m(n(m-1)+1)(\lfloor \frac{2n-j)}{2} \rfloor - \lfloor \frac{n-(j+1)}{2} \rfloor)$

$$-m^2((\lfloor \frac{2n-j}{2} \rfloor)^2 - (\lfloor \frac{n-(j+1)}{2} \rfloor)^2$$

Case 2: Consider the edge set $E\left(G\right) = u_{ij}u_{i+1j}\ , 1 \leq i \leq m{-}1\ , 1 \leq j \leq n$
Sub Case 2.1: $j = 1\ , 1 \leq j \leq n, 1 \leq i \leq m-1$
$f^*\left(u_{i1}u_{i+11}\right) = \left|\left(f\left(u_{i+11}\right)\right)^2 - \left(f\left(u_{i1}\right)\right)^2\right| \ = 2i-1$

Sub Case 2.2: $j = 2t\ , 1 \leq t \leq \lfloor \frac{n}{2} \rfloor\ \ , 1 \leq j \leq n, 1 \leq i \leq m-1$

$$f^*\left(u_{ij}u_{i+1j}\right) = 2(n(m-1) - \lfloor \frac{n-j}{2} \rfloor m) + 2i + 1$$

Sub Case 2.3:$j = 2t{+}1, 1 \leq t \leq \lfloor \frac{n}{2} \rfloor {-}1, 1 \leq j \leq n, 1 \leq i \leq m{-}1$

$$f^*\left(u_{ij}u_{i+1j}\right) = 2(n(m-1) - \lfloor \frac{2n-j}{2} \rfloor m) + 2i + 1$$

Case 3: Consider the edge set $E\left(G\right) = u_{i1}u_{in}\ , 1 \leq i \leq m\ , n = j$

$$f^*\left(u_{i1}u_{in}\right) = n(m-1)(n(m-1) + 2i) + 2i - 1$$

Case 4: Consider the edge set $E\left(G\right) = u_{1j}u_{mj}, 1 \leq j \leq n, m = i$
Sub Case 4.1: $j = 1, 1 \leq j \leq n, m = i$

$$f^*\left(u_{11}u_{m1}\right) = \left|\left(f\left(u_{m1}\right)\right)^2 - \left(f\left(u_{11}\right)\right)^2\right| = \left(m-1\right)^2.$$

Sub Case 4.2: $j = 2t\ , 1 \leq t \leq \lfloor \frac{n}{2} \rfloor, 1 \leq j \leq n\ , m = i$

$$f^*\left(u_{1j}u_{mj}\right) = 2(m-1)(n(m-1) - \lfloor \frac{n-j}{2} \rfloor m) + (m)^2 - 1$$

Sub Case 4.3: $j = 2t + 1\ , 1 \leq t \leq \lfloor \frac{n}{2} \rfloor - 1, 1 \leq j \leq n\ , m = i$

$$f^*\left(u_{1j}u_{mj}\right) = 2(m-1)(n(m-1) - \lfloor \frac{2n-j}{2} \rfloor m) + (m)^2 - 1$$

Therefore the induced edge labeling function are all distinct. Hence the Torus Network $T\left(m,n\right),\ m$ (odd) and n (even); $m \geq 5,\ n > 5$ is a square difference graph.

□

4 Conclusion

In this paper we have proved that the torus network $T(m, n)$ admits square difference labeling . Further we intend to study the square difference labeling of Torus network $T(m, m)$.

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