International Journal of Pure and Applied Mathematics
Volume 120 No. 8 2018, 177-185
ISSN: 1314-3395 (on-line version)
url: http://www.acadpubl.eu/hub/
Special Issue
http://www.acadpubl.eu/hub/

Fixed Point Results in Partially Ordered b-metric Spaces

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Abstract

In the present paper, we show the existence of a fixed point for a nondecreasing mapping in partially ordered complete b-metric space using sequential monotone property of the space. Moreover common fixed points for two mappings are also proved in the same space. In order to illustrate our main results, we give some examples.

AMS Subject Classification:47H10, 54H25

Key Words and Phrases: b**-metric space; partially ordered space; sequential monotone property; non-decreasing mappings.**

1 Introduction and Preliminaries

The Banach contraction principle is widely applied in different branches of mathematics and is regarded as the source of metric fixed point theory. It has been generalized in several directions, see for example [?] and [?]. Another recent direction of such generalizations, (see [?]-[?]), has been obtained by weakening the requirements in the contractive condition and, in compensation, by simultaneously enriching the metric space structure with a partial order. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity ([?]-[?]). The concept of b-metric space (also called quasimetric spaces) was introduced by Bakhtin in [?] and used by Czerwik in [?]. For previous results on b-metric spaces or extensions of this concept see also Bourbaki [?], Bakhtin [?], Blumenthal [?], among others.

Our technique of proof is essentially different. In our work, we show the existence of a fixed point for a nondecreasing mapping, not necessarily be continuous, in partially ordered complete b-metric space using a restrictive contractive condition.
Recall that if $(X, \preceq)$ is a partially ordered set and $T : X \to X$ such that for $x, y \in X, x \preceq y$ implies $Tx \preceq Ty$ then the mapping T is said to be nondecreasing.

Definition 1. (Bakhtin [?], Czerwik [?]): Let X be a set and let $s \geq 1$ be a given real number. A functional $d\colon X \times X \to R^{+}$ is said to be a b-metric if the following conditions are satisfied:

1. if $x, y \in X$, then $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$, for all $x, y] \in X$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space. It is easy to see that any metric space is a b-metric space with $s = 1$. Therefore, the class of b-metric spaces is larger than the class of metric spaces.

example 1. The set of real numbers together with the functional $d(x, y) = |x - y|^2$ for all $x, y \in R$ is a b-metric space with constant $s = 2$. Also, we obtain that d is not a metric on X.

Definition 2. [?]: Let $(X, \preceq)$ be a partially ordered set, and T and G be self-mappings of X. It is said that T is a G-isotone mapping if, for any $x, y \in X$, $Gx \preceq Gy \implies Tx \preceq Ty$.

Definition 3. [?]:A triple $(X, \tau, \preceq)$ is called a partially ordered topological space if is a Hausdorff topology on X and $\preceq$ is a partial order on X. A partially ordered topological space $(X, , \preceq)$ is said to have the sequential g-monotone property if it verifies:

1. If $\{x_n\}$ is a non-decreasing sequence and $\{x_n\} \to x$, then $gx_m \preceq gx$ for all m.

2. If $\{y_n\}$ is a non-increasing sequence and $\{y_n\} \to y$, then $gy_m \succeq gy$ for all m.

If g is the identity mapping, then X is said to have the sequential monotone property.

Some new features that our approach brings to the fixed point theory in b-metric spaces are :

1. We have used a restrictive contractive condition (only for comparable elements of the metric space i.e. if $x \preceq y$) rather than a contractive condition (for all members of metric space i.e. for all $x, y \in X$).

2. The result has shown the existence of fixed point for a mapping which is not continuous in b-metric spaces whereas most of the previous articles are for a continuous mapping.

2 Main Result

The first theorem shows the existence of a fixed point for a mapping which is not continuous.

Theorem 4. *Let (X, d) is an ordered complete b-metric space with $s \geq 1$ and " $\preceq$ " the partial order on X. Let $f\colon X \to X$ is a non-decreasing mapping on X with the following conditions:*

1. *There exist $x_0 \in X$ s.t. $x_0 \preceq f(x_0)$*

2. *$(X, \tau, \preceq)$ has the sequential monotone property.*

Then f has a fixed point in X.

Proof. let $x_0 \in X$ and define $x_{n+1} = f(x_n)$ for $n \in N$. Then using the condition (2), we have $x_0 \preceq x_1$ and so by (1) there exist $k \in [0, \frac{1}{s}]$ such that we have

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \leq kd(x_0, x_1)$$

Since $x_0 \preceq x_1$ we have $f(x_0) \preceq f(x_1)$ i.e. $x_1 \preceq x_2$ and so by (1) we have

$$d(x_2, x_3) = d(f(x_1), f(x_2)) \leq kd(x_1, x_2) \leq k^2 d(x_0, x_1)$$

Thus on the successive application of condition (1), we have

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) \text{ for all } n \in N$$

And

$$x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq x_4 \preceq ...(2.1)$$

Next we prove that the sequence $\{x_n\}$ is Cauchy. We have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^p d(x_{n+p-1}, x_{n+p}) \\ &\leq sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + \ + s^p k^{n+p-1} d(x_0, x_1) \\ &\leq sk^n [1 + sk + (sk)^2 + ... + (sk)^{p-1}] d(x_0, x_1) \\ &= sk^n . \frac{1-(sk)^p}{(1-sk)} . d(x_0, x_1) \to 0, asn \to \infty. \end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence. Since the b-metric space X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now since $\{x_n\}$ is a non-decreasing sequence and $x_n \to x^*$, by the sequential monotone property of $(X, \tau_m, \preceq)$, we have $x_n \to x^*$ for all n. And so by the condition (1) we have

$$d(f(x_n), f(x^*)) \leq kd(x_n, x^*) \text{ for } k \in [0, \tfrac{1}{s}]$$

Letting $n \to \infty$, we get $x_{n+1} \to f(x^*)$ as $n \to \infty$. By the uniqueness of the limit, we conclude that $f(x^*) = x^*$ and x^* is a fixed point of f.
For the uniqueness, let us assume that there exist $y^* \in X$ such that $f(y^*) = y^*$ and we see that

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq kd(x^*, y^*) \text{ for } k \in [0, \tfrac{1}{s}]$$

which gives $d(x^*, y^*) = 0$. Hence $x^* = y^*$. This completes the proof. □

example 2. Let $(X, \preceq)$ be the partially ordered set with $X = [0, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Let X be equipped with the b-metric $d(x, y) = |x - y|^2$ for all $x, y \in X$, $(s = 2)$ and let $f \colon X \to X$ and defined by $fx = \frac{x^2}{3} + \frac{2}{3}$. It is easy to verify the following:

1. f is a non-decreasing mapping.
2. There exist $x_0 = 0 \in X$ s.t. $g(x_0) = 0 \leq \frac{2}{3} = f(x_0)$.

Let $x, y \in X$ such that $x \preceq y$ i.e. $x \leq y$. Next we show that the inequality is satisfied with $k = \frac{4}{9} \in [0, \frac{1}{2}]$. If it is not so then

$$\begin{aligned} d(f(x), f(y)) &> kd(x, y) \\ i.e.\frac{1}{9}|x^2 - y^2|^2 &> \frac{4}{9}|x - y|^2 \\ i.e.|x + y|^2 &> 4 \end{aligned}$$

which is impossible since $x, y \in [0, 1]$. Hence the inequality holds. Therefore by the theorem 2.1, f has a unique common fixed point which is $t = 1$.

Theorem 5. *Let (X, d) is an ordered complete b-metric space with $s \geq 1$ and " $\leq$ " the partial order on X. Let $f, g \colon X \to X$ are two mappings such that f is g-isotone mapping with the following conditions:*

1. *there exist $k \in [0, \frac{1}{s})$ such that*

$$d(f(x), f(y)) \leq kd(g(x), g(y)) \text{ if } g(x) \preceq g(y).$$

2. *There exist $x_0 \in X$ s.t. $g(x_0) \preceq f(x_0)$,*
3. *$(X, \tau_m, \preceq)$ has the sequential monotone property.*
4. *$g(X)$ is closed.*

Then f and g have a coincidence point in X. Moreover, if g is a continuous idempotent mapping then f and g have a common fixed point.

Proof. Let $x_0 \in X$ and set $z_0 = g(x_0) \preceq f(x_0) = z_1$ and $z_{n+1} = f(x_n) = g(x_{n+1})$ for $n \in N$. Now we prove that the sequence $\{z_n\}$ is a non-decreasing sequence i.e. $z_n \preceq z_{n+1}$ $\forall n \in N$. We prove by using mathematical induction method. It is true for $n = 0$. We assume that $z_{n-1} \preceq z_n$ for some $n \in N$, i.e. $g(x_{n-1}) \preceq g(x_n)$. Since the f is g-isotone mapping we have $f(x_{n-1})) \preceq f(x_n)$ i.e. $z_n \preceq z_{n+1}$. This means that the sequence $\{z_n\}$ is a non-decreasing sequence i.e.

$$z_0 \preceq z_1 \preceq z_2 \preceq z_3 ...(2.1)$$

Since $g(x_0) \preceq g(x_1)$ by (1) we have

$$d(z_1, z_2) = d(f(x_0), f(x_1)) \leq kd(g(x_0), g(x_1)) = kd(z_0, z_1)$$

Again since $z_1 \preceq z_2$ or $g(x_1) \preceq g(x_2)$ by (1) we have

$$d(z_2, z_3) = d(f(x_1), f(x_2)) \leq kd(g(x_1), g(x_2)) = kd(z_1, z_2) = k^2 d(z_0, z_1)$$

Thus on the successive application of condition (1) and (2.1), we have

$$d(z_n, z_{n+1}) \leq k^n d(z_0, z_1) \text{ for all } n \in N$$

Next we prove that the sequence $\{z_n\}$ is Cauchy. We have

$$\begin{aligned} d(z_n, z_{n+p}) &\leq sd(z_n, z_{n+1}) + s^2 d(z(n+1, z_{n+2}) + ... + s^p d(z_{n+p-1}, z_{n+p}) \\ &\leq sk^n d(z_0, z_1) + s^2 k^{n+1} d(z_0, z_1) + ... + s^p k^{n+p-1} d(z_0, z_1) \\ &\leq sk^n [1 + sk + (sk)^2 + \ + (sk)^(p-1)] d(z_0, z_1) \\ &= sk^n.(1-(sk)^p)/(1-sk) d(z_0, z_1) \to 0, asn \to \infty. \end{aligned}$$

Hence the sequence $\{z_n\}$ is a Cauchy sequence. Since the b-metric space X is complete, there exists $t \in X$ such that $z_n \to t$ as $n \to \infty$ i.e. $f(x_n) \to t$ and $g(x_n) \to t$ as $n \to \infty$. Since $g(X)$ is closed, there is a point $w \in X$ such that $g(x_n) \to g(w)$ as $n \to \infty$. And so by the uniqueness of the limit, we get $g(w) = t$.
Now since z_n or $g(x_n)$ is a non-decreasing sequence and $z_n \to t$ or $g(x_n) \to t$, by the sequential monotone property of $(X, \tau_m, \preceq)$, we have $z_n \preceq t$ or $g(x_n) \preceq t$ for all n. And so by the condition (1) we have

$$d(f(x_n), f(w)) \leq kd(g(x_n), g(w)) \text{ for } k \in [0, \tfrac{1}{s})$$

Letting $n \to \infty$, we get $f(x_n) \to f(w)$ as $n \to \infty$. By the uniqueness of the limit, we conclude that $f(w) = t$ i.e. $f(w) = t = g(w)$. Therefore w is a coincidence point of f and g.
Next we prove that t is a fixed point of f and g. Since g is a continuous mapping and $g(x_n) \to t$ as $n \to \infty$, we have $gg(x_n) \to g(t)$ as $n \to \infty$. By the idempotent property of the mapping g we get $g(x_n) \to g(t)$ as $n \to \infty$. Therefore by the uniqueness of the limit, we get $g(t) = t$.
Again since $g(x_n)$ is a non-decreasing sequence and $g(x_n) \to g(t)$,by the sequential monotone property of $(X\tau_m, \preceq)$, we have $g(x_n) \preceq g(t)$ for all n. And so by the condition (1) we have

$$d(f(x_n), f(t)) \leq kd(g(x_n), g(t)) \text{ for } k \in [0, \tfrac{1}{s})$$

Letting $n \to \infty$, we get $f(x_n) \to f(t)$ as $n \to \infty$. By the uniqueness of the limit, we conclude that $f(t) = t$ i.e.$f(t) = t = g(t)$. Therefore t is fixed point of f and g. For the uniqueness, let us assume that there exist $t_1 \in X$ such that $f(t_1) = t_1 = g(t_1)$ and we see that

$$d(t, t_1) = d(f(t), f(t_1)) \leq kd(g(t), g(t_1)) = kd(t, t_1) \text{ for } k \in [0, \tfrac{1}{s})$$

which gives $d(t, t_1) = 0$. Hence $t = t_1$. □

example 3. Let $(X, \preceq)$ be the partially ordered set with $X = [0, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Let X be equipped with the b-metric $d(x, y) = |x - y|^2$ for all $x, y \in X$, $(s = 2)$ and let $f, g \colon X \to X$ and defined by $fx = \frac{x^2}{3} + \frac{2}{3}$ and $gx = x$ for all $x \in X$.It is easy to verify that the mappings are satisfying all the conditions of Theorem 5.
Let $x, y \in X$ such that $g(x) \preceq g(y)$ i.e. $x \leq y$. Next we show that the inequality is satisfied with $k = \frac{4}{9} \in [0, \frac{1}{2}]$. If it is not so then

$$\begin{aligned} d(f(x), f(y)) &> kd(g(x), g(y)) \\ i.e.\frac{1}{9}|x^2 - y^2|^2 &> \frac{4}{9}|x - y|^2 \\ i.e.|x + y|^2 &> 4 \end{aligned}$$

which is impossible since $x, y \in [0, 1]$. Hence the inequality holds. Moreover, we can see that the mapping g is a continuous idempotent mapping. Therefore by the theorem 2.2 f and g have a common fixed point which is $t = 1$.

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