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Lucky Labeling of Certain Cubic Graphs

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Abstract

We call a mapping $f : V(G) \rightarrow \{1, 2, \ldots, k\}$, a *lucky labeling* or *vertex labeling by sum* if for every two adjacent vertices u and v of G, $\sum_{(v,u)\in E(G)} f(v) \neq \sum_{(u,v)\in E(G)} f(u)$. The lucky number of a graph G, denoted by $\eta(G)$, is the least positive k such that G has a lucky labeling with $\{1, 2, \ldots, k\}$ as the set of labels. Ali Deghan et al. [2] have proved that it is NP-complete to decide whether $\eta(G) = 2$ for a given 3-regular graph G. In this paper, we show that $\eta(G) = 2$ for certain cubic graphs such as cubic diamond $k-$chain, petersen graph, generalized heawood graph, $G(2n, k)-$ cubic graph, pappus graph and dyck graph.

AMS Subject Classification: 05C78
Key Words: Lucky labeling, petersen, generalized heawood, pappus, dyck

1 Introduction

Graph coloring is one of the most studied subjects in graph theory. Recently, Czerwinski et al. [1] have studied the concept of lucky labeling as a vertex coloring problem. Ahadi et al. [3] have proved that computation of lucky number of planar graphs is NP-hard. Ali Dehghan et al. [4] have given the algorithmic complexity of various proper labeling problems. It was shown that for a given

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3-regular graph, a weighting of the edges from $\{a, b\}$, $(a \neq b)$ that induces a proper vertex coloring is NP-complete. Ali Deghan et al. [2] have proved that it is NP-complete to decide whether $\eta(G) = 2$ for a given 3-regular graph G. Lucky labeling has been studied extensively by several authors, for instance see [1, 2, 3, 5]. In this paper we show that $\eta(G) = 2$ for certain cubic graphs such as cubic diamond $k-$chain, petersen graph, generalized heawood graph, $G(2n, k)-$ cubic graph, pappus graph and dyck graph.

2 Main Results

We begin with the definition of lucky labeling.
For a vertex u in a graph G, let $N(u) = \{v \in V(G)/(u, v) \in E(G)\}$ and $N[u] = N(u) \bigcup \{u\}$.

Definition 1. [1] A lucky labeling of a graph G is a function $l : V(G) \to N$, such that for every two adjacent vertices u and v of G, $\sum_{w \sim u} l(w) \neq \sum_{w \sim v} l(w)$ ($x \sim y$ means that x is joined to y). The lucky number of G, denoted by $\eta(G)$, is the minimum number k such that G has a lucky labeling $l : V(G) \to \{1, 2, \ldots, k\}$.

Since the problem is NP- Complete for cubic graphs, we concentrate on certain cubic graphs for which lucky number can be determined.

2.1 Cubic Diamond k-chain

Definition 2. Cubic diamond k-chain is a graph obtained from k number of diamonds $D_1, D_2, \ldots, D_k$ such that a vertex of degree 2 in D_i is joined to a vertex of degree 2 in D_{i+1}, $1 \leq i \leq k-1$. In addition the remaining diametrically opposite vertices of degree 2 in each D_i are joined by an edge and is called the diagonal edge. To make the graph cubic, D_1 and D_k are further modified by subdividing the diagonal edge once and joining the new vertex to the left out vertex of degree 2 in the respective diamonds.

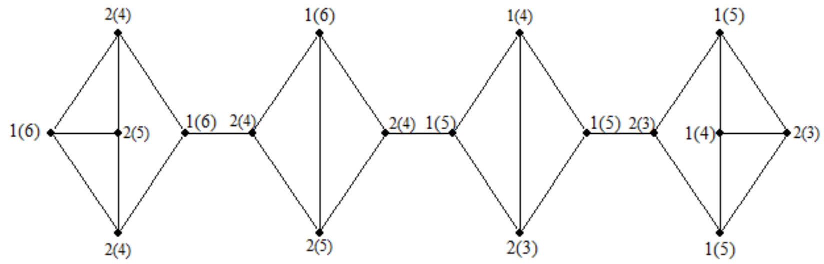

Figure 1: Lucky Labeling of cubic diamond 4-chain

2.1.1 Algorithm for lucky labeling of cubic diamond $k-$chain:

Input : Cubic Diamond k-chain.
Algorithm : We label the diamonds $D_1, D_2, \ldots, D_k$ in the following order:
Step 1. When k is even, $k \geq 2$,
(i). We label the vertices on the diagonal edge of D_1 as 2 and the left out vertices of D_1 are labeled as 1.
(ii). Further, we label the vertices on the diagonal edge of D_k as 1 and the left out vertices of D_k are labeled as 2.
(iii). We label the diagonal edge of D_i, for i even, as 1 and 2 and the left out vertices of D_i as 2.
(iv). We label the diagonal edge of D_i, for i odd, as 1 and 2 and the left out vertices of D_i as 1.
Step 2. When k is odd, $k \geq 2$,
(i). We label the vertices on the diagonal edge of D_1 and D_k as 2 and the left out vertices as 1.
(ii). Now, we follow the same labeling pattern as (iii) and (iv) in step 1. See Figure 1.
Output: $\eta(G) = 2$.
Proof of correctness: Let $e = (u, v) \in E(G)$. If u and v are two adjacent vertices in D_1 labeled 2, then $c(u) = 4$ and $c(v) = 5$. If u and v are two adjacent vertices in D_1 labeled 1 and 2, then $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 6$ and $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 4$. By labeling algorithm, we have $c(u) = 4$ or 5 and $c(v) = 6$ or 4. Therefore $c(u) \neq c(v)$, $\forall \ (u, v) \in E(G)$. Hence a cubic diamond k-chain admits lucky labeling and $\eta(G) = 2$. See Figure 1.

Theorem 3. *Let G be a cubic diamond k-chain. Then $\eta(G) = 2$.*

2.2 Petersen Graph

Definition 4. The Petersen graph shown in Figure 2 is an undirected graph with 10 vertices and 15 edges.

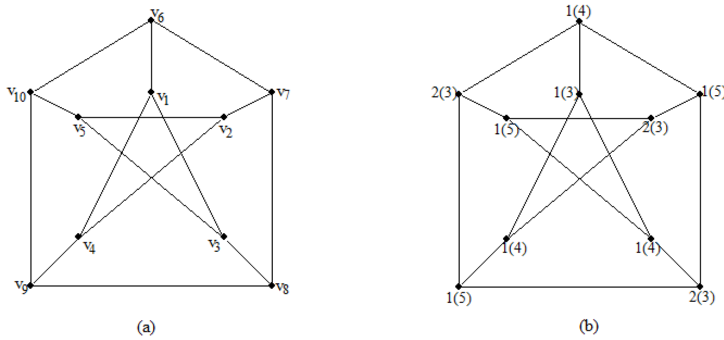

Figure 2: Lucky Labeling of Petersen Graph

2.2.1 Algorithm for lucky labeling of petersen graph:

Input: Petersen Graph.
Algorithm: We label the vertices as $v_1, v_2, \ldots, v_{10}$ as in Figure 2(a). We label the vertices of v_2, v_8 and v_{10} as 2 and the remaining vertices as 1. See Figure 2(b).
Output: $\eta(G) = 2$.
Proof of Correctness: Let $e = (u, v) \in E(G)$. If u and v are two adjacent vertices in the outer cycle, then $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 3$ or 4 and $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 5$. By labeling algorithm, we have $c(u) = 4$ or 5 and $c(v) = 3$ or 4. Therefore $c(u) \neq c(v)$, $\forall$ $(u, v) \in E(G)$. Hence the petersen graph admits lucky labeling and $\eta(G) = 2$. See Figure 2.

Theorem 5. *Let G be a petersen graph. Then $\eta(G) = 2$.*

2.3 Generalized Heawood Graph

Definition 6. Let C be a cycle on $7k$ vertices, $k \geq 1$. Beginning with a vertex v, join it to a vertex on C, skipping 4 vertices in between in the clockwise sense. Take a vertex adjacent to v and do the same in the anti-clockwise sense. Repeat till all vertices are visited.

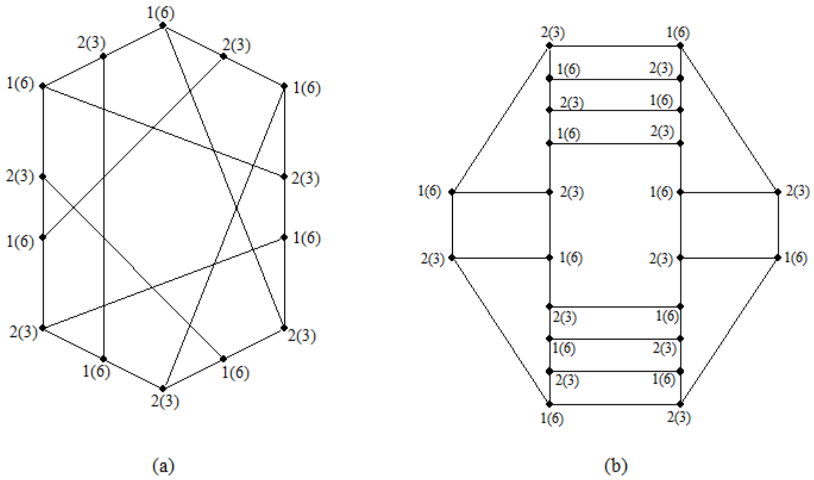

Figure 3: Lucky Labeling of (a) Generalized Heawood Graph and (b) $G(2n, k)-$cubic graph

2.3.1 Algorithm for lucky labeling of generalized heawood graph:

Input : Generalized Heawood Graph on $7k$ vertices, $k \geq 1$.
Algorithm: Alternately label all the vertices of the cycle on $7k$ vertices as 1 and 2. See Figure $3(a)$. *Output*: $\eta(G) = 2$
Proof of Correctness: Let $e = (u, v) \in E(G)$. If u and v are two adjacent vertices, since $l(u) = 1$, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 6$. Also since $l(v) = 2$, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 3$. Thus $c(u) \neq c(v)$, $\forall \ (u, v) \in E(G)$. Hence the generalized heawood graph admits lucky labeling and $\eta(G) = 2$. See Figure 3(a).

Theorem 7. *Let G be a generalized heawood graph. Then* $\eta(G) = 2$.

2.4 $G(2n, k)-$ cubic graph

Definition 8. Consider a circular ladder with even number of inner and outer cycle vertices. Select two diametrically opposite edges of the inner cycle and the corresponding diametrically opposite edges of the outer cycle. Subdivide the four ladder edges induced by these diametrically opposite edges equally. This is a bipartite cubic plane graph when k is even. If $2n$ is the number of vertices in each of the outer and inner cycle and the number of subdivisions is k, we call it a $(2n, k)-$cubic graph and denote it as $G(2n, k)$, $n \geq 4, k \geq 1$. We consider the case when k is even. Since $G(2n, k)$ is bipartite, let V_1 and V_2 be the bipartition of $G(2n, k)-$ cubic graph. See Figure 3(b) for $G(8, 2)-$ cubic graph.

2.4.1 Algorithm for lucky labeling of $G(2n, k)-$cubic graph:

Input: $G(2n, k)-$ cubic graph.
Algorithm: Label the vertices of V_1 as 1 and the vertices of V_2 as 2. See Figure 3(b).
Output: $\eta(G) = 2$
Proof of Correctness: Let $e = (u, v) \in E(G)$. If u and v are two adjacent vertices, since $l(u) = 1$, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 6$. Also since $l(v) = 2$, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 3$. Thus $c(u) \neq c(v)$, $\forall \; (u, v) \in E(G)$. Hence the $G(2n, k)-$cubic graph admits lucky labeling and $\eta(G) = 2$. See Figure 3(b).

Theorem 9. *Let G be a $G(2n, k)-$cubic graph. Then* $\eta(G) = 2$.

2.5 Pappus Graph

Definition 10. The pappus graph is a bipartite, 3-regular, undirected graph with 18 vertices and 27 edges. See Figure $4(a)$.

2.5.1 Algorithm for lucky labeling of pappus graph:

Input: Pappus Graph, G.
Algorithm: Let V_1 and V_2 be the bipartition of G. Label vertices of V_1 as 1 and vertices of V_2 as 2. See Figure 4(a). *Output*: $\eta(G) = 2$
Proof of Correctness: Let $e = (u, v) \in E(G)$. If u and v are two adjacent vertices, since $l(u) = 1$, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 6$. Also since $l(v) = 2$, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 3$. Thus $c(u) \neq c(v)$, $\forall \ (u, v) \in E(G)$. Hence the pappus graph admits lucky labeling and $\eta(G) = 2$. See Figure 4(a).

Theorem 11. *Let G be a pappus graph. Then $\eta(G) = 2$.*

2.6 Dyck Graph

Definition 12. The Dyck graph is the unique cubic symmetric graph on 32 nodes. See Figure 4(b).

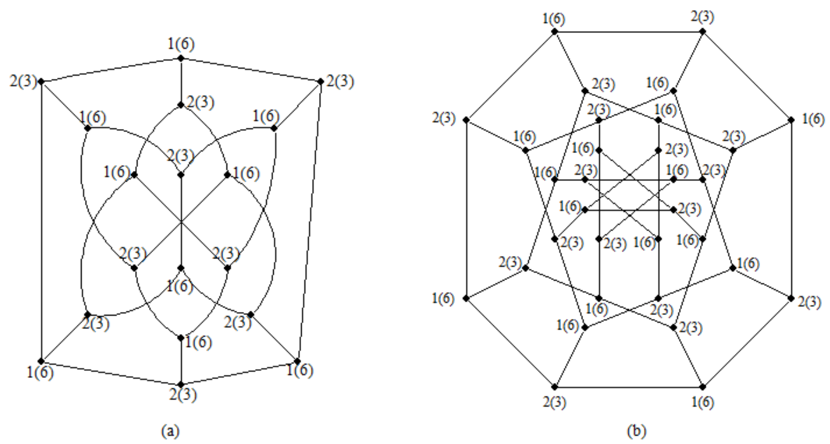

Figure 4: Lucky Labeling of (a) Pappus Graph and (b) Dyck Graph

2.6.1 Algorithm for lucky labeling of Dyck Graph:

Input: Dyck Graph, G.
Algorithm: Let V_1 and V_2 be the bipartition of G. Label vertices

of V_1 as 1 and vertices of V_2 as 2. See Figure 4 (b).
Output: $\eta(G) = 2$
Proof of Correctness: Let $e = (u, v) \in E(G)$. If u and v are two adjacent vertices, since $l(u) = 1$, we have $c(u) = \sum_{v \in N(u)} l(v) + d(u) = 6$. Also since $l(v) = 2$, we have $c(v) = \sum_{u \in N(v)} l(u) + d(v) = 3$. Thus $c(u) \neq c(v)$, $\forall \ (u, v) \in E(G)$. Hence the dyck graph admits lucky labeling and $\eta(G) = 2$. See Figure 4 (b).

Theorem 13. *Let G be a dyck graph. Then $\eta(G) = 2$.*

3 Conclusion

In this paper, we find the lucky number for certain cubic graphs such as cubic diamond $k-$chain, petersen graph, generalized heawood graph, $G(2n, k)-$ cubic graph, Pappus graph and Dyck graph.

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