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Total Coloring of Splitting Graph of Some Standard Graphs

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Abstract

A total coloring of a graph G is an assignment of colors to both the vertices and the edges of G such that no two adjacent or incident vertices and edges of G are assigned with the same colors. In this paper, we found that total chromatic number of $S'(B_{n,n}), S'(K_{m,n}), S'(F_{1,n}), S'(W_n)$.
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1 Introduction

In this paper, we have chosen finite, simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$, respectively. In 1965, the concept of total coloring was introduced by Behzad [1]. A total coloring of G is a function $f : S \to C$, where $S = V(G) \cup E(G)$ and C is a set of colors to satisfies the given conditions.
(i) no two adjacent vertices receive the same color.
(ii) no two adjacent edges receive the same color.
(iii) no edges and its end vertices receive the same color.

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The *total chromatic number* $\chi''(G)$ of a graph G is the minimum cardinality k such that G may have a total coloring by k colors. Behzad [1] and Vizing [7] Conjectured that for every simple graph G has $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of the graph G. This conjecture is called as the Total Coloring Conjecture(TCC). Rosenfeld [5] and Vijayaditya [6] verified the TCC, for every graph G with maximum degree $\leq$ 3. In Borodin [2] verified the total coloring conjecture(TCC) for the maximaum degree $\geq$ 9 in planar graphs. In recent era, total coloring have been extensively studied in different families of graph. Muthuramakrishnan et.al [3] prove that Total Chromatic Number of Star and Bistar graphs. In this paper, we found that the total chromatic number of splitting graph of Bistar graph, complete bipartite graph , Fan graph and wheel graph.

2 Total coloring of $S'(B_{n,n})$.

Definition 2.1. For a graph G, the *splitting graph* $S'(G)$ [4] of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 2.2. The *bistar* $B_{n,n}$ is graph obtained by joining the root vertices of two copies of $K_{1,n}$ by an edge.

Theorem 2.3. *Let $S'(B_{n,n})$ be the splitting graph of bistar graph. Then $\chi''(S'(B_{n,n})) = 2n + 3, \quad n \geq 2$.*

Proof. Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uu_i : 1 \leq i \leq n\} \cup \{vv_i : 1 \leq i \leq n\} \cup \{uv\}$, where $\{u_i, v_i : 1 \leq i \leq n\}$ are pendent vertices and $\{u, v\}$ are root vertices. Now we construct the splitting graph of Bistar. Adding the new vertices $\{u', v', u_i', v_i' : 1 \leq i \leq n\}$ corresponding to these vertices $\{u, v, u_i, v_i : 1 \leq i \leq n\}$ of $B_{n,n}$, which are added to obtain $S'(B_{n,n})$. In $S'(B_{n,n})$, the vertex and the edge set are given by $V(S'(B_{n,n})) =$ $\left\{\left(\bigcup_{i=1}^{n}\{u_i \cup u_i' \cup v_i \cup v_i'\}\right) \cup \{u\} \cup \{v\} \cup \{u'\} \cup \{v'\}\right\}$ and

$$E(S'(B_{n,n})) = \begin{cases} \left(\bigcup_{i=1}^{n}\{uu_i \cup uu_i' \cup vv_i \cup vv_i' \cup u'u_i \cup v'v_i\}\right) \cup \\ \{uv\} \cup \{u'v\} \cup \{uv'\}. \end{cases}$$

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Now we construct the total coloring f, such that $f : S \to C$, where $S = V(S'(B_{n,n})) \cup E(S'(B_{n,n}))$ and $C = \{1, 2, 3, \ldots 2n + 3\}$. Now we assign the total coloring to these vertices and edges as follows:
$f(u) = 2n + 2, \quad f(v) = 2n + 1, \quad f(u') = 1, \quad f(v') = 2n + 3$

$$f(u_i) = f(v_i) = f(uu_i') = f(vv_i') = \begin{cases} 2i, & \text{if } 2i \not\equiv 0(mod\ 2n) \\ 2n, & \text{otherwise} \quad \text{for } 1 \le i \le n \end{cases}$$

$$f(u_i') = f(v_i') = \begin{cases} i, & \text{if } i \not\equiv 0(mod\ n) \\ n, & \text{otherwise} \quad \text{for } 1 \le i \le n \end{cases}$$

$$f(uu_i) = f(vv_i) = \begin{cases} 2i-1, & \text{if } 2i-1 \not\equiv 0(mod\ 2n-1) \\ 2n-1, & \text{otherwise} \quad \text{for } 1 \le i \le n \end{cases}$$

$$f(u'u_i) = \begin{cases} 2i+1, & \text{if } 2i+1 \not\equiv 0(mod\ 2n+1) \\ 2n+1, & \text{otherwise} \quad \text{for } 1 \le i \le n \end{cases}$$

$$f(v'v_i) = \begin{cases} 2i+2, & \text{if } 2i+2 \not\equiv 0(mod\ 2n+2) \\ 2n+2, & \text{otherwise} \quad \text{for } 1 \le i \le n \end{cases}$$

$f(uv) = 2n + 3, \qquad f(uv') = 2n + 1, \qquad f(u'v) = 2n + 2$
Based on the above coloring pattern, the graph $S'(B_{n,n})$ is properly total colored with $2n+3$ colors. Hence, the total chromatic number of $S'(B_{n,n})$, $\qquad \chi''(S'(B_{n,n})) = 2n + 3$. □

3 Total coloring of $S'(K_{m,n})$.

Definition 3.1. The *Complete bipartite graph* is a simple bipartite graph in which each vertex of X is joined to each vertex of Y. If $|X| = m, |Y| = n$, such a graph is denoted by $K_{m,n}$.

Theorem 3.2. *Let $S'(K_{m,n})$ be the splitting graph of $K_{m,n}$. Then $\chi''(S'(K_{m,n})) = \Delta(S'(K_{m,n})) + 1$, for $m \le n$*

Proof. Let $V(K_{m,n}) = \{x_i : 1 \le i \le m\} \cup \{y_j : 1 \le j \le n\}$ and $E(K_{m,n}) = \bigcup_{i=1}^{m} \{x_i y_j : j > i,\ i \le j \le n\}$. Now we construct the splitting graph of $K_{m,n}$. Adding the new vertices $\{x_i' : 1 \le i \le m\}$ and $\{y_j' : 1 \le j \le n\}$ corresponding to these vertices $\{x_i : 1 \le i \le m\}$ and $\{y_j : 1 \le j \le n\}$ of $K_{m,n}$, which are added to obtain

$S'(K_{m,n})$. In $S'(K_{m,n})$, the vertex set and the edge set are given by $V(S'(K_{m,n})) = V(K_{m,n}) \cup \{x_i{}' : 1 \le i \le m\} \cup \{y_j{}' : 1 \le j \le n\}$ and $E(S'(K_{m,n})) = \left\{ \left(\bigcup_{i=1}^{m} (x_i y_j \cup x_i y_j{}' \cup x_i{}' y_j) : i \le j \le n \right) \right\}$. Now we construct the total coloring f, such that $f : S \rightarrow C$, where $S = V(S'(K_{m,n})) \cup E(S'(K_{m,n}))$ and C is set of positive integers. Now we assign the total coloring to these vertices and edges as follows. we consider the following three cases.

Case (i): When $m < n$,

For all $1 \le i \le m, \quad 1 \le j \le n$

$$f(x_i) = i, \quad f(x_i{}') = m+1, \quad \text{for } 1 \le i \le m$$
$$f(y_j) = 2n+1, \quad f(y_j{}') = m+1, \quad \text{for } 1 \le j \le n$$

$$f(x_i y_j) \equiv \begin{cases} i+j, & \text{if } \ i+j \not\equiv 0 \ (mod \ m+n) \\ m+n, & \text{otherwise} \qquad \text{for } i \le j \le n \end{cases}$$

$$f(x_i y_j{}') \equiv \begin{cases} n+i+j, & \text{if } \ n+i+j \not\equiv 0 \ (mod \ 2n+1) \\ 2n+1, & \text{otherwise} \qquad \text{for } i \le j \le n \end{cases}$$

$$f(x_i{}' y_j) \equiv \begin{cases} m+i+j, & \text{if } \ m+i+j \not\equiv 0 \ (mod \ 2n) \\ 2n, & \text{otherwise} \qquad \text{for } i \le j \le n \end{cases}$$

Case (ii): When $m = n$,

For all $1 \le i \le m, \quad 1 \le j \le n$

$$f(x_i) = i, \quad f(x_i{}') = m+1, \quad \text{for } 1 \le i \le m$$
$$f(y_j) = m+n+1, \quad f(y_j{}') = m+1 \text{ for } 1 \le j \le n$$

$$f(x_i y_j) \equiv \begin{cases} i+j, & \text{if } \ (i+j) \not\equiv 0 \ (mod \ m+n) \\ m+n, & \text{otherwise} \qquad \text{for } i \le j \le n \end{cases}$$

$$f(x_i y_j{}') \equiv \begin{cases} n+i+j, & \text{if } \ (i+j+n) \not\equiv 0 \ (mod \ 2n+1) \\ 2n+1, & \text{otherwise} \qquad \text{for } i \le j \le n \end{cases}$$

$$f(x_i{}' y_j) \equiv \begin{cases} n+i+j, & \text{if } \ (i+j+m) \not\equiv 0 \ (mod \ 2n) \\ 2n, & \text{otherwise} \qquad \text{for } i \le j \le n \end{cases}$$

It is clear that, the graph $S'(K_{m,n})$ is properly total colored with $\Delta(S'(k_{m,n})) + 1$ colors. Hence the total chromatic number of $S'(K_{m,n}), \chi''(S'(K_{m,n})) = \Delta(S'(k_{m,n})) + 1$. □

4 Total coloring of $S'(F_{1,n})$.

Definition 4.1. The *Fan graph* $F_{1,n}$ is defined as the graph join $K_1 + P_n$, where K_1 is the complete graph with one vertex and P_n is the path graph on n vertices.

Theorem 4.2. *Let $S'(F_{1,n})$ be the splitting graph of Fan graph. Then $\chi''(S'(F_{1,n})) = 2n + 1, \quad n \geq 4$*

Proof. Let $V(F_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$ and $E(F_{1,n}) = \{vv_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$. Now we construct the splitting graph of $F_{1,n}$. Adding the new vertices $\{v', v_i' : 1 \leq i \leq n\}$ corresponding to these vertices $\{v, v_1, v_2, v_3, \ldots, v_n\}$ of $F_{1,n}$, which are added to obtain $S'(F_{1,n})$. In $S'(F_{1,n})$, the vertex set and the edge set are given by $V(S'(F_{1,n}) = \left\{ \left(\bigcup_{i=1}^{n} \{v_i \cup v_i'\} \cup \{v\} \cup \{v'\} \right) \right\}$ and
$E(S'(F_{1,n}) = \left\{ \bigcup_{i=1}^{n} \{vv_i \cup vv_i' \cup v'v_i\} \right\} \cup \{v_iv_{i+1}' : 1 \leq i \leq n-1\} \cup \{v_i'v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$. Now we define the total coloring f, such that $f : S \to C$, where $S = V(S'(F_{1,n})) \cup E(S'(F_{1,n})$ and $C = \{1, 2, 3, \ldots 2n+1\}$. Now we assign the total coloring to these vertices and edges as follows:

$$f(v) = 2n + 1, \quad f(v') = f(v_i') = 2n, \quad \text{for } 1 \leq i \leq n$$

$$f(v_i) = i : 1 \leq i \leq n$$

$$f(vv_i') \equiv \begin{cases} 2i - 1, & \text{if } \ 2i \not\equiv 0 \ (mod \ 2n-1) \text{ for } 1 \leq i \leq n \\ 2n - 1, & \text{otherwise} \end{cases}$$

$$f(v'v_i) \equiv \begin{cases} 2i + 1, & \text{if } \ 2i \not\equiv 0 \ (mod \ 2n+1) \text{ for } 1 \leq i \leq n \\ 2n + 1, & \text{otherwise} \end{cases}$$

$$f(vv_i) \equiv \begin{cases} 2i, & \text{if } \ 2i \not\equiv 0 \ (mod \ 2n) \text{ for } 1 \leq i \leq n \\ 2n, & \text{otherwise} \end{cases}$$

$$f(v_iv_{i+1}) \equiv \begin{cases} 4 + 2i, & \text{if } \ 4 + 2i \not\equiv 0 \ (mod \ 2n) \text{ for } 1 \leq i \leq n-1 \\ 2n, & \text{otherwise} \end{cases}$$

$$f(v_iv_{i+1}') = 2n + 1 : 1 \leq i \leq n-1$$

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$$f({v_i}'v_{i+1}) = 1, \quad \text{for } 2 \leq i \leq n-1$$
$$f({v_1}'v_2) = 3, \qquad f(v_{n-1}v_n) = 2$$

It is clear that, the above rule of total coloring, the graph $S'(F_{1,n})$ is properly total colored with $2n+1$ colors. Hence the total chromatic number of $S'(F_{1,n})$, $\chi''(S'(F_{1,n})) = 2n+1$. □

5 Total coloring of $S'(W_n)$.

Definition 5.1. For any integer $n \geq 4$, the *Wheel graph* W_n is the $n-$ vertex graph obtained by joining a vertex v to each of the $n-1$ vertices $\{v_1, v_2, v_3, \ldots, v_n\}$ of the cycle graph C_{n-1}.

Theorem 5.2. *Let $S'(W_n)$ be the splitting graph of Wheel graph. Then $\chi''(S'(W_n)) = 2n+1, n \geq 4$*

Proof. Let $V(W_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$ and $E(W_n) = \{vv_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. Now we construct the splitting graph of W_n. Adding the new vertices $\{v', {v_1}', {v_2}', {v_3}', \ldots, {v_n}'\}$ corresponding to these vertices $\{v, v_1, v_2, v_3, \ldots, v_n\}$ of W_n, which are added to obtain $S'(W_n)$. In $S'(W_n)$, the vertex set and the edge set are given by $V(S'(W_n)) = \{\{v_i : 1 \leq i \leq n\} \cup \{{v_i}' : 1 \leq i \leq n\}) \cup \{v\} \cup \{v'\}\}$ and
$E(S'(W_n)) = \left(\bigcup_{i=1}^{n} \{vv_i \cup v{v_i}' \cup v'v_i\}\right) \cup$
$\left(\bigcup_{i=1}^{n-1} \{{v_i}'v_{i+1} \cup v_iv_{i+1} \cup v_i{v_{i+1}}'\}\right) \cup \{{v_n}'v_1\} \cup \{v_nv_1\} \cup \{v_n{v_1}'\}$. Now we construct the total coloring f, such that $f : S \to C$, where $S = V(S'(W_n)) \cup E(S'(W_n))$ and $C = \{1, 2, 3, \ldots 2n+1\}$. Now we assign the total coloring to these vertices and edges as follows. we consider the following two cases.

Case(i): when n is even $(n \geq 4)$

$$f(v) = 2n+1, \quad f(v') = 3, \quad f({v_i}') = n+1, \text{for } 1 \leq i \leq n$$

$$f(v_i) \equiv \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 2) \quad \text{for } 1 \leq i \leq n \\ 2, & \text{if } i \equiv 0 \ (mod\ 2) \end{cases}$$

$$f(vv_i) \equiv \begin{cases} n+i, & \text{if } (n+i) \not\equiv 0 \ (mod\ 2n) \text{ for } 1 \leq i \leq n \\ 2n, & \text{otherwise} \end{cases}$$

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$$f(vv_i{}') \equiv \begin{cases} i, & \text{if } \ i \not\equiv 0 \ (mod \ n) \text{ for } 1 \leq i \leq n \\ n, & \text{otherwise} \end{cases}$$

$$f(v_iv_{i+1}) \equiv \begin{cases} 3, & \text{if } \ i \equiv 1 \ (mod \ 2) \text{ for } 1 \leq i \leq n-1 \\ 4, & \text{if } \ i \equiv 0 \ (mod \ 2) \end{cases}$$

$$f(v_nv_1) \ = \ 4$$

$$f(v_iv_{i+1}{}') \equiv \begin{cases} n+i+1, & \text{if } (n+i+1) \not\equiv 0 \ (mod \ 2n) \\ 2n, & \text{otherwise} \qquad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$$f(v_nv_1{}') \ = \ 2n+1,$$

$$f(v_i{}'v_{i+1}) \equiv \begin{cases} n+i+3, & \text{if } (n+i+3) \not\equiv 0 \ (mod \ 2n) \\ 2n, & \text{otherwise} \qquad \text{for } 1 \leq i \leq n-2 \end{cases}$$

$$f(v_n{}'v_1) \ = \ n+3, \qquad f(v_{n-1}{}'v_n) \ = \ 1$$

$$f(v'v_i) \equiv \begin{cases} i+3, & \text{if } (i+3) \not\equiv 0 \ (mod \ 2n) \text{ for } 2 \ \leq \ i \ \leq \ n \\ 2n, & \text{otherwise} \end{cases}$$

$$f(v'v_1) \ = \ 2$$

Case(ii): when n is odd $(n \ > \ 4)$

$$f(v) = 2n+1, \qquad f(v') = 3 \qquad f(v_i{}') = n+1, \text{for } 1 \leq i \leq n$$

$$f(v_i) \equiv \begin{cases} 1, & \text{if } \ i \equiv 1 \ (mod \ 2) \text{ for } 1 \leq i \leq n-1 \\ 2, & \text{if } \ i \equiv 0 \ (mod \ 2) \end{cases}$$

$$f(v_n) = n+4$$

$$f(vv_i) \equiv \begin{cases} n+i, & \text{if } n+i \not\equiv 0 \ (mod \ 2n) \text{ for } 1 \leq i \leq n \\ 2n, & \text{otherwise} \end{cases}$$

$$f(vv_i{}') \equiv \begin{cases} i, & \text{if } \ i \not\equiv 0 \ (mod \ n) \text{ for } 1 \leq i \leq n \\ n, & \text{otherwise} \end{cases}$$

$$f(v_iv_{i+1}) \equiv \begin{cases} 3, & \text{if } \ i \equiv 1 \ (mod \ 2) \text{ for } 1 \leq i \leq n-1 \\ 4, & \text{if } \ i \equiv 0 \ (mod \ 2) \end{cases}$$

$$f(v_nv_1) \ = \ 5$$

$$f(v_i v_{i+1}{}') \equiv \begin{cases} n+i+1, & \text{if } n+i+1 \not\equiv 0 \ (mod\ 2n) \\ 2n, & \text{otherwise} \quad \text{for } 1 \leq i \leq n-1 \end{cases}$$

$$f(v_n v_1{}') = 2n+1,$$

$$f(v_i{}' v_{i+1}) \equiv \begin{cases} n+i+3, & \text{if } n+i+3 \not\equiv 0 \ (mod\ 2n) \\ 2n, & \text{otherwise} \quad \text{for } 1 \leq i \leq n-2 \end{cases}$$

$$f(v_n{}' v_1) = n+3, \quad f(v_{n-1}{}' v_n) = 1$$

$$f(v' v_i) \equiv \begin{cases} i+3, & \text{if } i+3 \not\equiv 0 \ (mod\ 2n) \text{ for } 2 \leq i \leq n \\ 2n, & \text{otherwise} \end{cases}$$

$$f(v' v_1) = 2$$

We observe that, the above total coloring pattern, the graph $S'(W_n)$ is properly total colored with $2n+1$ colors. Hence the total chromatic number of $S'(W_n), \chi''(S'(W_n)) = 2n+1$. □

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