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T-coloring of Certain Graphs

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Abstract

For a given finite set T of non-negative integers including zero, a proper vertex coloring is called a T-coloring if the distance of the colors of adjacent vertices is not an element of T. The span of T-coloring is the difference between the largest and smallest colors and the T-span of G is the minimum span over all T-colorings of G. In this paper, we compute T-span and T-edge span of swing graph, theta mesh and shadow graph of a cycle.

AMS Subject Classification: 05C15, 05C38, 05C76.

Key Words and Phrases: T-coloring; T-span; T-edge span; swing graph; theta mesh; shadow graph of a cycle.

1 Introduction

Hale's graph theoretic formulation of the channel assignment problem is as follows [3]: Let V be the set of transmitters $\{x_1, x_2, \ldots, x_n\}$ and G be a graph in which $V(G) = V$ and there is an edge between transmitters x_i and x_j if and only if they interfere. Let T be a subset of the non-negative integers, containing zero, which represents the forbidden set. Such a T will be called a T-set. For a given graph G and a T-set T, a *T-coloring* of G is a function f from $V(G)$ to the set of non-negative integers such that $(x, y) \in E(G)$ implies $|f(x) - f(y)| \notin T$. The *order* of a T-coloring f of G denoted by $\chi_T^f(G)$ is the number of distinct values of $f(x)$, $x \in V(G)$. The *span* of a T-coloring f of G, $sp_T^f(G) = \max_{x,y\in V(G)} |f(x) - f(y)|$. The *edge span* of a T-coloring f of G, $esp_T^f(G) = \max_{(x,y)\in E(G)} |f(x) - f(y)|$. The *$T$-chromatic number* $\chi_T(G)$ is defined as $\chi_T(G) = \min \chi_T^f(G)$ where the minimum is taken over all T-colorings f of G. Similarly the *T-span* $sp_T(G)$ is defined as $sp_T(G) = \min sp_T^f(G)$ and *T-edge span* $esp_T(G)$ is defined as $esp_T(G) = \min esp_T^f(G)$, where the minimum is taken over all T-colorings f of G. If $T = \{0\}$, then the T-coloring of G is the same as a proper coloring of G. In this case, $sp_T(G) = \chi(G) - 1$

where $\chi(G)$ is the chromatic number of G. In general the problem of finding $sp_T(G)$ is NP-complete [2, 4].

The T-coloring problem has been studied for over two decades. It was first introduced by Hale [3] who formulated several frequency assignment problems in graph-theoretic terms. Cozzens and Wang [2], Raychaudhuri [9] and Tesman [13, 14] extended the work of Cozzens and Roberts [1] by considering various frequency interference constraints. The number of facts concerning its computational complexity has been discovered in [9]. T-span and T-edge span for graphs has been studied extensively in [1, 5, 7, 11, 13]. In this paper we compute T-span and T-edge span of swing graph, theta mesh, shadow graph of a cycle.

2 T-coloring

In the case of radio frequency assignment, the forbidden T-sets can be very complex and difficult to model. We focus on a special family of T-sets called the *k multiple of s sets* which has the form $T = \{0, s, 2s, \ldots, ks\} \cup S$, where $s, k \geq 1$ and $S \subseteq \{s+1, s+2, \ldots, ks-1\}$. When $s = 1$, the set $T = \{0, 1, 2, \ldots, k\}$ is also called a *k-initial set*. Some practical forbidden sets, such as those that arise in Ultra High Frequency television problem are very similar to k multiple of s sets [9, 14].

Lemma 1. [1] *If T is k-initial, then for all graphs G, $sp_T(G) = sp_T(K_{\chi(G)}) = (k+1)(\chi(G)-1)$.*

Definition 2. [1] A graph G is *weakly γ perfect*, if $\chi(G) = \omega(G)$ where $\chi(G)$ is the chromatic number of G and $\omega(G)$ is the maximum size of a clique in G.

Lemma 3. [1] *If G is weakly γ perfect, then for all sets T, $esp_T(G) = sp_T(G) = sp_T(K_{\chi(G)})$.*

Lemma 4. [1] *For all graphs G and sets T, (1) $\chi_T(G) = \chi(G)$, (2) $\chi(G) - 1 \leq esp_T(G) \leq sp_T(G)$, (3) $sp_T(K_{\omega(G)}) \leq esp_T(G) \leq sp_T(K_{\chi(G)})$.*

Lemma 5. [9] *If T is a k multiple of s set, then for all graphs G,*

$$sp_T(G) = sp_T(K_{\chi(G)}) = \begin{cases} st + skt - sk - 1, & \text{if } \chi(G)=st; \\ st + skt + m - 1, & \text{if } \chi(G)=st+m,\ 1 \leq m \leq s-1. \end{cases}$$

Lemma 6. [5] *If T is a k multiple of s set and $\chi(G) \leq s$, then $sp_T(G) = esp_T(G) = \chi(G) - 1$.*

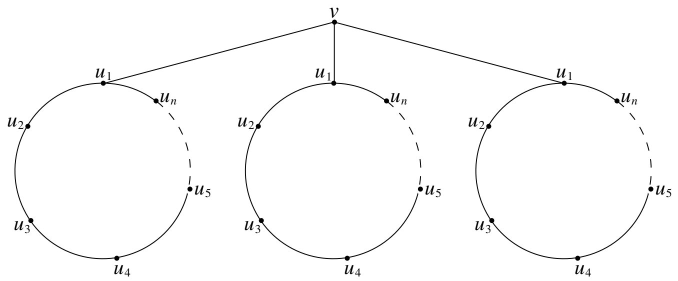

Figure 1: : Swing Graph S_3^n

3 Swing Graphs

Definition 7. [6] A graph obtained by taking k copies of cycle C_n and one vertex of each C_n joined to a common vertex v, is called a swing graph denoted by S_k^n. See Figure 1.

Lemma 8. [8] *For any odd cycle* C_n *and* $T = \{0, 1, \ldots, k-1\}$, $esp_T(C_n) = \lceil \frac{(n+1)}{n-1} k \rceil$.

Corollary 9. *Let* C_n *be an odd cycle and* T *be a k-initial set with* $n \geq 2k+3$. *Then* $esp_T(C_n) = k+2$.

Lemma 10. [12] *Let* G *be an odd cycle and* T, *a* k *multiple of* s *sets with* $s = 2$. *Then* $esp_T(G) \geq 2k+2$.

Theorem 11. *Let* S_k^n *be the swing graph with* n *odd and* T, *a* k *multiple of* s *sets.*
(i) *If* T *is a* k*-initial set, then* $sp_T(S_k^n) = 2(k+1)$ *and* $esp_T(S_k^n) = k+2$, $\forall n \geq 2k+3$.
(ii) *For* $s = 2$, $sp_T(S_k^n) = esp_T(S_k^n) = 2k+2$.
(iii) *For* $s \geq 3$, $sp_T(S_k^n) = esp_T(S_k^n) = 2$.

Proof. We denote the vertices of C_n as $u_1, u_2, \ldots, u_n$.
Proof of (i). Here $T = \{0, 1, 2, \ldots, k\}$.

In C_n label u_1 as 1, the vertices u_i as $2k+3-(\lfloor \frac{i}{2} \rfloor - 1)$ for i odd, $3 \leq i \leq 2k+3$ and as $k+2-(\frac{i}{2}-1)$ for i even, $2 \leq i \leq 2k+2$. Label the remaining vertices u_i are as 1 for $i \equiv 0 \pmod 2$ and as $k+2$ for $i \equiv 1 \pmod 2$, $2k+4 \leq i \leq n$. Label the common vertex v as $k+2$. Let f denote the resulting labeling. See Figure 2.

By Lemma 1, we obtain $sp_T(S_k^n) = sp_T(K_{\chi(S_k^n)}) = sp_T(K_3) = 2(k+1)$. By the definition of f, we have $|f(u_i) - f(u_{i+1})| = |(k+2-$

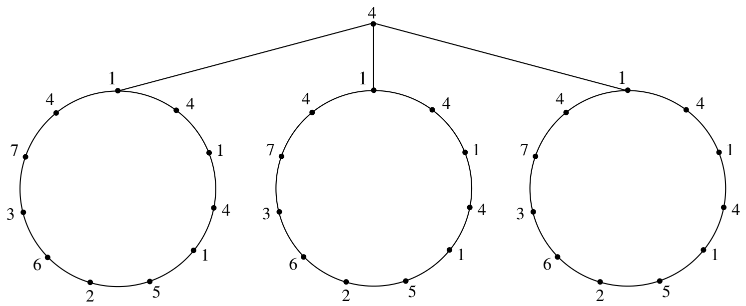

Figure 2: : T-coloring of S_3^9 when $T = \{0, 1, 2\}$

$(\frac{i}{2} - 1)) - (2k + 3 - (\lfloor \frac{i}{2} \rfloor - 1))| = k + 1 \notin T$ for i even, $2 \leq i \leq 2k+2$. Thus, we have $|f(u) - f(v)| \notin T$ for all edge (u, v) in $E(S_k^n)$. Hence f is a T-coloring function of S_k^n and $esp_T(S_k^n) \leq k+2$. By Corollary 9, we have $esp_T(S_k^n) \geq k + 2$. Hence $esp_T(S_k^n) = k + 2$.

Proof of (ii). Here $T = \{0, 2, 3, 4, \ldots, 2k\}$.

Label the vertices u_i as 1 for $1 \leq i \leq n - 2$, i odd and as 2 for $2 \leq i \leq n - 1$, i even. Label the last vertex u_n as $2k + 3$. Label the common vertex v as 2. From the labeling pattern defined, we get $esp_T(S_k^n) \leq 2k + 2$. By Lemmas 5 and 10, we obtain the required result.

Proof of (iii).

Label the vertices u_i as 1 for $1 \leq i \leq n - 2$, i odd and as 2 for $2 \leq i \leq n - 1$, i even. Label the last vertex u_n as 3. Label the common vertex v as 2. Since $s \geq 3 = \chi(S_k^n)$, the result follows from Lemma 6. □

4 Theta Mesh

Definition 12. [10] A graph obtained by taking 3 copies of uniform theta graph $\theta(l, m)$ and joining all the North poles to a common vertex u and all the South poles to a common vertex v, is called a theta mesh denoted by $\theta(m)$.

Theorem 13. *Let G be the theta mesh $\theta(m)$ with $\chi(G) \neq \omega(G)$ and T, a k multiple of s sets.*

(i) If T is a k-initial set, then $sp_T(G) = 2(k+1)$ and $esp_T(G) = k + 2$, $\forall m \geq 2k + 2$.

(ii) For $s = 2$, $sp_T(G) = esp_T(G) = 2k + 2$.

(iii) For $s \geq 3$, $sp_T(G) = esp_T(G) = 2$.

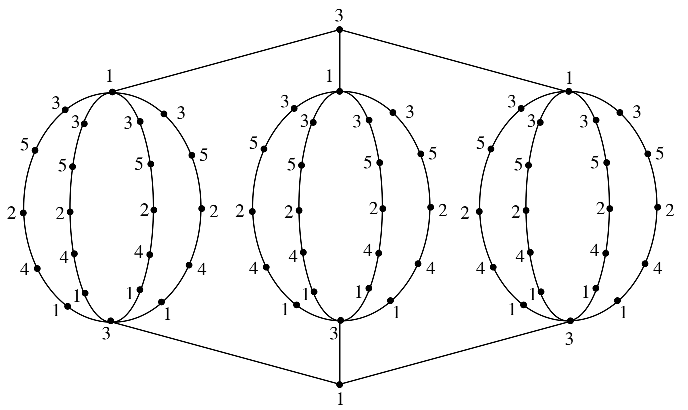

Figure 3: : T-coloring of $\theta(5)$ when $T = \{0, 1\}$

Proof. We denote the vertices of longitudes L_i from north to south pole by $u_1, u_2, \ldots, u_n$. The end vertices are denoted by x and y.

Proof of (i). Here $T = \{0, 1, 2, \ldots, k\}$.

Label the end vertices x and y as 1 and $k+2$ respectively. When m is odd, label the vertices of L_i, u_j as $k+2-(\lceil \frac{j}{2} \rceil - 1)$ for j odd, $1 \le j \le 2k+1$ and as $2k+3-(\frac{j}{2}-1)$ for j even, $2 \le j \le 2k+2$. Label the remaining vertices u_j, m odd as 1 for $j \equiv 1 \pmod 2$ and as $k+2$ for $j \equiv 0 \pmod 2$, $2k+3 \le j \le n$. When m is even, label the vertices of L_i, u_j as $k+2$ for $1 \le j \le n-1$, j odd and as 1 for $2 \le j \le n$, j even. Label the common vertices u and v as $k+2$ and 1 respectively. Let f denote the resulting labeling. See Figure 3.

By Lemma 1, we obtain $sp_T(G) = sp_T(K_{\chi(G)}) = sp_T(K_3) = 2(k+1)$. By the definition of f, we have $|f(u_j) - f(u_{j+1})| = |2k + 3 - (\frac{j}{2} - 1) - (k + 2 - (\lceil \frac{j}{2} \rceil - 1))| = k + 2 \notin T$, j even, $2 \le j \le 2k$, m odd. Also, $|f(u_j) - f(u_{j+1})| = |1 - (k+2)| = k+1 \notin T$, j even, $2 \le j \le n-2$, m even. Thus, we have $|f(u) - f(v)| \notin T$ for all edge (u, v) in $E(G)$. Hence f is a T-coloring function of G and $esp_T(G) \le k+2$. By Corollary 9, we have $esp_T(G) \ge k+2$. Hence $esp_T(G) = k+2$.

Proof of (ii). Here $T = \{0, 2, 3, 4, \ldots, 2k\}$.

The vertices u_j are labeled as 1 for $j \equiv 0 \pmod 2$ and as 2 for $j \equiv 1 \pmod 2$, $1 \le j \le n$ and the end vertices x and y are labeled as 1 and $2k+3$ respectively. The common vertices u and v are labeled as 2 and 1 respectively. From labeling pattern defined,

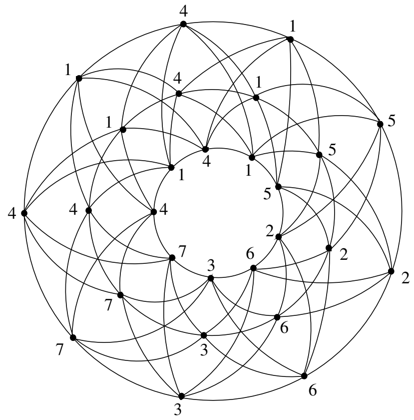

Figure 4: : T-coloring of $D_3(C_9)$ when $T = \{0, 1, 2\}$

we get $esp_T(G) \leq 2k + 2$. By Lemmas 5 and 10, we obtain the required result.

Proof of (iii).

The vertices u_j are labeled as 1 for $j \equiv 0 \pmod 2$ and as 2 for $j \equiv 1 \pmod 2$, $1 \leq j \leq n$ and the end vertices x and y are labeled as 1 and 3 respectively. The common vertices u and v are labeled as 2 and 1 respectively. Since $s \geq 3 = \chi(G)$, the result follows from Lemma 6. □

5 Shadow Graph

Definition 14. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say $G'$and G'' and joining each vertex u' in G' to the neighbours of the corresponding vertex v' in G''.

Definition 15. The m-shadow graph $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, G_3, \ldots, G_m$ and joining each vertex u in G_i to the neighbours of the corresponding vertex v in G_j, $1 \leq i, j \leq m$.

Theorem 16. *Let G be the m-shadow graph of a cycle graph $D_m(C_n)$ with n odd and T, a k multiple of s sets.*

(i) If T is a k-initial set, then $sp_T(G) = 2(k+1)$ and $esp_T(G) = k + 2$, $\forall n \geq 2k + 3$.

(ii) For $s = 2$, $sp_T(G) = esp_T(G) = 2k + 2$.

(iii) For $s \geq 3$, $sp_T(G) = esp_T(G) = 2$.

Proof. We denote the vertices of jth-copy of C_n in $D_m(C_n)$ as u_1^j, u_2^j, ..., u_n^j, $1 \leq j \leq m$. Let $f : V \rightarrow \mathbb{Z}^+ \cup \{0\}$ be a T-coloring.
Proof of (i). Here $T = \{0, 1, 2, \ldots, k\}$. In C_n, label u_1^j as 1, the vertices u_i^j as $2k+3-(\lfloor\frac{i}{2}\rfloor - 1)$ for i odd, $3 \leq i \leq 2k+3$ and as $k+2-(\frac{i}{2}-1)$ for i even, $2 \leq i \leq 2k+2$, $1 \leq j \leq m$. Label the remaining vertices u_i^j as 1 for $i \equiv 0 \pmod 2$ and as $k+2$ for $i \equiv 1 \pmod 2$, $2k+4 \leq i \leq n$, $1 \leq j \leq m$. See Figure 4.

By Lemma 1, we obtain $sp_T(G) = sp_T(K_{\chi(G)}) = sp_T(K_3) = 2(k+1)$. From the above defined labeling pattern, we find $|f(u_i^j) - f(u_{i+1}^j)| = |(k+2-(\frac{i}{2}-1)) - (2k+3-(\lfloor\frac{i}{2}\rfloor - 1))| = k+1 \notin T$ for i even, $2 \leq i \leq 2k+2$. Thus, we have $|f(u) - f(v)| \notin T$ for all edge (u, v) in $E(G)$. Hence f is a T-coloring function of G and $esp_T(G) \leq k+2$. By Corollary 9, we have $esp_T(G) \geq k+2$. Hence $esp_T(G) = k+2$.
Proof of (ii). Here $T = \{0, 2, 3, 4, \ldots, 2k\}$. Label the vertices u_i^j as 1 for $1 \leq i \leq n-2$, i odd and as 2 for $2 \leq i \leq n-1$, i even, $1 \leq j \leq m$. Further, label the vertices u_n^j as $2k+3$ for $1 \leq j \leq m$. From the labeling pattern defined, we get $esp_T(G) \leq 2k+2$. By Lemmas 5 and 10, we obtain the required result.
Proof of (iii). Label the vertices u_i^j as 1 for $1 \leq i \leq n-2$, i odd and as 2 for $2 \leq i \leq n-1$, i even, $1 \leq j \leq m$. Label the vertices u_n^j as 3 for $1 \leq j \leq m$. Since $s \geq 3 = \chi(G)$, the result follows from Lemma 6. □

6 Conclusion

In this paper we have obtained the T-span and T-edge span for the swing graph, theta mesh and shadow graph of a cycle. Finding T-span and T-edge span for other interconnection networks such as cube connected cycles and honeycomb networks are under investigation and are quite challenging.

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