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SHELL BUTTERFLY GRAPHS ARE ELEGANT

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Abstract

Deb and Limaye [4] have defined a shell graph is a cycle C_n with $(n-3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n, n-3)$. A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence a double shell consists of two edge disjoint shells with a common apex. A shell-butterfly is defined as a double shell graph with exactly two pendant edges at the apex. In this paper we prove that shell butterfly graphs are elegant.

AMS Subject Classification: 05C78

Key Words: shell graph; double shell; shell-butterfly graph; elegant labeling.

1 Introduction

A graph labeling is an assignment of integers)ususally non negative) to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the late 1960s.

In 1967 Rosa[7] introduced the labeling method called β-valuation as a tool for decomposing the complete graph into isomorphic sub graphs. Later on, this β-valuation was renamed as graceful labeling by Golomb [6]. A graceful labeling of a graph G with q edges and vertex set V is an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with vertices u and v is assigned the label $|f(u)-f(v)|$. A graph which admits a graceful labeling is called a graceful graph. Various kinds of graphs, in particular many cycle related graphs, are shown to be graceful. After this labeling was invented, many other related labelings came into existence which include almost graceful labeling, k- graceful labeling, one modulo three graceful labeling, ρ-labeling. Other type of labelings like prime labeling, cordial labeling, magic labeling, vertex prime labelings were also invented and thousands of papers have been published.

In 1981 Chang, Hsu and Rogers [3] defined an elegant labeling f of a graph with q edges as an injective function from the vertices of G to the set$\{0, 1, 2, ..., q\}$ such that when each edge xy is assigned the label $f(x) + f(y)(mod(q + 1))$, the resulting edge labels are distinct and nonzero. Chang et al.[3] proved that the cycle C_n is elegant when $n \equiv$0 or 3 (mod 4) . Chang et al. [3]further showed that the shell graphs are elegant and paths P_n are elegant for all n not congruent to 0 (mod 4). Cahit [2] then showed that P4 is the only path that is not elegant. Balakrishnan et al. [1] have proved that the complete bipartitie graphs $K_{m,n}$ are elegant. They have also proved that bistar $B_{n,n}n$ are elegant if and only if n is even. For an exhaustive survey on elegant labeling, refer to the dynamic survey by Gallian [5].

Deb and Limaye [4] have defined a shell graph is a cycle C_n with $(n - 3)$ chords sharing a common end point called the apex. Shell graphs are denoted as $C(n, n - 3)$. A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence a double shell consists of two edge disjoint shells with a common apex. A shell-butterfly is defined as a double shell graph with exactly two pendant edges at the apex.

2 Main Result

In this section, we prove that shell-butterfly graphs have elegant labeling.

Theorem 1. *All shell-butterfly graphs with path orders m and l of the shells are elegant when $m = l$.*

Proof. Let G be a shell-Butterfly graph with n vertices and q edges. Let $m(m \geq 3)$be the path order of the shells of G . We describe G as follows: Denote the apex of G as v_0. Denote the vertices in the path of the right shell of G from bottom to top as $v_1, v_2, \ldots, v_m$. The vertices in the path of the left shell of G are denoted from top to bottom as $v_{m+1}, v_{m+2}, \ldots, v_{(2m-1)}, v_{2m}$. The two pendant vertices of G are denoted as $v_{(2m+1)}, v_{(2m+2)}$. Note that G has $2m + 3$ vertices and $4m$ edges. To prove that G has elegant labeling we consider two cases namely, when m is odd and when m is even. In both the cases we define $f(v_0) = 0$.

Case 1: When m is odd.

Define the labeling $f : V(G) \to \{0, 1, 2, \ldots, q\}$ by

$$f(v_{2i-1}) = \begin{cases} 3m + 2i - 1, & for\ 1 \leq i \leq \frac{m+1}{2} \\ 2m - 2i + 3, & for\ \frac{m+3}{2} \leq i \leq m \\ 2m + 1, & for\ i = m + 1 \end{cases} \quad (1)$$

$$f(v_{2i}) = \begin{cases} m + 2i, & for\ 1 \leq i \leq \frac{m-1}{2} \\ 4m - 2i, & for\ \frac{m+1}{2} \leq i \leq m \\ 1, & for\ i = m + 1 \end{cases} \quad (2)$$

The above equations (1) and (2) show that all the vertices have been covered and they are distinct. If any two vertex labels are same, then we would get a contradiction to the fact that m is a positive integer.

The Edge labels are computed as follows.

By definition of elegant labeling, each edge xy is assigned the label $f(x) + f(y)\ (mod(q+1))$.

$$f(v_0)+f(v_{2i-1}) = \begin{cases} 3m+2i-1, & for\ 1 \le i \le \frac{m+1}{2} \\ 2m-2i+3, & for\ \frac{m+3}{2} \le i \le m \\ 2m+1, & for\ i = m+1 \end{cases} \quad (3)$$

$$f(v_0)+f(v_{2i}) = \begin{cases} m+2i, & for\ 1 \le i \le \frac{m-1}{2} \\ 4m-2i, & for\ \frac{m+1}{2} \le i \le m \\ 1, & for\ i = m+1 \end{cases} \quad (4)$$

$$f(v_{2i-1})+f(v_{2i}) = \begin{cases} 4m+4i-1(mod(q+1)), & for\ 1 \le i \le \frac{m-1}{2} \\ 6m-4i+3(mod(q+1)), & for\ \frac{m+3}{2} \le i \le m \end{cases} \quad (5)$$

$$f(v_{2i})+f(v_{2i+1}) = \begin{cases} 4m+4i+1(mod(q+1)), & for\ 1 \le i \le \frac{m-1}{2} \\ 6m-4i+1(mod(q+1)), & for\ \frac{m+1}{2} \le i \le m \end{cases} \quad (6)$$

By the above calculations one can easily observe that all edge labels are distinct and they are ranging from 1 to q. Hence the shell butterfly graphs are elegant when m is odd.

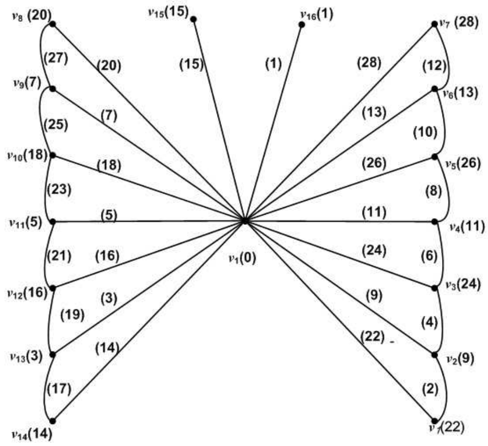

Figure 1: An elegant shell-butterfly graph, when m =7

Case 2: When m is even.

Define the labeling $f : V(G) \to \{0, 1, 2, \ldots, q\}$ by

$$f(v_{2i-1}) = \begin{cases} m + 2i - 1, & for\ 1 \leq i \leq \frac{m}{2} \\ 4m - 2i + 2, & for\ \frac{m+2}{2} \leq i \leq m \\ 2i - 1, & for\ i = m + 1 \end{cases} \quad (7)$$

$$f(v_{2i}) = \begin{cases} 3m + 2i, & for\ 1 \leq i \leq \frac{m}{2} \\ 2m - 2i + 1, & for\ \frac{m+2}{2} \leq i \leq m \\ 2i - 2, & for\ i = m + 1 \end{cases} \quad (8)$$

From the above equations it is clear that the function f is an injective function.

Edge labelings can be computed as follows.

$$f(v_0) + f(v_{2i-1}) = \begin{cases} m + 2i - 1, & for\ 1 \leq i \leq \frac{m}{2} \\ 4m - 2i + 2, & for\ \frac{m+2}{2} \leq i \leq m \\ 2i - 1, & for\ i = m + 1 \end{cases} \quad (9)$$

$$f(v_0) + f(v_{2i}) = \begin{cases} 3m + 2i, & for\ 1 \leq i \leq \frac{m}{2} \\ 2m - 2i + 1, & for\ \frac{m+2}{2} \leq i \leq m \\ 2i - 2, & for\ i = m + 1 \end{cases} \quad (10)$$

$$f(v_{2i-1}) + f(v_{2i}) = \begin{cases} 4m + 4i - 1(mod(q + 1)), & for\ 1 \leq i \leq \frac{m}{2} \\ 6m - 4i + 3(mod(q + 1)), & for\ \frac{m+2}{2} \leq i \leq m \end{cases} \quad (11)$$

$$f(v_{2i}) + f(v_{2i+1}) = \begin{cases} 4m + 4i + 1(mod(q + 1)), & for\ 1 \leq i \leq \frac{m}{2} \\ 6m - 4i + 1(mod(q + 1)), & for\ \frac{m+2}{2} \leq i \leq m - 1 \end{cases} \quad (12)$$

By the above calculations one can easily observe that all edge labels are distinct and they are ranging from 1 to q. No two of them are same, in which case we would get a contradiction to the fact that m is a positive integer. Hence the shell butterfly graphs satisfy elegant labeling in this case also.

Hence shell-butterfly graphs are elegant when the path orders of the shells are equal.

□

An illustration for Case 2, when $m = 8$ is given in the Figure 3.

Graph labelings have many applications in various fields such as Coding theory, X-ray crystallography, radar, Astronomy, Circuit design, Communication networks. Being one vertex union of shell graphs, the shell butterfly graphs might be used in Radar and Communication networks extensively.

3 Conclusion

In this paper, we have proved that shell-butterfly graphs are elegant. We would like to find other shell related graphs that admit elegant labeling and find out necessary and sufficient conditions for graphs to have elegant labeling.

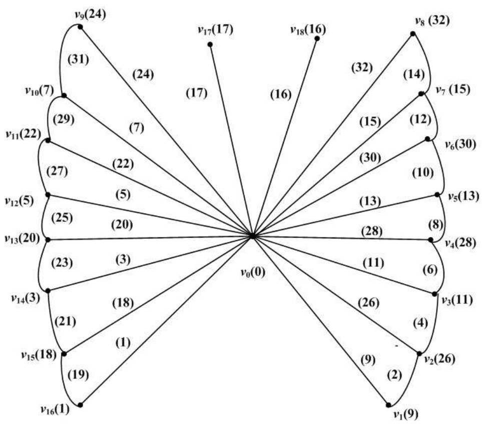

Figure 2: An elegant shell-butterfly graph

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