THE TOTAL RESTRAINED MONOPHONIC DOMINATION NUMBER OF A GRAPH

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Abstract

In this paper the concept of total restrained monophonic domination number of a graph is introduced. A restrained monophonic dominating set $M$ of a graph $G$ is a monophonic dominating set such that either $M = V$ or the subgraph induced by $V - M$ has no isolated vertices. The minimum cardinality of a restrained monophonic dominating set of $G$ is the restrained monophonic domination number of $G$ and is denoted by $\gamma_{m_r}(G)$. A total restrained monophonic dominating set $M$ of a graph $G$ is a restrained monophonic dominating set such that the subgraph $G[M]$ induced by $M$ has no isolated vertices. The minimum cardinality of a total restrained monophonic dominating set of $G$ is the total restrained monophonic domination number of $G$ and is
denoted by $\gamma_{m_{tr}}(G)$. We investigate total restrained monophonic domination number of complete graphs, Petersen graph and Wheel graph. If for every pair of positive integers $k$ and $p$ with $3 \leq k \leq p$, then there exists a connected graph $G$ of order $p$ such that $\gamma_{m_{tr}}(G) = k$. It is shown that for any positive integers $3 < a < b < c < d$, then there exists a connected graph $G$ such that $m(G) = a, \gamma_{m}(G) = b, \gamma_{m_{r}}(G) = c$ and $\gamma_{m_{tr}}(G) = d.$

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1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$, respectively. For basic graph theoretic terminology, we refer to Harary [2], [3]. The neighbourhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. A vertex $v$ is an extreme vertex if the subgraph induced by its neighbourhood is complete. A vertex with degree 1 is called an end vertex. A vertex $v$ of a connected graph $G$ is called a support vertex of $G$ if it is adjacent to an end vertex of $G$. A chord of a path $u_1, u_2, \cdots, u_k$ in $G$ is an edge $u_iu_j$ with $j \geq i + 2$. A path $P$ is called a monophonic path if it is a chordless path. For a subset $D$ of vertices, we call $D$ a dominating set if for each $x \in V - D$, $x$ is adjacent to at least one vertex of $D$. The domination number of $D$ is the minimum cardinality of a dominating set of $G$ and is denoted by $\gamma(G)$ [4]. A monophonic set of $G$ is a set $M \subseteq V$ such that every vertex of $G$ is contained in a monophonic path joining some pair of vertices in $M$. The minimum cardinality of a monophonic set of $G$ is the monophonic number of $G$ and is denoted by $m(G)$ [6]. A set $M$ of vertices of a connected graph $G$ is a monophonic dominating set if $M$ is both a monophonic set and a dominating set. The minimum
cardinality of a monophonic dominating set of $G$ is a monophonic domination number of $G$ and is denoted by $\gamma_m(G)$ \cite{5}. A connected restrained monophonic set of $G$ is a restrained monophonic set $M$ such that the subgraph $G[M]$ induced by $M$ is connected. The minimum cardinality of a connected restrained monophonic set of $G$ is the connected restrained monophonic number of $G$ and is denoted by $mcr(G)$. The connected restrained monophonic number of a graph was introduced and studied in \cite{7}.

The following theorems will be used in the sequel.

**Theorem 1.** \cite{8} Every cut vertex of a connected graph $G$ belongs to every connected restrained monophonic set of $G$.

**Theorem 2.** \cite{5} Let $G$ be a connected graph with cut vertices and let $M$ be a monophonic dominating set of $G$. If $v$ is a cut vertex of $G$, then every component $G - v$ contains an element of $M$.

**Theorem 3.** \cite{5} Each extreme vertex of a connected graph $G$ belongs to every monophonic dominating set of $G$.

**Theorem 4.** \cite{7} Each extreme vertex and each support vertex of a connected graph $G$ belongs to every total restrained monophonic set of $G$. If the set $M$ of all extreme vertices and support vertices form a total restrained monophonic set, then it is the unique minimum total restrained monophonic set of $G$.

**Theorem 5.** \cite{7} Every connected restrained monophonic set of $G$ is a total restrained monophonic set of $G$.

## 2 Total Restrained Monophonic Domination Number

**Definition 6.** A restrained monophonic dominating set $M$ of a graph $G$ is a monophonic dominating set such that either $M = V$ or the subgraph induced by $V - M$ has no isolated vertices. The minimum cardinality of a restrained monophonic dominating set of $G$ is the restrained monophonic domination number of $G$ and is denoted by $\gamma_{mr}(G)$.

**Definition 7.** A total restrained monophonic dominating set
$M$ of a graph $G$ is a restrained monophonic dominating set such that the subgraph $G[M]$ induced by $M$ has no isolated vertices. The minimum cardinality of a total restrained monophonic dominating set of $G$ is the total restrained monophonic dominating number of $G$ and is denoted by $\gamma_{m_{tr}}(G)$.

**Example 8.** For the graph $G$ given in Figure 1, we have $M_1 = \{v_1, v_3, v_6\}$ is a minimum monophonic set of $G$ and so $m(G) = 3$, $M_2 = M_1 \cup \{v_8\}$ is a minimum monophonic dominating set of $G$ and so $\gamma_m(G) = 4$, $M_3 = \{v_1, v_3, v_6, v_7, v_8\}$ is minimum restrained monophonic dominating set so that $\gamma_{m_{r}}(G) = 5$ and $M_4 = M_3 \cup \{v_2\}$ is a total restrained monophonic dominating set of $G$ and so $\gamma_{m_{tr}}(G) = 6$.

![Figure 1: G](image)

**Theorem 9.** Each extreme vertex of a connected graph $G$ belongs to every total restrained monophonic dominating set.

*Proof.* Since each extreme vertex of a connected graph $G$ belongs to every monophonic dominating set, obviously these extreme vertices also belong to every total restrained monophonic dominating set.

**Theorem 10.** Each extreme vertex and each support vertex of a connected graph $G$ belongs to every total restrained monophonic dominating set of $G$. If the set $M$ of all extreme vertices and support vertices form a total restrained monophonic dominating set, then it is the unique minimum total restrained monophonic dominating set of $G$.

*Proof.* Since every total restrained monophonic dominating set is a monophonic dominating set, by Theorem 9, each extreme vertex belongs to every total restrained monophonic dominating set of $G$.
Since a total restrained monophonic dominating set of $G$ contains no isolated vertices, it follows that each support vertex of $G$ also belongs to every total restrained monophonic dominating set. The second part of the theorem is clear from Theorems 1, 3 and 4.

**Corollary 11.** For the complete graph, $K_p$, ($p \geq 2$), $\gamma_{m_{tr}}(G) = p$.

**Theorem 12.** Let $G$ be a connected graph with cut vertices and let $M$ be the total restrained monophonic dominating set of $G$. If $u$ is a cut vertex of $G$, then every component of $G - u$ contains an element of $M$.

**Theorem 13.** For a connected graph $G$ of order $p$, $2 \leq m(G) \leq \gamma_m(G) \leq \gamma_{m_{tr}}(G) \leq \gamma_{m_r}(G) \leq \gamma_{m_{tr}}(G) \leq p$.

**Proof.** Any monophonic dominating set needs at least 2 vertices and so $\gamma_{m_{tr}}(G) \geq 2$. Since every total restrained monophonic dominating set is a restrained monophonic dominating set it follows that $\gamma_{m_r}(G) \leq \gamma_{m_{tr}}(G)$. Since every restrained monophonic dominating set is a monophonic dominating set, it follows that $\gamma_{m_r}(G) \leq \gamma_{m_{tr}}(G)$. Therefore $\gamma_m(G) \leq \gamma_{m_r}(G) \leq \gamma_{m_{tr}}(G)$. Since $m(G)$ is a connected monophonic number of $G$, it is clear that $\gamma_{m_{tr}}(G) \leq p$. Hence $2 \leq m(G) \leq \gamma_{m_r}(G) \leq \gamma_{m_{tr}}(G) \leq \gamma_{m_{tr}}(G) \leq p$.

**Corollary 14.** Let $G$ be a connected graph. If $\gamma_{m_{tr}}(G) = 2$, then $\gamma_{m}(G) = 2$.

**Remark 15.** The converse of the above corollary need not be true. We notice that for any cycle of order 4, the monophonic domination number is 2, whereas the total restrained monophonic domination number is 3.

**Theorem 16.** For any non-trivial tree $T$, the set of all end vertices and support vertices of $T$ is the unique minimum total restrained monophonic dominating set of $G$.

**Theorem 17.** For any connected graph $G$, $\gamma_{m_{tr}}(G) = 2$, if and only if $G = K_2$.

**Proof.** If $G = K_2$, then $\gamma_{m_{tr}}(G) = 2$. Conversely, let $\gamma_{m_{tr}}(G) = 2$. Let $M = \{u, v\}$ be a minimum total restrained monophonic
dominating set of $G$. Then $uv$ is an edge. It is clear that a vertex different from $u$ and $v$ cannot lie on a $u-v$ monophonic path and so $G = K_2$.

**Remark 18.** For the complete graph $G = K_2, \gamma_{mtr}(G) = 2$ and for the complete graph $K_p, \gamma_{mtr}(G) = p$. So that the total restrained monophonic domination number of a graph attains its least value 2 and largest value $p$.

**Result 19.** For the wheel $G = W_n (n \geq 4), \gamma_{mtr}(G) = 3$.

**Result 20.** For the Petersen graph, $\gamma_{mtr}(G) = 4$.

### 3 Realization Results

**Theorem 21.** If for every pair of positive integers $k$ and $p$ with $3 \leq k \leq p$, then there exists a connected graph $G$ of order $p$ such that $\gamma_{mtr}(G) = k$.

**Proof.** Let $u_1, u_2$ be two vertices. Add new vertices $v_1, v_2, \ldots, v_{k-3}$ and $w_1, w_2, \ldots, w_{p-k+1}$ and join each $w_i (1 \leq i \leq p - k + 1)$ with $u_1$ and $u_2$ and join each $v_i (1 \leq i \leq k - 3)$ with $u_2$. Also join $w_i$ with $w_{i+1} (1 \leq i \leq p - k + 1)$ thereby obtaining the graph $G$. The graph $G$ is shown in Figure 2. Then $G$ has order $p$. Clearly $M = \{u_1, v_1, v_2, \ldots, v_{k-3}\}$ is the monophonic set and dominating set of $G$. It is clear that $M_1 = M \cup \{u_2, w_1\}$ is the minimum total restrained monophonic dominating set of $G$ and so $\gamma_{mtr}(G) = k - 3 + 2 + 1 = k$. 

![Figure 2: G](image-url)
Theorem 22. For any positive integers $3 < a < b < c < d$, then there exists a connected graph $G$ such that $m(G) = a, \gamma_m(G) = b, \gamma_{m_1}(G) = c$ and $\gamma_{m_1}(G) = d$.

Proof. For $a < b < c < d$. Let $d = c + b - a + 1$. Let $H$ be a graph on 8 vertices $v_1, v_2, \cdots, v_8$ as shown in the Figure 3. Let $P : u_1, u_2, \cdots, u_p$ be a path of order $p$. Let $H_1$ be a graph obtained from $H$ and $P$ by joining $u_1$ to $v_5$ and $u_p$ to $v_7$. Now, add a new vertex $q$ to $H_1$ and join each $u_i (1 \leq i \leq p)$ to $q$ and obtained the graph $H_2$. Let $G$ be a graph obtained from $H_2$ by adding each $w_i (1 \leq i \leq a - 3)$ to $v_6$ and each $h_i (1 \leq i \leq c - b - 1)$ to $v_1$ and $v_5$.

Clearly $M = \{v_1, v_3, v_7, w_1, w_2, \cdots, w_{a-3}\}$ is the monophonic set and so that the monophonic number $m(G) = a$. Obviously $M_1 = M \cup \{v_5, q, z_i (1 \leq i \leq b - a - 2)\}$ be the monophonic domi-
nating set and so that the monophonic domination number $\gamma_m(G) = b$. It is clear that $M_2 = M_1 \cup \{v_8, h_i \mid 1 \leq i \leq c - b - 1\}$ is the restrained monophonic dominating set and so that $\gamma_{mr}(G) = c$. Clearly $M_3 = M_2 \cup \{v_2, v_6, u_1, y_i \mid 1 \leq i \leq b - a - 2\}$ is the total restrained monophonic dominating set and so that $\gamma_{mtr}(G) = c + b - a + 1 = d$. Hence $m(G) = a, \gamma_m(G) = b, \gamma_{mr}(G) = c$ and $\gamma_{mtr}(G) = d$.

References


