Upper Monophonic Domination Number of a Graph

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Abstract

In this paper, the concept of Minimal Monophonic Dominating Set (\textit{mmd - set}) and Upper Monophonic Domination Number (\textit{umd - number}) of a connected graph $G$ are introduced. The \textit{umd-number} $\gamma_m^+(G)$ of a connected graph $G$ is the maximum cardinality of the \textit{mmd - set} of $G$. The \textit{umd-number} of certain classes are determined and some of its general properties are studied. It is shown that for any two integers $a$ and $b$ such that $2 \leq a \leq b$, there exists a connected graph $G$ such that $\gamma(G) = a$, $m(G) = b$ and $\gamma_m^+(G) = a + b$. Also, realizes the fact that, for $a, b \in \mathbb{Z}$ with $2 \leq a < b$, there exists a connected graph $G$ such that $\gamma_m^+(G) = a$ and $\gamma_g^+(G) = b$ where $\gamma_g^+(G)$ is the Upper Geodetic Domination Number of $G$.

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1 Introduction

For the domination problem in graphs, multiple edges and loops are irrelevant. So throughout this paper we consider only undirected connected finite simple graphs (without loops or multiple edges). The concept of dominating set in graphs introduced by O. Ore, is currently receiving much attention in the literature of graph theory. A unique handling of fundamentals of domination in graphs is given by T.W. Haynes et. al[5].

For any graph $G$, set of vertices is denoted by $V$ and set of edges by $E$. Define order of $G$ by $p = |V|$ and size of $G$ by $q = |E|$. For basic graph theoretic notations and terminology we suggest the reader to refer [1]. A subset $D$ of $V$ is said to be a dominating set of $G$ if every vertex in $V$ is either an element of $D$ or is adjacent to an element of $D$, or $N[D] = V$. The domination number $\gamma(G)$ is the minimum cardinalities of the dominating sets of $G$. For further reference, see[5].

A chord of a path $P: [u_1u_2...u_n]$ in a graph $G$ is an edge $u_iu_j$ with $j \geq i + 2$. The distance $d(u, v)$ between two vertices $u$ and $v$ is the length of a shortest $u - v$ path in $G$. A $u - v$ path is a monophonic path or induced path if it is chordless path. For any two vertices $u$ and $v$ in a graph $G$, the monophonic distance $d_m(u, v)$ from $u$ to $v$ is the length of the longest $u - v$ monophonic path in $G$. The monophonic eccentricity of a vertex $v$ in $G$ is defined by $e_m(v) = \max\{d_m(v, u) : u \in V\}$. The monophonic radius of any graph $G$ is denoted by $r_m$ or $\text{rad}_m(G)$, and is defined by $r_m = \min\{e_m(v) : v \in V\}$ and the monophonic diameter of $G$ is denoted by $d_m$ or $\text{diam}_m(G)$, and is defined by $d_m = \max\{e_m(v) : v \in V\}$. For any connected graph $G$, we have an inequality $r_m \leq d_m \leq 2r_m$.

A subset $M$ of $V$ is said to be a monophonic set of a graph $G$ if each vertex $v$ of $G$ lies on an $x - y$ monophonic path in $G$ for some $x, y \in M$. The monophonic number $m(G)$ is the minimum cardinalities of monophonic sets of $G$. A monophonic set of order $m(G)$ is called $m - \text{set}$ of $G$. The concepts of monophonic sets and monophonic numbers of $G$ are appeared in [1], studied by I M. Pelayo and further studied in [3],[7].

A subset $M$ of $V$ is said to be a Monophonic Dominating set of a graph $G$ if $M$ is both a monophonic set and a dominating set.
The minimum of the cardinalities of monophonic dominating sets of $G$ is called the monophonic domination number, and is denoted by $\gamma_m(G)$. The concept of monophonic domination number of a graph was introduced by J.John et.al[6] in the year 2012.

The geodetic number of a graph was introduced in [2]. A geodetic dominating set $S \subseteq V$ in $G$ is called minimal geodetic dominating set if no proper subset of $S$ is a geodetic dominating set of $G$. The maximum cardinality of the minimal geodetic dominating set of $G$ is the upper geodetic dominating number of $G$, and is denoted by $\gamma^+_g(G)$. The geodetic domination number of a graph is studied by A.Hansberg et.al [4].

2 umd-number of a Graph

Definition 1. A Monophonic dominating set $M$ of $G$ is called a minimal monophonic dominating set (mmd-set) of $G$ if no proper subset of $M$ is a monophonic dominating set of $G$. The maximum cardinalities of the mmd-sets of $G$ is called upper monophonic dominating number (umd-number) of $G$, and is denoted by $\gamma^+_m(G)$. A mmd-set of a graph $G$ with order $\gamma^+_m(G)$ is called $\gamma^+_m$-set.

We follow this definition by giving an example

Example 2. For the graph $G$ given in the Figure 1,

![Figure 1](image.png)

Figure 1: A graph $G$ with $\gamma^+_m(G) = 3$

$M_1 = \{v_2, v_4\}$, $M_2 = \{v_4, v_6\}$ and $M_3 = \{v_2, v_5\}$ are the only minimum monophonic dominating sets of the graph $G$ so that
\( \gamma_m(G) = 2 \). Also \( M_4 = \{v_1, v_3, v_5\} \) and \( M_5 = \{v_1, v_3, v_6\} \) are the mmd-sets of \( G \). It is clear that umd-number \( \gamma_m^+(G) \geq 3 \) and easily verified that no 4-element subsets or 5-element subsets of \( V \) is a mmd-sets of \( G \). Hence umd-number \( \gamma_m^+(G) = 3 \).

For unexplained term and symbols refer to [1],[3],[4]. In the next section, we cite some basic observations related to umd-number, which are to be used in the sequel

3 Some Basic Observations

Observation 3. Each simplicial vertex of a graph \( G \) belongs to every monophonic dominating set of \( G \).

Observation 4. For any graph \( G \) with vertex count \( p \), we have
\[
2 \leq \max\{m(G), \gamma(G)\} \leq \gamma_m^+(G) \leq p.
\]

Proof. By Definitions, we have \( 2 \leq m(G) \) and \( 2 \leq \gamma(G) \). Clearly \( 2 \leq \max\{m(G), \gamma(G)\} \). Also \( m(G) \leq \gamma_m(G) \) and \( \gamma(G) \leq \gamma_m(G) \). Thus \( \gamma_m(G) \geq 2 \). Since every mmd-set is a monophonic dominating set of \( G \), \( 2 \leq \gamma_m(G) \leq \gamma_m^+(G) \). Also, \( \gamma_m(G) \leq |V| = p \). Hence \( 2 \leq \max\{m(G), \gamma(G)\} \leq \gamma_m^+(G) \leq p \).

Observation 5. For a graph \( G \) with vertex count \( p \), we have
\[
2 \leq \gamma_m(G) \leq \gamma_m^+(G) \leq p
\]

4 Main Results

Theorem 6. If a connected graph \( G \) has a unique monophonic dominating set, then \( \gamma_m(G) = \gamma_m^+(G) \)

Proof. Let \( M \) be the monophonic dominating set of \( G \). Since \( M \) is unique, clearly \( M \) is the mmd-set of \( G \). Thus \( \gamma_m(G) = \gamma_m^+(G) \).

Theorem 7. If \( G \) is a non-complete connected graph such that it has a minimum cut set, then \( \gamma_m^+(G) \leq p - k(G) \), where \( k(G) \) is the vertex connectivity of \( G \).

Proof. Since \( G \) is non-complete, it is clear that \( 1 \leq k(G) \leq p - 2 \). Let \( U = \{u_1, u_2, u_3, \ldots, u_k\} \) be a minimum cut set of \( G \). Then
\(G_1, G_2, \cdots, G_r \ (r \geq 2)\) be the components of \(G - U\). Choose \(M = V - U\), it follows that every vertex \(u_i \ (1 \leq i \leq k)\) is adjacent to at least one vertex of \(G_j\) for \(1 \leq j \leq r\). It is clear that \(M\) is a monophonic dominating set of \(G\) and \(\gamma_m(G) \leq p - k(G)\). Since no proper subset of \(M\) is a monophonic dominating set of \(G\), \(M\) is the mmd-set of \(G\). Hence \(\gamma_m^+(G) \leq p - k(G)\).

**Theorem 8.** Let \(G\) be a connected graph of order \(p \leq 2\). Then \(\gamma_m^+(G) = 2 \iff \exists\) a monophonic dominating set \(M = \{u, v\}\) of \(G\) such that monophonic distance \(d_m(u, v) \leq 3\).

**Proof.** Assume that \(\gamma_m^+(G) = 2\). Let \(M\) be a minimal monophonic dominating set of \(G\). Suppose that \(d_m(u, v) \geq 4\), then monophonic diametrical \(u - v\) path contain at least three internal vertices. Therefore mmd-set contains three vertices, that is \(\gamma_m^+(G) = 3\) which is a contradiction to our assumption. Thus \(d_m(u, v) \leq 3\). Hence there exists a monophonic dominating set \(M = \{u, v\}\) of \(G\) such that \(d_m(u, v) \leq 3\). Conversely assume that there exists a monophonic dominating set \(M = \{u, v\}\) of \(G\) such that \(d_m(u, v) \leq 3\). Clearly \(M\) is the mmd-set of \(G\). Hence \(\gamma_m^+(G) = 2\).

The upper monophonic domination number of standard graphs can be easily found which are given below.

1. For a complete graph \(K_p\), \(\gamma_m^+(K_p) = p\) where \(p \geq 2\).
2. For a cycle graph \(C_p\), \(\gamma_m^+(C_p) = \lceil \frac{p}{3} \rceil\) where \(p \geq 4\).
3. For a complete bipartite graph \(K_{p,q}\), \(\gamma_m^+(K_{p,q}) = \min\{p, q, 4\}\) where \(p, q \geq 2\). In particular, \(\gamma_m^+(K_{1,1}) = 2\).
4. For a star graph \(K_{1,q}\), \(\gamma_m^+(K_{1,q}) = q\) where \(q \geq 2\).

In the next section, we shall realize a **mmd-set** in terms of upper monophonic domination number and upper geodetic domination number of graphs.

### 5 Realization Problems

**Theorem 9.** For \(a, b \in \mathbb{Z}\) with \(2 \leq a \leq b\), then \(\exists\) a connected graph \(G\) such that \(\gamma(G) = a\), \(m(G) = b\) and \(\gamma_m^+(G) = a + b\).
Proof. Let $F : [r, s, u, v, t]$ be a copy of $C_5$ where $C_5$ is a cycle of length 5. Let $H$ be a graph obtained from $F$ by adding the new vertices $z_1, z_2, \ldots, z_{b-1}$ and join each to the vertex $r$. Let $G$ be the graph obtained from $H$ by adding a path on $3(a-2) + 1$ vertices $y_0, y_1, \ldots, y_{3(a-2)}$ and joining $y_0$ to the vertex $u$ as shown in the given Figure 2.

Let $M_1 = \{r, u, y_2, y_5, \ldots, y_{3(a-2)-1}\}$. Then we can see that set $M_1$ is minimum dominating set of $G$ so that $\gamma(G) = a$, where $a = |M_1|$. Let $M_2 = \{z_1, z_2, \ldots, z_{b-1}, y_{3(a-2)}\}$. Then clearly, $M_2$ is a subset of every monophonic set of $G$ and so that $m(G) \geq b$. It is clear that $M_2$ is the minimum monophonic set of $G$ so that $m(G) = b$, where $b = |M_2|$. Now let us consider $M = M_1 \cup M_2$. Since no proper subset of $M$ is a monophonic dominating set of $G$, $M$ is the minimal monophonic dominating set of $G$, also $M$ is unique. Hence by the Theorem[6], $\gamma_m^+(G) = a + b$.

**Theorem 10.** For $a, b \in \mathbb{Z}$ with $2 \leq a < b$, then $\exists$ a connected graph $G$ such that $\gamma_m^+(G) = a$ and $\gamma_g^+(G) = b$.

Proof. Let $P : [x, y, z]$ be a path on three vertices, let $P_i : [u_i, v_i]$ where $1 \leq i \leq (b-a+2)$ be a path on two vertices. Let $H$ be a graph obtained from $P$ and $P_i$ by joining each $u_i$ where $1 \leq i \leq (b-a+2)$ with $x$ and each $v_i$ where $1 \leq i \leq (b-a+2)$ with $z$. Let $G$ be a graph obtained from $H$ by adding the vertices $z_1, z_2, \ldots, z_{a-2}$ and joining each $z_i$ where $1 \leq i \leq a-2$ with $x$ and $y$ as shown in the given Figure 3.
First show that $\gamma^+_m(G) = a$. Let $M_1 = \{z_1, z_2, \ldots, z_{a-2}\}$ be the set of all simplicial vertices of $G$. It follows that $M_1$ is a subset of every monophonic dominating subset of $G$. It is clear that $M_1$ is not a monophonic dominating set of $G$. If adding the vertex $v \not\in M_1$ to the set $M_1$, then we can clearly see that $M_1 \cup \{v\}$ is not a monophonic dominating set of $G$ so that $\gamma^+_m(G) \geq a$. However $M_2 = M_1 \cup \{x, z\}$ is a monophonic dominating set of $G$. Since $M_2$ is the minimal monophonic dominating set, $\gamma^+_m(G) = a$.

Next show that $\gamma^+_g(G) = b$. It is easily observe that $M_1$ is a subset of every geodetic dominating set of $G$ and it can be verified that $M_1$ is not a geodetic dominating set of $G$. Now consider the set $M_2 = M_1 \cup \{v_1, v_2, \ldots, v_{b-a+2}\}$. Clearly $M_2$ is a geodetic dominating set of $G$ so that $\gamma^+_g(G) \leq b$. Since $M_2$ is the minimal geodetic dominating set, $\gamma^+_g(G) \leq b$.

Let $H_i = \{u_i, v_i\}$ where $1 \leq i \leq b-a+2$, let $S$ be a geodetic dominating set of $G$. If $z \in S$, then $S$ contains at least one element of each $H_i$ for all $i$. If not, suppose that $u_1, v_1 \not\in S$. Then $u_1, v_1$ do not lie on a geodetic joining a pair of vertices of $S$, which is a contradiction. Therefore $S$ contains at least one element of each $H_i$ for all $i$. Thus it follows that $\gamma^+_g(G) \leq (a - 2) + 1 + (b - a + 2) = b + 1$, which is a contradiction. Hence $z \not\in S$. Let $G_i = \{u_i, v_{i+1}\}$ where $1 \leq i \leq b-a+2$, $Q_i = \{v_i, u_{i+1}\}$ for all $i$. If
and \( S_1 = \{v_1, v_2, \ldots, v_{b-a+2}\} \). Then easily observe that \( S_1 \) contain at least one element from each \( G_i \) or \( Q_i \) or \( S_1 \subseteq S \). Hence it follows the fact that \( \gamma^*_d(G) = (a - 2) + (b - a + 2) = b. \)

6 Conclusion

This work initiates the study of upper monophonic domination number and minimal monophonic dominating set of a graph. These concepts have many applications in communications networks, MANET, Military battle fields and location theory etc..

References


