A Hybrid Scheme for Solving Singularly Perturbed Delay Differential Equations and its Applications to MADM Problems

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Abstract

In Multiple Attribute Decision Making (MADM) problems, weights of decision makers play an important role. In this paper, we present a new approach for finding the weights for decision making process based on the weights proposed by the decision maker in the form of singularly perturbed delay differential problem. The decision maker weights are derived using a new proposed hybrid scheme which is applied at the point where the interior layer takes place. The classical finite difference scheme in the rest of the domain is constructed and this scheme is applied on a piecewise uniform Shishkin mesh. For the decision making process we utilize a class of Ordered Weighted Averaging (OWA) operators and the newly calculated decision maker weights through numerical method are used in the computations of identifying the best alternative from the available alternatives. Numerical illustrations are presented to support the developed theory.

Keywords: MADM, Intuitionistic Fuzzy Sets, Singular perturbation problem, Numerical methods, OWA operator, Delay differential equation, Shishkin mesh, Finite difference scheme.
1 Introduction
Multi Attribute Decision Making (MADM) problems are widely spread in many real life situations. Zadeh, [14] introduced the concept of fuzzy set, in which each object is assigned a single real value, called the grade of membership, between zero and one. Using this concept Atanassov[1,2] introduced the concept of Intuitionistic Fuzzy Set (IFS) characterized by a membership function and a non-membership function, which is a generalization of the concept of fuzzy set whose basic component is only a membership function. Yager, [12-13] introduced some families of OWA operators. The Indeed OWA(IOWA) operator was introduced by Yager & Filev, [11] which is the extension of the OWA operator. Robinson & Amirtharaj, [7,8], Robinson & Jeeva, [9] and Zeng & Li, [15] proposed correlation coefficients for different higher order intuitionistic fuzzy sets and utilized them in MADM problems for ranking the alternatives. Robinson & Poovarasam, [10] contributed novel approaches to the field MADM. In this work, singularly perturbed delay differential equations are used for weight determination for decision maker in MADM problems. A differential equation in which the derivative of the unknown function at a certain time is expressed in terms of the values of the function at previous times is known as a Delay Differential Equation or a Differential-Difference Equation. A delay differential equation in which the highest order derivative is multiplied by a small parameter $\varepsilon$ is a singularly perturbed delay differential equation. Delay differential equations play an important role in the mathematical modeling of various practical phenomena in the subjects of bioscience and control theory. Singularly Perturbed Delay Differential Equations(SPDDEs) arise frequently in the modeling of human pupil-light reflex, the study of bistable devices and variational problems in control theory. manikandan et.al., [4] and Paramasivam et.al., [6] presented a numerical method composed of a classical finite difference scheme applied on a piecewise-uniform Shishkin mesh is suggested to solve the problems involving ordinary differential equations. But in [4] the method is proved to be a first-order convergent and in [6] the method is essentially second order convergent in the maximum norm uniformly in the perturbation parameter. Cen, [3] gave a numerical method which is suggested for
the initial value problem for a class of delay differential equations. The method uses a hybrid difference scheme on a fitted mesh and is proved to be second-order convergent under the assumption that $CN^{-1}$, where $\varepsilon$ is a perturbation parameter and $N$ is the number of mesh points. Miller et. al. [5] have devoted their work on singular perturbation problems (SPP) in two dimensions. The parameter-uniform method can be achieved by using both a finite difference operator method and a piecewise uniform Shishkin mesh method. The plan of the paper is as follows. In section 2 and 3, Intuitionistic Fuzzy Set and aggregation operators of IFS are presented. In section 4, SPDDEs are introduced. In section 5, a new method of deriving decision maker weights using SPDDEs is proposed. In section 6, Algorithm for MADM are presented and in section 7, numerical illustrations are presented.

2 Intuitionistic Fuzzy Set (IFS)

Definition 1 (Intuitionistic Fuzzy Set [1-2]). Let $X$ be a set which is fixed. An IFS $\tilde{A}$ in $X$ is an object having the form

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \gamma_{\tilde{A}}(x)), x \in X\}$$

where $\mu_{\tilde{A}}(x) : X \rightarrow [0, 1]$ and $\gamma_{\tilde{A}}(x) : X \rightarrow [0, 1]$ define the degree of membership and degree of non-membership respectively, of the element $x \in X$ to the set $\tilde{A}$, which is a subset of $X$, for every element $x \in X$, $0 \leq \mu_{\tilde{A}}(x) + \gamma_{\tilde{A}}(x) \leq 1$.

3 Different Classes of Aggregation Operators

Definition 2. Let $\tilde{a}_j = (\mu_j, \gamma_j), 1 \leq j \leq n$ be a collection of intuitionistic fuzzy values. The Intuitionistic Fuzzy Weighted Averaging (IFWA) operator, $IFWA : Q^n \rightarrow Q$ is defined as:

$$IFWA(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} \tilde{a}_j \omega_j = \left[1 - \prod_{j=1}^{n} (1 - \mu_j)^{\omega_j}, \prod_{j=1}^{n} (\gamma_j)^{\omega_j}\right],$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{a}_j$ for all $j = 1, 2, \ldots, n$, such that $\omega_j > 0$ & $\sum_{j=1}^{n} w_j = 1$. 

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**Definition 3.** Let \( \tilde{a}_j = (\mu_j, \gamma_j) \), \( 1 \leq j \leq n \) be a collection of intuitionistic fuzzy values. An Induced Intuitionistic Fuzzy Ordered Weighted Averaging (I-IFOWA) operator, \( I - \text{IFOWA} : Q^n \rightarrow Q \) is defined as:

\[
I - \text{IFOWA}_w \left[ (v_1, \tilde{a}_1), (v_2, \tilde{a}_2), ..., (v_n, \tilde{a}_n) \right] = \sum_{j=1}^{n} \tilde{g}_j w_j = \left[ 1 - \prod_{j=1}^{n} (1 - \tilde{\mu}_j)^{w_j}, \prod_{j=1}^{n} \tilde{\gamma}_j^{w_j} \right],
\]

where \( w = (w_1, w_2, ..., w_n)^T \) is the associated weighting vector such that \( w_j \in [0, 1] \), \( \sum_{j=1}^{n} w_j = 1 \), \( j = 1, 2, ..., n \), \( \tilde{g}_j = (\mu_j, \gamma_j) \) is the \( \tilde{a}_j \) value of the I-IFOWA pair \((v_i, \tilde{a}_i)\) having the \( j^{th} \) largest \( v_i \), \( v_i \in [0, 1] \), and \( v_i \) in \((v_i, \tilde{a}_i)\) is referred to as the order inducing variable and \( \tilde{a}_i, \tilde{a}_i = (\mu_i, \gamma_i) \) are the intuitionistic fuzzy values.

The I-IFOWA operator has the Commutativity, Idempotency and Monotonicity properties.

4 Singularity Perturbed Delay Differential Equations (SPDDEs)

In this paper, a boundary value problem for a singularly perturbed delay differential equation of reaction-diffusion type is considered. A numerical method is proposed and constructed using a classical finite difference scheme and a hybrid scheme on an appropriate Shishkin mesh, which resolves not only the usual boundary layers but also the interior layers arising from the delay term. More precisely, the singularly perturbed boundary value problem is

\[
Lu(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x - 1) = f(x) \quad \text{on} \quad (0, 2) \quad (1)
\]

with \( u = \varphi \) on \([-1, 0]\) and \( u(2) = l \) \( (2) \)

where \( \varphi \) is sufficiently smooth on \([-1, 0]\). For all \( x \in [0, 2] \), it is assumed that \( a(x) \) and \( b(x) \) satisfy, \( a(x) + b(x) > 2\alpha \) and \( b(x) < 0 \), for some real number \( \alpha > 0 \). Furthermore, the functions \( a(x), b(x) \), and \( f(x) \) are assumed to be in \( C^3[0, 2] \).
4.1 Analytical Results

The continuous linear operator of boundary value problems L satisfies the following maximum principle.

**Lemma 1:** Let \( a(x) \) and \( b(x) \) satisfy \( a(x) + b(x) > 2\alpha \) and \( b(x) < 0 \). Let \( \psi \) be in \( C \) such that \( \psi(0) \geq 0 \), \( \psi(2) \geq 0 \), \( L\psi(0) \geq 0 \) on \((0, 2)\) then \( \psi \geq 0 \) on \([0, 2] \).

As a consequence of the maximum principle, the stability result for the problem (1) and (2) is established in the following:

**Lemma 2:** Let \( a(x) \) and \( b(x) \) satisfy \( a(x) + b(x) > 2\alpha \) and \( b(x) < 0 \). If \( \psi \) is any function in \( C \), then for all \( x \in [0, 2] \),

\[
|\psi(x)| \leq \max\{|\psi(0)|, |\psi(2)|, \frac{1}{\alpha} \| L\psi \| \}.
\]

4.2 The Shishkin Mesh

A piecewise-uniform Shishkin mesh with \( N \) mesh intervals is now constructed on \( \Omega=[0, 2] \) as follows. Let \( \Omega^N=\Omega_1^N \cup \Omega_2^N \) where \( \Omega_1^N = \{x_j\}_{j=1}^{\frac{N}{2}-1}, \quad \Omega_2^N = \{x_j\}_{j=\frac{N}{2}+1}^{N} \) and \( x_{\frac{N}{2}} = 1 \). Then \( \overline{\Omega}_1^N = \{x_j\}_{j=0}^{\frac{N}{2}}, \quad \overline{\Omega}_2^N = \{x_{\frac{N}{2}}, x_j\}_{j=\frac{N}{2}+1}^{N} \) \( \overline{\Omega}^N = \overline{\Omega}_1^N \cup \overline{\Omega}_2^N = \{x_j\}_{j=0}^{N} \) and \( \Gamma^N = \{0, 2\} \) The interval \([0, 1]\) is divided into three subintervals as follows:

\[
[0, \tau] \cup (\tau, 1 - \tau] \cup (1 - \tau, 1].
\]

The parameter \( \tau \), which determine the points separating the uniform meshes, is defined by \( \tau = \min \left\{ \frac{1}{4}, 2\sqrt{\frac{1}{\alpha} \ln N} \right\} \).

Then, on the subinterval \((\tau, 1 - \tau)\), a uniform mesh of \( N/4 \) mesh points is placed and on each of the subintervals \([0, \tau]\) and \((1 - \tau, 1]\), a uniform mesh of \( N/8 \) mesh points is placed.

Similarly, the interval \((1, 2]\) is also divided into 3 subintervals \((1, 1 + \tau], (1 + \tau, 2 - \tau]\), and \((2 - \tau, 2]\), using the same parameter \( \tau \). In particular, when the parameter \( \tau \) takes on its left-hand value, the Shishkin mesh \( \overline{\Omega}^N \) becomes a classical uniform mesh throughout from 0 to 2.
In practice, it is convenient to take $N = 8k$, $k \geq 2$. From the above construction of $\Omega^N$, it is clear that the transition points \{\tau, 1 - \tau, 1 + \tau, 2 - \tau\} are the only points at which the mesh size can change and that it does not necessarily change at each of these points.

The following notations are introduced: $h_j = x_j - x_{j-1}$, $h_{j+1} = x_{j+1} - x_j$, and if $x_j = \tau$, then $h_j = x_j - x_{j-1}$, $h_{j+1} = x_{j+1} - x_j$. For each point $x_j$ in the mesh intervals $[0, \tau]$ and $(1 - \tau, 1]$, 

$$x_j - x_{j-1} = 8N^{-1}\tau,$$

and for $x_j \in (\tau, 1 - \tau]$, $x_j - x_{j-1} = 4N^{-1}(1 - 2\tau)$.

4.3 The Discrete Problem

In this section, an hybrid finite difference scheme operator with an appropriate Shishkin mesh is used to construct a numerical method for (1) and (2).

The discrete two-point boundary value problem is now defined to be

$$L^N U(x_j) = -\varepsilon \delta^2 U(x_j) + a(x_j) U(x_j) + b(x_j) U(x_j - 1) = f(x_j), \text{ on } \Omega_N,$$

$$U = u \text{ on } \Gamma_N. \quad (3)$$

The problem (3) can be rewritten as

$$L_1^N U(x_j) = -\varepsilon \delta^2 U(x_j) + a(x_j) U(x_j) = g(x_j), \text{ on } \Omega^-_N,$$

$$L_2^N U(x_j) = -\varepsilon \delta^2 U(x_j) + a(x_j) U(x_j) + b(x_j) U(x_j - 1) = f(x_j), \text{ on } \Omega^+_N,$$

$$U = u \text{ on } \Gamma_N,$$

$$D^-_0 U(x_{N/2}) = D^+_0 U(x_{N/2}). \quad (4)$$

where
Table 1: Values of $\varepsilon = 2^{-10}$, $N = 2048$ and $\alpha = 0.9$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Values of $\tau$</th>
<th>Normalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>0.0730641261</td>
<td>0.0189586297</td>
</tr>
<tr>
<td>$1 - \tau$</td>
<td>0.853871763</td>
<td>0.221562064</td>
</tr>
<tr>
<td>$\frac{2 + \tau}{2}$</td>
<td>1.07306409</td>
<td>0.27843794</td>
</tr>
<tr>
<td>$2 - \tau$</td>
<td>1.85387170</td>
<td>0.48104136</td>
</tr>
</tbody>
</table>

$$
\delta^2 U(x_j) = \left( \frac{D^+ - D^-}{h} \right) U(x_j), \quad h = \frac{h_j + h_{j+1}}{2}
$$

$$
D^+ U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{h_{j+1}}
$$

$$
D^- U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j}
$$

Hybrid scheme:

$$
D_0^+ U(x_j) = \frac{3U(x_{j+1}) - 4U(x_j) + U(x_{j-1})}{2h_{j+1}}
$$

$$
D_0^- U(x_j) = \frac{3U(x_j) - 4U(x_{j-1}) + U(x_{j-2})}{2h_j}
$$

5 Decision maker weights from SPDDEs

Problem proposed by the decision maker:
The following problem is proposed by the decision maker to construct weights to the decision variables which is solved through a coding in FORTRAN in the LINUX environment.
The boundary value problem given by the decision maker is as follows:

$$
-\varepsilon u''(x) + 3u(x) - u(x-1) = 1 \text{ for } x \in (0, 2)
$$

$$
u(x) = x^2 \text{ for } x \in [-1, 0], \quad u(2) = 0
$$
Table 2: Values of $\varepsilon = 2^{-10}$, $N = 2048$ and $\alpha = 0.9$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\tau$ values of $\tau$</th>
<th>Normalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>0.146128252</td>
<td>0.037756589</td>
</tr>
<tr>
<td>$1 - 2\tau$</td>
<td>0.926935852</td>
<td>0.15081622</td>
</tr>
<tr>
<td>$\frac{1 + \tau}{2}$</td>
<td>1.146128388</td>
<td>0.186479733</td>
</tr>
<tr>
<td>$4 - 2\tau$</td>
<td>3.926935917</td>
<td>0.638928389</td>
</tr>
</tbody>
</table>

Figure 1: Numerical solution of $u$ for $\varepsilon = 2^{-10}$

6 Algorithm for MADM Problem using SPDDEs

The algorithm for MAGDM problem using I-IFOWA and IFWA operator is as follows:

**Step:1** Use I-IFOWA operator for the decision matrix $R_{(k)}$, to derive the individual overall preference IFS values.

**Step:2** Use IFWA operator, to derive the collective overall preference IFS values of the alternatives $A_i$.

**Step:3** Calculate the correlation coefficient [15] between the collective overall values and the positive ideal value $\tilde{s}^+$, where $\tilde{s}^+ = (1, 0)$.

**Step:4** Finally rank the alternatives and find the most desirable one.
7 Numerical Illustration

The construction companies of management in a particular city wants to select the prospect construction projects. There are five such prospect construction projects out of which the best has to be chosen. The decision maker has made strict evaluation for five prospect construction projects $A_i (i = 1, 2, 3, 4, 5)$ according to the following four attributes: $G_1$ is the safety; $G_2$ is the product quality; $G_3$ is the duration; $G_4$ is the cost. The five possible alternatives $A_i$ are to be evaluated using intuitionistic fuzzy numbers by the decision maker whose weighting vector is obtained by singular perturbed delay differential problem as $\omega = (0.0189586297, 0.221562064, 0.27843794, 0.48104136)^T$ and $w = (0.0237756589, 0.15081622, 0.186479733, 0.638928389)^T$, from table 1 and table 2. The decision matrices $R_{(k)}$ are listed below:

$$R_1 = \begin{pmatrix}
(0.4, 0.3) & (0.5, 0.2) & (0.2, 0.5) & (0.1, 0.6) \\
(0.6, 0.2) & (0.6, 0.1) & (0.6, 0.1) & (0.3, 0.4) \\
(0.6, 0.3) & (0.5, 0.3) & (0.6, 0.2) & (0.4, 0.2) \\
(0.1, 0.7) & (0.3, 0.5) & (0.2, 0.6) & (0.4, 0.5) \\
(0.3, 0.3) & (0.6, 0.1) & (0.4, 0.5) & (0.6, 0.1) \\
\end{pmatrix}$$

$$R_2 = \begin{pmatrix}
(0.5, 0.4) & (0.5, 0.3) & (0.2, 0.6) & (0.4, 0.4) \\
(0.7, 0.3) & (0.7, 0.3) & (0.6, 0.2) & (0.6, 0.2) \\
(0.5, 0.4) & (0.6, 0.4) & (0.6, 0.2) & (0.5, 0.3) \\
(0.8, 0.2) & (0.7, 0.2) & (0.4, 0.2) & (0.5, 0.2) \\
(0.4, 0.3) & (0.4, 0.2) & (0.4, 0.5) & (0.4, 0.6) \\
\end{pmatrix}$$

$$R_3 = \begin{pmatrix}
(0.4, 0.5) & (0.6, 0.2) & (0.5, 0.4) & (0.5, 0.3) \\
(0.5, 0.4) & (0.6, 0.2) & (0.6, 0.3) & (0.7, 0.3) \\
(0.4, 0.5) & (0.3, 0.5) & (0.4, 0.4) & (0.2, 0.6) \\
(0.5, 0.4) & (0.7, 0.2) & (0.4, 0.4) & (0.6, 0.2) \\
(0.6, 0.3) & (0.7, 0.2) & (0.4, 0.2) & (0.7, 0.2) \\
\end{pmatrix}$$

$$R_4 = \begin{pmatrix}
(0.4, 0.4) & (0.3, 0.4) & (0.4, 0.3) & (0.3, 0.3) \\
(0.5, 0.3) & (0.5, 0.3) & (0.3, 0.5) & (0.5, 0.2) \\
(0.6, 0.2) & (0.6, 0.4) & (0.4, 0.4) & (0.6, 0.3) \\
\end{pmatrix}$$

By using the algorithm, we obtain:

$$\rho_{ZL}(\tilde{s}^{(1)}, \tilde{s}^+) = 0.78788; \rho_{ZL}(\tilde{s}^{(2)}, \tilde{s}^+) = 0.60536;$$

$$\rho_{ZL}(\tilde{s}^{(3)}, \tilde{s}^+) = 0.54911; \rho_{ZL}(\tilde{s}^{(4)}, \tilde{s}^+) = 0.53356;$$

$$\rho_{ZL}(\tilde{s}^{(5)}, \tilde{s}^+) = 0.57134.$$ 

Rank all the alternatives $A_i, (i = 1, 2, 3, 4, 5)$.

$A_1 > A_2 > A_3 > A_4 > A_5.$

Hence the best alternative is $A_1$. 

9
8 Conclusion

In this work, the decision maker presents the weighting vector of the decision problem in the form of SPDDEs. A new method and a scheme based on numerical methods is proposed which calculates the decision maker weights and further used in the MADM problem. A class of OWA operators was used to aggregate intuitionistic fuzzy information and correlation coefficient was used to rank the alternatives. The hybrid scheme proposed in this paper to solve SPDDEs is a novel and efficient method based on the order of convergence.

References


