



ON $\delta\mathcal{I}_s$ -OPEN SETS AND $\delta\mathcal{I}_s$ -CONTINUOUS FUNCTIONS

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Abstract: In this paper, we introduce and investigate the notion of $\delta\mathcal{I}_s$ -open sets and $\delta\mathcal{I}_s$ -continuous functions in ideal topological spaces. Also we introduce $R\mathcal{I}_s$ -open and semi- $\delta\mathcal{I}_s$ -continuous functions and we obtain several characterization and their properties.

1 Introduction and Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ) , $cl(A)$ and $int(A)$ denote the closure and interior of A in (X, τ) respectively.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies: (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$ (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

If (X, τ) is a topological space and $\mathcal{I}$ is an ideal on X , then $(X, \tau, \mathcal{I})$ is called an ideal topological space or an ideal space.

Let $P(X)$ be the power set of X . Then the operator $()^* : P(X) \rightarrow P(X)$ called a local function [7] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every open set } U \text{ containing } x\}$. We simply write A^* instead of $A^*(\mathcal{I}, \tau)$ in case there is no confusion. For every ideal topological space $(X, \tau, \mathcal{I})$ there exists topology τ^* finer than τ , generated by $\beta(\mathcal{I}, \tau) = \{U \setminus J : U \in \tau \text{ and } J \in \mathcal{I}\}$ but in general $\beta(\mathcal{I}, \tau)$ is not always a topology. Additionally $cl^*(A) = A \cup A^*$ defines Kuratowski closure operator for a topology τ^* finer than τ . Throughout this paper X denotes the ideal topological space $(X, \tau, \mathcal{I})$ and also $int^*(A)$ denotes the interior of A with respect to τ^* .

DEFINITION 1.1. Let (X, τ) be a topological space. A subset A of X is said to be semi-open [8] if there exists an open set U in X such that $U \subseteq A \subseteq cl(U)$. The complement of a semi-open set is said to be semi-closed. The collection of all semi-open (resp. semi-closed) sets in X is denoted by $SO(X)$ (resp. $SC(X)$). The semi-closure of A in (X, τ) is denoted by the intersection of all semi-closed sets containing A and is denoted by $scl(A)$.

DEFINITION 1.2. For $A \subseteq X$, $A_*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in SO(X)\}$ is called the semi-local function [5] of A with respect to $\mathcal{I}$ and τ , where $SO(X, x) = \{U \in SO(X) : x \in U\}$. We simply write A_* instead of $A_*(\mathcal{I}, \tau)$ in this case there is no ambiguity.

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It is given in [1] that $\tau^{**}(\mathcal{I})$ is a topology on X , generated by the sub basis $\{U - E : U \in SO(X) \text{ and } E \in \mathcal{I}\}$ or equivalently $\tau^{**}\mathcal{I} = \{U \subseteq X : cl^{**}(X - U) = X - U\}$. The closure operator cl^{**} for a topology $\tau^{**}(\mathcal{I})$ is defined as follows: for $A \subseteq X$, $cl^{**}(A) = A \cup A_*$ and int^{**} denotes the interior of the set A in $(X, \tau^{**}, \mathcal{I})$. It is known that $\tau \subseteq \tau^*(\mathcal{I}) \subseteq \tau^{**}(\mathcal{I})$. A subset A of $(X, \tau, \mathcal{I})$ is called semi- $*$ -perfect [6] if $A = A_*$. $A \subseteq (X, \tau, \mathcal{I})$ is called $*$ -semi dense in-itself [6](resp. semi- $*$ -closed [6]) if $A \subset A_*$ (resp. $A_* \subseteq A$).

LEMMA 1.3. [5] *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B be subsets of X . Then for the semi-local function the following properties hold:*

1. *If $A \subseteq B$ then $A_* \subseteq B_*$.*
2. *If $U \in \tau$ then $U \cap A_* \subseteq (U \cap A)_*$*
3. *$A_* = scl(A_*) \subseteq scl(A)$ and A_* is semi-closed in X .*
4. *$(A_*)_* \subseteq A_*$.*
5. *$(A \cup B)_* = A_* \cup B_*$.*
6. *If $\mathcal{I} = \{\phi\}$, then $A_* = scl(A)$.*

DEFINITION 1.4. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

1. $\delta\mathcal{I}$ -open [2] if $int(cl^*(A)) \subseteq cl^*(int(A))$,
2. $\alpha\mathcal{I}_s$ -open [9] if $A \subseteq int(cl^{**}(int(A)))$,
3. pre- $\mathcal{I}_s$ -open [9] if $A \subseteq int(cl^{**}(A))$,
4. semi- $\mathcal{I}_s$ -open [9] if $A \subseteq cl^{**}(int(A))$.

2 $\delta\mathcal{I}_s$ -open sets

DEFINITION 2.1. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called $\delta\mathcal{I}_s$ -open if $int(cl^{**}(A)) \subseteq cl^{**}(int(A))$. The family of all $\delta\mathcal{I}_s$ -open (resp. $\delta\mathcal{I}$ -open) sets in an ideal topological space $(X, \tau, \mathcal{I})$ is denoted by $\delta ISO(X)$ (resp. $\delta IO(X)$).

DEFINITION 2.2. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $R\mathcal{I}_s$ -open (resp. $R\mathcal{I}$ -open and regular open) set if $int(cl^{**}(A)) = A$ (resp. $int(cl^*(A)) = A$ and $int(cl(A)) = A$). We call a subset A of $R\mathcal{I}_s$ -closed if its complement is $R\mathcal{I}_s$ -open.

PROPOSITION 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Every $\delta\mathcal{I}_s$ -open set is $\delta\mathcal{I}$ -open.*

Proof. It is obvious

REMARK 2.4. *The following example shows that the converse of the Proposition 2.3 need not be true.*

EXAMPLE 2.5. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then $A = \{a, b, c\}$ is $\delta\mathcal{I}$ -open set which is not $\delta\mathcal{I}_s$ -open.*

PROPOSITION 2.6. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is a semi- $\mathcal{I}_s$ -open, then A is $\delta\mathcal{I}_s$ -open.

Proof: Let A is semi- $\mathcal{I}_s$ -open. Then $A \subseteq cl^{*s}(int(A))$. Now, $int(cl^{*s}(A)) \subseteq cl^{*s}(A) \subseteq cl^{*s}(cl^{*s}(int(A))) = cl^{*s}(int(A))$. Therefore A is $\delta\mathcal{I}_s$ -open.

PROPOSITION 2.7. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then a subset of X is $\alpha\mathcal{I}_s$ -open if and only if it is both $\delta\mathcal{I}_s$ -open and pre- $\mathcal{I}_s$ -open.

Proof. Necessity Let A be an $\alpha\mathcal{I}_s$ -open set. Since every $\alpha\mathcal{I}_s$ -open set is semi- $\mathcal{I}_s$ -open, by Proposition 2.6 A is a $\delta\mathcal{I}_s$ -open set. Since A is an $\alpha\mathcal{I}_s$ -open set, we have $A \subseteq int(cl^{*s}(int(A))) \subseteq int(cl^{*s}(A))$. Hence A is pre- $\mathcal{I}_s$ -open set.

Sufficiency Let A be a $\delta\mathcal{I}_s$ -open and pre- $\mathcal{I}_s$ -open set. Then we have $int(cl^{*s}(A)) \subseteq cl^{*s}(int(A))$ and hence $int(cl^{*s}(A)) \subseteq int(cl^{*s}(int(A)))$. Since A is pre- $\mathcal{I}_s$ -open, $A \subseteq int(cl^{*s}(A))$. Therefore we obtain that $A \subseteq int(cl^{*s}(int(A)))$ and hence A is an $\alpha\mathcal{I}_s$ -open set.

REMARK 2.8. $\delta\mathcal{I}_s$ -open and pre- $\mathcal{I}_s$ -open are independent of each other as seen from the following examples.

EXAMPLE 2.9. In Example 2.5, $A = \{a, b, c\}$ is pre- $\mathcal{I}_s$ -open set which is not $\delta\mathcal{I}_s$ -open.

EXAMPLE 2.10. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Then $A = \{a, b, c\}$ is $\delta\mathcal{I}_s$ -open set which is not pre- $\mathcal{I}_s$ -open.

REMARK 2.11. We obtained the following diagram:

$$\begin{array}{ccccc} \text{open} & \longrightarrow & \alpha\mathcal{I}_s\text{-open} & \longrightarrow & \text{pre-}\mathcal{I}_s\text{-open} \\ & & \downarrow & & \\ & & \text{semi-}\mathcal{I}_s\text{-open} & \longrightarrow & \delta\mathcal{I}_s\text{-open} \end{array}$$

PROPOSITION 2.12. Let A, B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If $A \subseteq B \subseteq cl^{*s}(A)$ and $A \in \delta ISO(X)$, then $B \in \delta ISO(X)$.

Proof. Suppose that $A \subseteq B \subseteq cl^{*s}(A)$ and $A \in \delta ISO(X)$. Then, since $A \in \delta ISO(X)$, we have $int(cl^{*s}(A)) \subseteq cl^{*s}(int(A))$. Since $A \subseteq B$, $cl^{*s}(int(A)) \subseteq cl^{*s}(int(B))$ and $int(cl^{*s}(A)) \subseteq cl^{*s}(int(B))$. Since $B \subseteq cl^{*s}(A)$, we have $cl^{*s}(B) \subseteq cl^{*s}(cl^{*s}(A)) = cl^{*s}(A)$ and $int(cl^{*s}(B)) \subseteq int(cl^{*s}(A))$. Therefore we obtain that $int(cl^{*s}(B)) \subseteq cl^{*s}(int(B))$. This shows that B is a $\delta\mathcal{I}_s$ -open set.

COROLLARY 2.13. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $A \subseteq X$ is $\delta\mathcal{I}_s$ -open and $*$ -semi-dense, then every subset of X containing A is $\delta\mathcal{I}_s$ -open.

Proof. The proof is obvious by Proposition 2.12.

PROPOSITION 2.14. Let A, B and C be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. If A is a $\delta\mathcal{I}_s$ -open set, then $A = B \cup C$, where $B \in \alpha ISO(X)$, $int(cl^{*s}(C)) = \phi$ and $B \cap C = \phi$.

Proof. Suppose that $A \in \delta\text{ISO}(X)$. Then we have $\text{int}(cl^{**}(A)) \subseteq cl^{**}(\text{int}(A))$ and $\text{int}(cl^{**}(A)) \subseteq \text{int}(cl^{**}(\text{int}(A)))$. Now we have $A = [\text{int}(cl^{**}(A)) \cap A] \cup [A \setminus \text{int}(cl^{**}(A))]$. Now, we set $B = \text{int}(cl^{**}(A)) \cap A$ and $C = A \setminus \text{int}(cl^{**}(A))$. We first show that $B \in \alpha\text{ISO}(X)$, that is $B \subseteq \text{int}(cl^{**}(\text{int}(B)))$.

Now we have $\text{int}(cl^{**}(\text{int}(B))) = \text{int}(cl^{**}(\text{int}[\text{int}(cl^{**}(A)) \cap A])) = \text{int}(cl^{**}[\text{int}(cl^{**}(A)) \cap \text{int}(A)]) = \text{int}(cl^{**}(\text{int}(A)))$. Since $A \in \delta\text{ISO}(X)$, $\text{int}(cl^{**}(\text{int}(A))) \supseteq \text{int}(cl^{**}(A)) \supseteq B$ and thus $B \in \alpha\text{ISO}(X)$. Next we show that $\text{int}(cl^{**}(C)) = \phi$. Since $\tau \subseteq \tau^{**}$, $cl^{**}(S) \subseteq cl(S)$ for any subset S of X . Therefore, we have

$$\begin{aligned} \text{int}(cl^{**}(C)) &\subseteq \text{int}(cl^{**}[A \cap (X \setminus \text{int}(cl^{**}(A))])) \subseteq \text{int}(cl^{**}(A)) \cap \text{int}(cl^{**}(X \setminus \text{int}(cl^{**}(A)))) \\ &\subseteq \text{int}(cl^{**}(A)) \cap \text{int}(cl(X \setminus \text{int}(cl^{**}(A)))) \subseteq \text{int}(cl^{**}(A)) \cap (X \setminus \text{int}(cl^{**}(A))) = \phi. \end{aligned}$$

It is obvious that $B \cap C = [\text{int}(cl^{**}(A)) \cap A] \cap [A \setminus \text{int}(cl^{**}(A))] = \phi$.

PROPOSITION 2.15. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If A is $\delta\mathcal{I}_s$ -open and semi-closed, then A is a δ -set.

Proof. Since A is semi-closed, $\text{int}(cl(A)) \subseteq A$ and hence $\text{int}(cl(A)) \subseteq \text{int}(A)$. Thus $\text{int}(cl(A)) \subseteq \text{int}(A) \subseteq \text{int}(cl^{**}(A)) \subseteq cl^{**}(\text{int}(A)) = \text{int}(A) \cup (\text{int}(A))_*$. By [5] $(\text{int}(A))_* \subseteq scl(\text{int}(A)) \subseteq cl(\text{int}(A))$. Hence $\text{int}(cl(A)) \subseteq cl(\text{int}(A))$. Therefore A is a δ -set.

REMARK 2.16. $\delta\mathcal{I}_s$ -openness and δ -openness are independent of each other as seen from the following examples.

EXAMPLE 2.17. In Example 2.10, $A = \{b, c, d\}$ is a $\delta\mathcal{I}_s$ -open set which is not δ -open.

EXAMPLE 2.18. In Example 2.5, $A = \{a, b, c\}$ is a δ -open set which is not $\delta\mathcal{I}_s$ -open.

Now, we introduce the following definitions,

DEFINITION 2.19. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, S a subset of X and x a point of X .

1. x is called a $\delta\mathcal{I}_s$ -cluster point of S if $S \cap \text{int}(cl^{**}(U)) \neq \phi$ for each open neighborhood U of x ,
2. The family of all $\delta\mathcal{I}_s$ -cluster points of S is called the $\delta\mathcal{I}_s$ -closure of S and is denoted by $[S]_{\delta\mathcal{I}_s}$,
3. A subset S is said to be $\delta\mathcal{I}_s$ -closed if $[S]_{\delta\mathcal{I}_s} = S$. The complement of a $\delta\mathcal{I}_s$ -closed set of X is said to be $\delta\mathcal{I}_s$ -open.

LEMMA 2.20. Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. Then the following properties hold:

1. $\text{int}(cl^{**}(A))$ is $R\mathcal{I}_s$ -open,
2. If A and B are $R\mathcal{I}_s$ -open, then $A \cap B$ is $R\mathcal{I}_s$ -open,
3. If A is regular open, then it is $R\mathcal{I}_s$ -open,

4. If A is $R\mathcal{I}_s$ -open, then it is $\delta\mathcal{I}_s$ -open,
5. Every $\delta\mathcal{I}_s$ -open set is the union of a family of $R\mathcal{I}_s$ -open sets.

Proof. (1) Let A be a subset of X and $V = \text{int}(cl^{**}(A))$. Then we have $\text{int}(cl^{**}(V)) = \text{int}(cl^{**}(\text{int}(cl^{**}(A)))) \subseteq \text{int}(cl^{**}(cl^{**}(A))) = \text{int}(cl^{**}(A)) = V$ and also $V = \text{int}(V) \subseteq \text{int}(cl^{**}(V))$.

(2) Let A and B be $R\mathcal{I}_s$ -open. Then we have $A \cap B = \text{int}(cl^{**}(A)) \cap \text{int}(cl^{**}(B)) = \text{int}(cl^{**}(A) \cap cl^{**}(B)) \supseteq \text{int}(cl^{**}(A \cap B)) \supseteq \text{int}(A \cap B) = A \cap B$. Therefore we obtain $A \cap B = \text{int}(cl^{**}(A \cap B))$. This shows that $A \cap B$ is $R\mathcal{I}_s$ -open.

(3) Let A be a regular open set, since $\tau^* \supseteq \tau$, we have $A = \text{int}(A) \subseteq \text{int}(cl^{**}(A)) \subseteq \text{int}(cl(A)) = A$ and hence $A = \text{int}(cl^{**}(A))$. Therefore A is $R\mathcal{I}_s$ -open.

(4) Let A be any $R\mathcal{I}_s$ -open set. For each $x \in A$, $(X - A) \cap A = \phi$ and A is $R\mathcal{I}_s$ -open, $(X - A) \cap \text{int}(cl^{**}(A)) = \phi$. Hence $x \notin [X - A]_{\delta-\mathcal{I}_s}$ for each $x \in A$. This shows that $x \notin X - A$ implies $x \notin [X - A]_{\delta-\mathcal{I}_s}$. Therefore we have $[X - A]_{\delta-\mathcal{I}_s} \subseteq X - A$. Since in general $S \subseteq [S]_{\delta-\mathcal{I}_s}$ for any subset S of X , $[X - A]_{\delta-\mathcal{I}_s} = X - A$ and hence A is $\delta\mathcal{I}_s$ -open.

(5) Let A be a $\delta\mathcal{I}_s$ -open set. Then $(X - A)$ is $\delta\mathcal{I}_s$ -closed and hence $[X - A]_{\delta-\mathcal{I}_s} = X - A$. For each $x \in A$, $x \notin [X - A]_{\delta-\mathcal{I}_s}$ and there exists an open neighborhood V_x such that $\text{int}(cl^{**}(V_x)) \cap (X - A) = \phi$. Therefore, we have $x \in V_x \subseteq \text{int}(cl^{**}(V_x)) \subseteq A$ and hence $A = \bigcup \{\text{int}(cl^{**}(V_x)) / x \in A\}$. By (1), $\text{int}(cl^{**}(V_x))$ is $R\mathcal{I}_s$ -open for each $x \in A$.

LEMMA 2.21. Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. Then the following properties hold:

1. $A \subseteq [A]_{\delta-\mathcal{I}_s}$.
2. If $A \subseteq B$ then $[A]_{\delta-\mathcal{I}_s} \subseteq [B]_{\delta-\mathcal{I}_s}$.
3. $[A]_{\delta-\mathcal{I}_s} = \bigcap \{F \subseteq X / A \subseteq F \text{ and } F \text{ is a } \delta\mathcal{I}_s\text{-closed}\}$.
4. If A is a $\delta\mathcal{I}_s$ -closed set of X for each $\alpha \in \Delta$, then $\bigcap \{A_\alpha / \alpha \in \Delta\}$ is a $\delta\mathcal{I}_s$ -closed.
5. $[A]_{\delta-\mathcal{I}_s}$ is $\delta\mathcal{I}_s$ -closed.

Proof. (1) For any $x \in A$ and any open neighborhood V of x , we have $\phi \neq A \cap V \subseteq A \cap \text{int}(cl^{**}(V))$ and hence $x \in [A]_{\delta-\mathcal{I}_s}$. This shows that $A \subseteq [A]_{\delta-\mathcal{I}_s}$.

(2) Suppose that $x \notin [B]_{\delta-\mathcal{I}_s}$. There exists an open neighborhood V of x such that $\phi = \text{int}(cl^{**}(V)) \cap B$; hence $\text{int}(cl^{**}(V)) \cap A = \phi$. Therefore $x \notin [A]_{\delta-\mathcal{I}_s}$.

(3) Suppose that $x \in [A]_{\delta-\mathcal{I}_s}$. For any open neighborhood V of x and any $\delta\mathcal{I}_s$ -closed set F containing A , we have $\phi \neq A \cap \text{int}(cl^{**}(V)) \subseteq F \cap \text{int}(cl^{**}(V))$ and hence $x \in [F]_{\delta-\mathcal{I}_s} = F$. This shows that $x \in \bigcap \{F \subseteq X / A \subseteq F \text{ and } F \text{ is a } \delta\mathcal{I}_s\text{-closed}\}$.

Conversely, suppose that $x \notin [A]_{\delta-\mathcal{I}_s}$. There exists an open neighborhood V of x such that $\text{int}(cl^{**}(V)) \cap A = \phi$. By Lemma 2.20 $X - \text{int}(cl^{**}(V))$ is a $\delta\mathcal{I}_s$ -closed set which contains A and does not contain x . Therefore we obtain $x \notin \bigcap \{F \subseteq X / A \subseteq F \text{ and } F \text{ is a } \delta\mathcal{I}_s\text{-closed}\}$. Therefore $[A]_{\delta-\mathcal{I}_s} = \bigcap \{F \subseteq X / A \subseteq F \text{ and } F \text{ is a } \delta\mathcal{I}_s\text{-closed}\}$.

(4) For each $\alpha \in \Delta$, $\left[\bigcap_{\alpha \in \Delta} A_\alpha \right]_{\delta-\mathcal{I}_s} \subseteq [A_\alpha]_{\delta-\mathcal{I}_s} = A_\alpha$ and hence $\left[\bigcap_{\alpha \in \Delta} A_\alpha \right]_{\delta-\mathcal{I}_s} \subseteq \bigcap_{\alpha \in \Delta} A_\alpha$. By (1), we obtain $\left[\bigcap_{\alpha \in \Delta} A_\alpha \right]_{\delta-\mathcal{I}_s} = \bigcap_{\alpha \in \Delta} A_\alpha$. This shows that $\bigcap_{\alpha \in \Delta} A_\alpha$ is $\delta\mathcal{I}_s$ -closed.

(5) This follows immediately from (3) and (4).

THEOREM 2.22. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $\tau_{\delta-\mathcal{I}_s} = \{A \subseteq X / A \text{ is a } \delta\text{-}\mathcal{I}_s\text{-open set of } (X, \tau, \mathcal{I})\}$. Then $\tau_{\delta-\mathcal{I}_s}$ is a topology.

Proof. (1) It is obvious that $\phi, X \in \tau_{\delta-\mathcal{I}_s}$.
 (2) Let $V_\alpha \in \tau_{\delta-\mathcal{I}_s}$ for each $\alpha \in \Delta$. Then $X - V_\alpha$ is $\delta\text{-}\mathcal{I}_s$ -closed for each $\alpha \in \Delta$. By Lemma 2.21, $\bigcap_{\alpha \in \Delta} (X - V_\alpha)$ is $\delta\text{-}\mathcal{I}_s$ -closed and $\bigcap_{\alpha \in \Delta} (X - V_\alpha) = X - \bigcup_{\alpha \in \Delta} V_\alpha$. Hence $\bigcup_{\alpha \in \Delta} V_\alpha$ is $\delta\text{-}\mathcal{I}_s$ -open.
 (3) Let $A, B \in \tau_{\delta-\mathcal{I}_s}$. By Lemma 2.20, $A = \bigcup_{\alpha \in \Delta_1} A_\alpha$ and $B = \bigcup_{\beta \in \Delta_2} B_\beta$, where A_α, B_β are $R\text{-}\mathcal{I}_s$ -open sets for each $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. Thus $A \cap B = \bigcup \{A_\alpha \cap B_\beta / \alpha \in \Delta_1, \beta \in \Delta_2\}$. Since $A_\alpha \cap B_\beta$ is $R\text{-}\mathcal{I}_s$ -open, $A \cap B$ is a $\delta\text{-}\mathcal{I}_s$ -open set by Lemma 2.20.

3 $\delta\text{-}\mathcal{I}_s$ -continuous functions

DEFINITION 3.1. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be $\delta\text{-}\mathcal{I}_s$ -continuous if for each $x \in X$ and each open neighborhood V of $f(x)$, then there exists an open neighborhood U of x such that $f(\text{int}(cl^{*s}(U))) \subseteq \text{int}(cl^{*s}(V))$.

THEOREM 3.2. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent.

1. f is $\delta\text{-}\mathcal{I}_s$ -continuous,
2. For each $x \in X$ and each $R\text{-}\mathcal{I}_s$ -open set V containing $f(x)$, there exists an $R\text{-}\mathcal{I}_s$ -open set containing x such that $f(U) \subseteq V$,
3. $f([A]_{\delta-\mathcal{I}_s}) \subseteq [f(A)]_{\delta-\mathcal{I}_s}$ for every $A \subseteq X$,
4. $[f^{-1}(B)]_{\delta-\mathcal{I}_s} \subseteq f^{-1}([B]_{\delta-\mathcal{I}_s})$ for every $B \subseteq Y$,
5. For every $\delta\text{-}\mathcal{I}_s$ -closed set F of Y , $f^{-1}(F)$ is $\delta\text{-}\mathcal{I}_s$ -closed in X ,
6. For every $\delta\text{-}\mathcal{I}_s$ -open set V of Y , $f^{-1}(V)$ is $\delta\text{-}\mathcal{I}_s$ -open in X ,
7. For every $R\text{-}\mathcal{I}_s$ -open set V of Y , $f^{-1}(V)$ is $\delta\text{-}\mathcal{I}_s$ -open in X ,
8. For every $R\text{-}\mathcal{I}_s$ -closed set F of Y , $f^{-1}(F)$ is $\delta\text{-}\mathcal{I}_s$ -closed in X .

Proof. (1) $\Rightarrow$ (2) This follows immediately from Definition 2.1

(2) $\Rightarrow$ (3) Let $x \in X$ and $A \subseteq X$ such that $f(x) \in f([A]_{\delta-\mathcal{I}_s})$. Suppose that $f(x) \notin f([A]_{\delta-\mathcal{I}_s})$. Then there exists an $R\text{-}\mathcal{I}_s$ -open neighborhood V of $f(x)$ such that $f(A) \cap V = \phi$. By (2) there exists an $R\text{-}\mathcal{I}_s$ -open neighborhood U of x such that $f(U) \subseteq V$. Since $f(A) \cap f(U) \subseteq f(A) \cap V = \phi$, $f(A) \cap f(U) = \phi$. Hence, we get that $U \cap A \subseteq f^{-1}(f(U)) \cap f^{-1}(f(A)) = f^{-1}(f(U) \cap f(A)) = \phi$. Hence we have $U \cap A = \phi$ and $x \notin [A]_{\delta-\mathcal{I}_s}$. This shows that $f(x) \notin f([A]_{\delta-\mathcal{I}_s})$. This is a contradiction. Therefore we obtain that $f(x) \in f([A]_{\delta-\mathcal{I}_s})$.

(3) $\Rightarrow$ (4) Let $B \subseteq Y$ such that $A = f^{-1}(B)$. By (3), $f([f^{-1}(B)]_{\delta-\mathcal{I}_s}) \subseteq [f(f^{-1}(B))]_{\delta-\mathcal{I}_s} \subseteq [B]_{\delta-\mathcal{I}_s}$. We have $[f^{-1}(B)]_{\delta-\mathcal{I}_s} \subseteq f^{-1}([f(f^{-1}(B))]_{\delta-\mathcal{I}_s}) \subseteq f^{-1}([B]_{\delta-\mathcal{I}_s})$. Thus we obtain that $[f^{-1}(B)]_{\delta-\mathcal{I}_s} \subseteq f^{-1}([B]_{\delta-\mathcal{I}_s})$.

(4) $\Rightarrow$ (5) Let $F \subseteq Y$ be $\delta\mathcal{I}_s$ -closed. By (4) $[f^{-1}(F)]_{\delta-\mathcal{I}_s} \subseteq f^{-1}([F]_{\delta-\mathcal{I}_s}) = f^{-1}(F)$ and always $f^{-1}(F) \subseteq [f^{-1}(F)]_{\delta-\mathcal{I}_s}$. Hence we obtain that $[f^{-1}(F)]_{\delta-\mathcal{I}_s} = f^{-1}(F)$. This shows that $f^{-1}(F)$ is $\delta\mathcal{I}_s$ -closed.

(5) $\Rightarrow$ (6) Let $V \subseteq Y$ be $\delta\mathcal{I}_s$ -open. Then $Y - V$ is $\delta\mathcal{I}_s$ -closed. By (5) $f^{-1}(Y - V) = X - f^{-1}(V)$ is $\delta\mathcal{I}_s$ -closed. Therefore $f^{-1}(V)$ is $\delta\mathcal{I}_s$ -open.

(6) $\Rightarrow$ (7) Let $V \subseteq Y$ be $R\mathcal{I}_s$ -open. Since every $R\mathcal{I}_s$ -open is $\delta\mathcal{I}_s$ -open, V is a $\delta\mathcal{I}_s$ -open. By (6) $f^{-1}(V)$ is $\delta\mathcal{I}_s$ -open.

(7) $\Rightarrow$ (8) Let $F \subseteq Y$ be an $R\mathcal{I}_s$ -closed set. Then $Y - F$ is $R\mathcal{I}_s$ -open. By (7), $f^{-1}(Y - F) = X - f^{-1}(F)$ is $\delta\mathcal{I}_s$ -open. Therefore $f^{-1}(F)$ is $\delta\mathcal{I}_s$ -closed.

(8) $\Rightarrow$ (1) Let $x \in X$ and V be an open set containing $f(x)$. Now, set $V_\circ = \text{int}(cl^{**}(V))$, then by Lemma 2.20, $Y - V_\circ$ is an $R\mathcal{I}_s$ -closed set. By (8), $f^{-1}(Y - V_\circ) = X - f^{-1}(V_\circ)$ is a $\delta\mathcal{I}_s$ -closed set. Thus we have $f^{-1}(V_\circ)$ is a $\delta\mathcal{I}_s$ -open. Since $x \in f^{-1}(V_\circ)$, by Lemma 2.20 there exists an open neighborhood U of x such that $x \in U \subseteq \text{int}(cl^{**}(U)) \subseteq f^{-1}(V_\circ)$. Hence we obtain that $f(\text{int}(cl^{**}(U))) \subseteq \text{int}(cl^{**}(V))$.

COROLLARY 3.3. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is $\delta\mathcal{I}_s$ -continuous if and only if $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is continuous.

Proof. This is obvious from Theorem 2.22.

THEOREM 3.4. [6] Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq Y \subseteq X$, where Y is a α -open in X . Then $A_*(\mathcal{I}, \tau|_Y) = A_*(\mathcal{I}, \tau) \cap Y$.

THEOREM 3.5. Let $(X, \tau, \mathcal{I})$ be an ideal topological space, $A, X_\circ$ are subsets of X such that $A \subseteq X_\circ$ and $X_\circ$ is open in X .

1. If A is $R\mathcal{I}_s$ -open in $(X, \tau, \mathcal{I})$, then A is $R\mathcal{I}_s$ -open in $(X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ})$,
2. If A is $\delta\mathcal{I}_s$ -open in $(X, \tau, \mathcal{I})$, then A is $\delta\mathcal{I}_s$ -open in $(X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ})$,

Proof. (1) Let A be $R\mathcal{I}_s$ -open in $(X, \tau, \mathcal{I})$. Then $A = \text{int}(cl^{**}(A))$ and $cl_{X_\circ}^{**}(A) = A \cup A_*(\tau|_{X_\circ}, \mathcal{I}_{X_\circ}) = A \cup [A_*(\tau, \mathcal{I}) \cap X_\circ] = (A \cap X_\circ) \cup (A_* \cap X_\circ) = (A \cup A_*) \cap X_\circ = cl^{**}(A) \cap X_\circ$. Hence we have $\text{int}_{X_\circ}(cl_{X_\circ}^{**}(A)) = \text{int}(cl_{X_\circ}^{**}(A)) = \text{int}(cl^{**}(A) \cap X_\circ) = \text{int}(cl^{**}(A)) \cap X_\circ = A$. Therefore A is $R\mathcal{I}_s$ -open in $(X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ})$.

(2) Let A is $\delta\mathcal{I}_s$ -open set in $(X, \tau, \mathcal{I})$. By Lemma 2.20(5) $A = \bigcup_{\alpha \in \Delta} A_\alpha$, where A_α is $R\mathcal{I}_s$ -open set of $(X, \tau, \mathcal{I})$ for each $\alpha \in \Delta$. By (1), A is $R\mathcal{I}_s$ -open in $(X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ})$ for each $\alpha \in \Delta$ and hence A is $\delta\mathcal{I}_s$ -open in $(X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ})$.

THEOREM 3.6. If $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is a $\delta\mathcal{I}_s$ -continuous function and $X_\circ$ is a $\delta\mathcal{I}_s$ -open set of $(X, \tau, \mathcal{I})$, then the restriction $f|_{X_\circ} : (X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ}) \rightarrow (Y, \sigma, \mathcal{J})$ is $\delta\mathcal{I}_s$ -continuous.

Proof. Let V be any is a $\delta\mathcal{I}_s$ -open set of $(Y, \sigma, \mathcal{J})$. Since f is $\delta\mathcal{I}_s$ -continuous, $f^{-1}(V)$ is $\delta\mathcal{I}_s$ -open in $(X, \tau, \mathcal{I})$. Since $X_\circ$ is $\delta\mathcal{I}_s$ -open, by Theorem 2.22, $X_\circ \cap f^{-1}(V)$ is $\delta\mathcal{I}_s$ -open in $(X, \tau, \mathcal{I})$ and hence by Theorem 3.4 $X_\circ \cap f^{-1}(V)$ is $\delta\mathcal{I}_s$ -open in $(X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ})$. This shows that $(f|_{X_\circ})^{-1}(V)$ is $\delta\mathcal{I}_s$ -open in $(X_\circ, \tau|_{X_\circ}, \mathcal{I}_{X_\circ})$ and hence $f|_{X_\circ}$ is $\delta\mathcal{I}_s$ -continuous.

THEOREM 3.7. If $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \longrightarrow (Z, \eta, \mathcal{K})$ are $\delta\text{-}\mathcal{I}_s$ -continuous, then so is $g \circ f : (X, \tau, \mathcal{I}) \longrightarrow (Z, \eta, \mathcal{K})$.

Proof. It is obvious from corollary 3.3.

THEOREM 3.8. If $f, g : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ are $\delta\text{-}\mathcal{I}_s$ -continuous and Y is a Hausdorff space, then $A = \{x \in X : f(x) = g(x)\}$ is a $\delta\text{-}\mathcal{I}_s$ -closed set of $(X, \tau, \mathcal{I})$

Proof. We prove that $X - A$ is $\delta\text{-}\mathcal{I}_s$ -open set. Let $x \in X - A$. Then, $f(x) \neq g(x)$. Since Y is Hausdorff, there exists open sets V_1 and V_2 containing $f(x)$ and $g(x)$, respectively, such that $V_1 \cap V_2 = \phi$. From here we have $\text{int}(cl(V_1)) \cap \text{int}(cl(V_2)) = \phi$. Thus we obtain that $\text{int}(cl^{**}(V_1)) \cap \text{int}(cl^{**}(V_2)) = \phi$. Since f and g are $\delta\text{-}\mathcal{I}_s$ -continuous, there exists an open neighborhood U of x such that $f(\text{int}(cl^{**}(U))) \subseteq \text{int}(cl^{**}(V_1))$ and $g(\text{int}(cl^{**}(U))) \subseteq \text{int}(cl^{**}(V_2))$.

Hence we obtain that $\text{int}(cl^{**}(U)) \subseteq f^{-1}(\text{int}(cl^{**}(V_1)))$ and $\text{int}(cl^{**}(U)) \subseteq g^{-1}(\text{int}(cl^{**}(V_2)))$. We have $\text{int}(cl^{**}(U)) \subseteq f^{-1}(\text{int}(cl^{**}(V_1))) \cap g^{-1}(\text{int}(cl^{**}(V_2)))$. More over $f^{-1}(\text{int}(cl^{**}(V_1))) \cap g^{-1}(\text{int}(cl^{**}(V_2))) \cap A = \phi$. Suppose that $f^{-1}(\text{int}(cl^{**}(V_1))) \cap g^{-1}(\text{int}(cl^{**}(V_2))) \cap A \neq \phi$. Hence there exists a point z such that $z \in f^{-1}(\text{int}(cl^{**}(V_1))) \cap g^{-1}(\text{int}(cl^{**}(V_2))) \cap A$. Thus $f(z) \in \text{int}(cl^{**}(V_1))$, $g(z) \in \text{int}(cl^{**}(V_2))$ and $z \in A$. Since $z \in A$, $f(z) = g(z)$. Therefore $f(z) \in \text{int}(cl^{**}(V_1)) \cap \text{int}(cl^{**}(V_2))$ and $\text{int}(cl^{**}(V_1)) \cap \text{int}(cl^{**}(V_2)) \neq \phi$. This is a contraction to our assumption.

Hence we obtain that $f^{-1}(\text{int}(cl^{**}(V_1))) \cap g^{-1}(\text{int}(cl^{**}(V_2))) \cap A = \phi$. Thus $f^{-1}(\text{int}(cl^{**}(V_1))) \cap g^{-1}(\text{int}(cl^{**}(V_2))) \subseteq X - A$. Since $\text{int}(cl^{**}(U)) \subseteq f^{-1}(\text{int}(cl^{**}(V_1))) \cap g^{-1}(\text{int}(cl^{**}(V_2)))$, we have that there exists an open neighborhood of x such that $x \in U$, $\text{int}(cl^{**}(U)) \subseteq X - A$. Therefore $X - A$ is $\delta\text{-}\mathcal{I}_s$ -open set. This shows that A is $\delta\text{-}\mathcal{I}_s$ -closed.

DEFINITION 3.9. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is called semi- $\delta\text{-}\mathcal{I}_s$ -continuous (resp. semi- $\delta\text{-}\mathcal{I}$ -continuous [2]) if for every $V \in \sigma$, $f^{-1}(V) \in \delta\text{ISO}(X)$ (resp. $\delta\text{IO}(X)$).

THEOREM 3.10. For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$, the following properties are equivalent:

1. f is semi- $\mathcal{I}_s$ -continuous.
2. f is semi- $\delta\text{-}\mathcal{I}_s$ -continuous.

Proof. The proof is obvious by Proposition 2.6.

THEOREM 3.11. For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$, the following properties are equivalent:

1. f is $\alpha\text{-}\mathcal{I}_s$ -continuous.
2. f is pre- $\mathcal{I}_s$ -continuous and semi- $\mathcal{I}_s$ -continuous
3. f is pre- $\mathcal{I}_s$ -continuous and semi- $\delta\text{-}\mathcal{I}_s$ -continuous.

Proof. The proof is obvious by Proposition 2.6 and Proposition 2.7.

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