



EXISTENCE AND UNIQUENESS OF NON-LOCAL CAUCHY PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATION ON BANACH SPACE

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July 25, 2018

Abstract

This article determines sufficient conditions for existence and uniqueness of solution for impulsive fractional integro-differential system with classical and non-local condition on Banach space using fixed point theorem. The system taken by the authors is more general system taken so far, as it is more relevant to physical system encountered. An example is added to illustrate the efficacy of the method.

Keywords: Existence and Uniqueness, Cauchy problem, Impulsive conditions, Fixed point theory.

1 Introduction

There are various problems in science and engineering like seepage flow in porous media [1], anomalous diffusion [2, 3], and transport [4], the nonlinear oscillations of earthquake [5], fluid dynamics traffic model [6] etc are well modeled in fractional differential equations. This is because of its non local property [8] which mean, that the

next state of the system depends upon its entire historical states. Due to this nonlocal property fractional dynamical systems are considered as an alternative model for highly nonlinear integer order systems [7]. Using xed point theory existence and uniqueness of fractional dynamical semilinear and nonlinear systems have been studied by Delbosco and Rodino [5], Cheng and Guozhu [8] and El-Borai [9]. Byszewski [10] initiated study of existence of non-local Cauchy problem using xed point theorem and extended it by various researchers [11]. The study of existence and uniqueness of mild solution of nonlocal Cauchy problem of fractional dynamical systems with xed point theory was initiated by N Guerekata [12] followed by Balachandran and Park [13].

On the other hand, rapid development of impulsive differential equations played very important role in modeling of many problems of population dynamics, chemical technology and biotechnology [14]. This motivates many researchers to study existence and uniqueness solutions of the impulsive differential equations [15]. Determination of sufficient conditions for existence and uniqueness of mild solution of impulsive fractional Cauchy problem with classical conditions with the use of xed point theory and semigroup theory given by Benchohra and Slimani [16], Mophou [17], Ravichandran and Arjunan [18]. Sufficient conditions for the mild solution for semilinear impulsive fractional nonlocal Cauchy problem using various xed point theorem was studied by Benchohra and Slimani [16]. Balachandran et. al. [19, 20] and Gao et. al. [21] studied existence and uniqueness of mild solution of impulsive fractional nonlocal quasilinear Cauchy problem with nonlocal conditions. Motivated by the work of Balachandran et. al. [19] this paper presents the existence and uniqueness of solutions of the quasilinear impulsive fractional order Volterra-Fredholm kind of integro-differential equation of the form over the interval $[0, T]$.

$$\begin{aligned} {}^c D^\alpha x(t) &= A(t, x)x(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k, \quad k = 1, 2, \dots, p \\ \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, \dots, p \\ x(0) &= x_0 - g(x), \end{aligned}$$

2 Preliminaries

Some basic definitions and properties of fractional calculus and fractional differential equations used in this article, are as follows:

Definition 2.1. [19, 20] The Riemann-Liouville fractional integral operator of $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is gamma function. **Definition 2.2.** [19, 20] The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

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where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

This derivative has singularity at zero and also requires special initial condition which lacks physical interpretation. To overcome this difficulty, Caputo [22] interchanged the role of operators and defined the fractional derivatives as follows:

Definition 2.3. [19, 20] The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$${}^c D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n f(s)}{ds^n} ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Moreover if $0 < \alpha < 1$, then

$${}^c D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{df(s)}{ds} ds.$$

The Riemann-Liouville integral I_α and Caputo derivative ${}^cD_{0+}^\alpha$ satisfies following properties which is mentioned in Kilbas et. al. [23] and Samko et. al. [24].

Theorem 2.1. For $\alpha, \beta > 0$ and f has absolutely continuous derivatives up to suitable order then

$$\begin{aligned} (1) & I_{0+}^\alpha I_{0+}^\beta f(t) = I_{0+}^{\alpha+\beta} f(t) \\ (2) & I_{0+}^\alpha I_{0+}^\beta f(t) = I_{0+}^\beta I_{0+}^\alpha f(t) \\ (3) & I_{0+}^\alpha (f(t) + g(t)) = I_{0+}^\alpha f(t) + I_{0+}^\alpha g(t) \\ (4) & I_{0+}^\alpha {}^cD_{0+}^\alpha f(t) = f(t) - f(0), \quad 0 < \alpha < 1 \\ (5) & {}^cD_{0+}^\alpha I_{0+}^\alpha f(t) = f(t) \\ (6) & {}^cD_{0+}^\alpha f(t) = I_{0+}^{1-\alpha} f'(t), \quad 0 < \alpha < 1 \\ (7) & {}^cD_{0+}^\alpha {}^cD_{0+}^\beta f(t) \neq {}^cD_{0+}^{\alpha+\beta} f(t) \\ (8) & {}^cD_{0+}^\alpha {}^cD_{0+}^\beta f(t) \neq {}^cD_{0+}^\beta {}^cD_{0+}^\alpha f(t) \end{aligned}$$

2.1 Notations

(N1) X = Banach space.

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0, T_0], X) = \{x : [0, T_0] \rightarrow X/x \text{ is continuous}\}$ with norm $\|x\| = \sup_t \|x(t)\|$

(N4) $PC([0, T_0], X) = \{x : [0, T_0] \rightarrow X; x \in C([t_{k-1}, t_k], X), \text{ and } x(t_k^-) \text{ and } x(t_k^+) \text{ exist, } k = 1, 2, \dots, p \text{ with } x(t_k^-) = x(t_k)\}$ with norm $\|x\|_{PC} = \sup_{t \in [0, T_0]} \|x(t)\|$

(N5) $AC([0, T_0], X) = \{x : [0, T_0] \rightarrow X/x \text{ is absolutely continuous}\}$ with norm $\|x\| = \sup_t \|x(t)\|$

(N6) $B(X) = \{A : X \rightarrow X/A \text{ is bounded and linear}\}$ with norm $\|A\|_{B(X)} = \sup\{\|A(y)\|; y \in X \text{ and } \|y\| \leq 1\}$

For convenience, ${}^cD_{0+}^\alpha$ is taken as ${}^cD^\alpha$ and with these denitions and properties, sucient conditions for existence and uniqueness of solutions are derived as follows:

3 Equation with classical condition

This section presents the study of the existence and uniqueness of the solution of impulsive fractional diifferential equation with classical condition of the form

$$\begin{aligned}
{}^c D^\alpha x(t) &= A(t, x)x(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k, \quad k = 1, 2, \dots, p \\
\Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, \dots, p \\
x(0) &= x_0
\end{aligned} \tag{3.1}$$

over the interval $[0, T]$. Where A is bounded linear operator on X and $f : [0, T] \times X \times X \times X \rightarrow X$, $T, S : X \rightarrow X$ are defined by $Tx(t) = \int_0^t h(t, s, x(s))ds$ and $Sx(t) = \int_0^t k(t, s, x(s))ds$. Where $h : D_0 \times X \rightarrow X$, $D_0 = \{(t, s); 0 \leq s \leq t \leq T\}$ and $k : D_1 \times X \rightarrow X$, $D_1 = \{(t, s); 0 \leq t, s \leq T\}$ are continuous.

This type of nonlinear equations arise in many physical situation like mathematical problems concerned with heat flow in materials with viscoelastic problems [25].

Applying fractional integral operator both side of the equation (3.1) to get

$$\begin{aligned}
x(t) &= \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x(s))x(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s, x(s))ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s))ds + \sum_{0 < t_k < t} Ix(t_k^-). \end{cases}
\end{aligned} \tag{3.2}$$

The following conditions are assumed to show the existence and uniqueness of the solution (3.1).

- (H1) $A : [0, T] \times X \rightarrow X$ is continuous bounded linear operator and there exists a positive constant M , such that $\|A(t, x) - A(t, y)\|_{B(X)} \leq M\|x - y\|$, for all $x, y \in X$.
- (H2) $f : [0, T] \times X \times X \times X \rightarrow X$ is continuous and there exists positive constants L_1, L_2 and L_3 , such that $\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq L_1\|x_1 - y_1\| + L_2\|x_2 - y_2\| + L_3\|x_3 - y_3\|$ for all x_1, x_2, x_3, y_1, y_2 and y_3 in X .

(H3) $h : D_0 \times X \rightarrow X$ and $k : D_1 \times X \rightarrow X$ are continuous and there exists positive constants H and K , such that $\|h(t, s, x) - h(t, s, y)\| \leq H\|x - y\|$ and $\|k(t, s, x) - k(t, s, y)\| \leq K\|x - y\|$ for all x and y in X .

(H4) The functions $I_k : X \rightarrow X$ are continuous and there exist positive constants I_k^* for all $k = 1, 2, \dots, p$, such that $\|I_k x - I_k y\| \leq I_k^* \|x - y\|$ for all x and y in X .

Set, $\gamma = \frac{T^\alpha}{\Gamma(\alpha+1)}$ and further assume that,

(H5) $q = \gamma[(p+1)[M + L_1 + THL_2 + TKL_3] + \sum I_k^*] < 1$

Define $F : PC([0, T], X) \rightarrow PC([0, T], X)$ by

$$Fx(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x(s)) x(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s, x(s)) x(s) ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s)) ds + \sum_{0 < t_k < t} I_k x(t_k^-). \end{cases} \quad (3.3)$$

Then equation (3.2) has unique solution if F defined by (3.3) has unique fixed point. This means F is well defined bounded operator on $PC([0, T], X)$ and F is contraction [26].

Lemma 3.1. If the operators A, f, T, S and I_k for $k = 1, 2, \dots, p$ are continuous then F is bounded operator on $PC([0, T], X)$.

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0, T], X)$. Therefore $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Consider,

$$\begin{aligned} \|Fx_n - Fx\|_{PC} &\leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s, x_n(s)) x_n(s) - A(s, x(s)) x(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(s, x_n(s)) x_n(s) - A(s, x(s)) x(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, x_n(s), Tx_n(s), Sx_n(s)) - f(s, x(s), Tx(s), Sx(s))\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|f(s, x_n(s), Tx_n(s), Sx_n(s)) - f(s, x(s), Tx(s), Sx(s))\| ds \\ &\quad + \sum_{0 < t_k < t} \|I_k x_n(t_k^-) - I_k x(t_k^-)\| \end{aligned}$$

Assuming the continuity of A, f, T, S and I_k for $k = 1, 2, \dots, p$ the right side of above expression tends to zero as $n \rightarrow \infty$. Therefore F is continuous on $PC([0, T], X)$ and hence F is bounded.

Now we derive sufficient conditions for existence and uniqueness of the solution of equation (3.1).

Theorem 3.2. If the hypotheses (H1)-(H5) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (3.1) has unique solution in $PC([0, T], X)$ for $0 < \alpha \leq 1$.

Proof. To show equation (3.1) has unique solution it is sufficient to show F defined (3.3) is contraction. Let x and y in $PC([0, T], X)$ and consider,

$$\begin{aligned} \|Fx - Fy\|_{PC} &\leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s, x(s))x(s) - A(s, y(s))y(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(s, x(s))x(s) - A(s, y(s))y(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))\| ds \\ &\quad + \sum_{0 < t_k < t} \|I_k x(t_k^-) - I_k y(t_k^-)\| \end{aligned}$$

Applying hypotheses (H1)-(H4) we obtain,

$$\begin{aligned}
 \|Fx - Fy\| &\leq \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} M \|x - y\| ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} M \|x - y\| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \{L_1 + THL_2 + TKL_3\} \|x - y\| ds \\
 &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} ds \{L_1 + THL_2 + TKL_3\} \|x - y\| ds + \sum_{0 < t_k < t} I_k^* \|x - y\| \\
 &\leq \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} [(p+1)[M + L_1 + THL_2 + TKL_3]] + \sum I_k^* \right\} \|x - y\| \\
 &= \left\{ \gamma [(p+1)[M + L_1 + THL_2 + TKL_3]] + \sum I_k^* \right\} \|x - y\|
 \end{aligned}$$

Assuming hypotheses (H5) to obtain, $\|Fx - Fy\|_{PC} \leq q \|x - y\|$ with $q < 1$. Hence by Banach fixed point theorem the equation (3.1) has unique solution.

4 Equation with nonlocal condition

This section determines sufficient conditions for existence of mild solution of fractional impulsive system with nonlocal condition of the form

$$\begin{aligned}
 {}^c D^\alpha x(t) &= A(t, x)x(t) + f(t, x(t), Tx(t), Sx(t)) \quad t \neq t_k, \quad k = 1, 2, \dots, p \\
 \Delta x(t_k) &= I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, \dots, p \\
 x(0) &= x_0 - g(x)
 \end{aligned} \tag{4.1}$$

over the interval $[0, T]$. Where A is bounded linear operator on X and $f : [0, T] \times X \times X \times X \rightarrow X$, $T, S : X \rightarrow X$ are defined by $Tx(t) = \int_0^t h(t, s, x(s)) ds$ and $Sx(t) = \int_0^t k(t, s, x(s)) ds$. Where $h : D_0 \times X \rightarrow X$, $D_0 = \{(t, s); 0 \leq s \leq t \leq T\}$ and $k : D_1 \times X \rightarrow X$, $D_1 = \{(t, s); 0 \leq t, s \leq T\}$ are continuous and $g : X \rightarrow X$ is given function.

The equivalent integral equation of (4.1) is given by

$$x(t) = \begin{cases} x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x(s)) x(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s, x(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s)) ds + \sum_{0 < t_k < t} I_k x(t_k^-). \end{cases} \quad (4.2)$$

The following hypotheses are to be assumed.

(H6) $g : X \rightarrow X$ is continuous and there exist a positive constant g^* , such that $\|g(x) - g(y)\| \leq g^* \|x - y\|$ for each x and y in X .

(H7) $q^* = g^* + \gamma[(p+1)[M + L_1 + THL_2 + TKL_3]] + \sum I_k^* < 1$.

Define $G : PC([0, T], X) \rightarrow PC([0, T], X)$ by

$$Gx(t) = \begin{cases} x_0 - g(x) + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} A(s, x(s)) x(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} A(s, x(s)) x(s) ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} f(s, x(s), Tx(s), Sx(s)) ds + \sum_{0 < t_k < t} I_k x(t_k^-). \end{cases} \quad (4.3)$$

Lemma 4.1. If the operators A, f, T, S and I_k for $k = 1, 2, \dots, p$ are continuous then G is bounded operator on $PC([0, T], X)$.

Proof. Let a sequence $\{x_n\}$ be converges to x in $PC([0, T], X)$. Therefore $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Consider,

$$\begin{aligned} \|Gx_n - Gx\|_{PC} &\leq \|g(x_n(s)) - g(x(s))\| \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s, x_n(s)) x_n(s) - A(s, x(s)) x(s)\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(s, x_n(s)) x_n(s) - A(s, x(s)) x(s)\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, x_n(s), Tx_n(s), Sx_n(s)) - f(s, x(s), Tx(s), Sx(s))\| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|f(s, x_n(s), Tx_n(s), Sx_n(s)) - f(s, x(s), Tx(s), Sx(s))\| ds \\ &+ \sum_{0 < t_k < t} \|I_k x_n(t_k^-) - I_k x(t_k^-)\|. \end{aligned}$$

Assuming the continuity of A, f, T, S, g and I_k for $k = 1, 2, \dots, p$ the right side of above expression tends to zero as $n \rightarrow \infty$. Therefore G is continuous on $PC([0, T], X)$ and hence G is bounded.

Now the sufficient conditions are derived as under for existence and uniqueness of the solution of equation (4.1).

Theorem 4.2. If the hypotheses (H1)-(H4) and (H6)-(H7) are satisfied, then the fractional quasi-linear impulsive integro-differential equation (4.1) has unique solution in $PC([0, T], X)$ for $0 < \alpha \leq 1$.

Proof. To show equation (4.1) has unique solution it is sufficient to show F defined in (4.3) is contraction. Let x and y in $PC([0, T], X)$ and consider,

$$\begin{aligned} \|Gx - Gy\|_{PC} &\leq \|g(x) - g(y)\| + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|A(s, x(s))x(s) - A(s, y(s))y(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|A(s, x(s))x(s) - A(s, y(s))y(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} \|f(s, x(s), Tx(s), Sx(s)) - f(s, y(s), Ty(s), Sy(s))\| ds \\ &\quad + \sum_{0 < t_k < t} \|I_k x(t_k^-) - I_k y(t_k^-)\| \end{aligned}$$

Applying hypotheses (H1)-(H4) and (H6) the result is,

$$\begin{aligned}
\|Gx - Gy\| &\leq g^* \|x - y\| + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} M \|x - y\| ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} M \|x - y\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \{L_1 + THL_2 + TKL_3\} \|x - y\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha-1} ds \{L_1 + THL_2 + TKL_3\} \|x - y\| + \sum_{0 < t_k < t} I_k^* \|x - y\| \\
&\leq \left\{ g^* \frac{T^\alpha}{\Gamma(\alpha+1)} [(p+1)(M + L_1 + THL_2 + TKL_3)] + \sum I_k^* \right\} \|x - y\| \\
&= \left\{ g^* + \gamma [(p+1)(M + L_1 + THL_2 + TKL_3)] + \sum I_k^* \right\} \|x - y\|
\end{aligned}$$

Assuming hypotheses (H7) to obtain, $\|Gx - Gy\|_{PC} \leq q^* \|x - y\|$ with $q^* < 1$. Hence by Banach fixed point theorem the equation (4.1) has unique solution.

Example 4.2.1. Consider the following fractional integro-differential equation with the impulsive condition,

$$\begin{aligned}
{}^c D^\alpha x(t) &= \frac{1}{9} \cos x(t) x(t) + \frac{1}{(t+3)^4} \frac{|x|}{1+|x|} + \frac{1}{9} \int_0^t s e^{\frac{-x(s)}{4}} + \frac{1}{9} \int_0^1 (t-s) e^{-x(s)} ds \\
\Delta x(1/2) &= \frac{|x(1/2^-)|}{18 + |x(1/2^-)|} \\
x(0) &= x_0 - \frac{x}{18}
\end{aligned} \tag{4.4}$$

where $0 < \alpha < 1$ over the interval $[0, 1]$.

Since, $A(t, x) = \frac{1}{9} \cos x I$ therefore $\|A(t, x)x - A(t, y)y\| \leq \frac{1}{9} \|(\cos x I)x - (\cos y I)y\| \leq \frac{1}{9} \|x - y\|$,
 $\|Tx - Ty\| \leq \frac{1}{9} \int_0^t s \|e^{\frac{-x(s)}{4}} - e^{\frac{-y(s)}{4}}\| \leq \frac{1}{36} \|x - y\|$, $\|Sx - Sy\| \leq \frac{1}{9} \int_0^1 |(t-s)| e^{-x(s)} - e^{-y(s)} \|ds \leq \frac{1}{18} \|x - y\|$, $\|f(t, x, Tx, Sx) - f(t, y, Ty, Sy)\| \leq \frac{7}{48} \|x - y\|$, $\|g(x) - g(y)\| \leq \frac{1}{18} \|x - y\|$ and $q^* = g^* + \gamma[(p+1)[M + L_1 + THL_2 + TKL_3]] + \sum I_k^* = \frac{1}{18} + \gamma \frac{29}{72} + \frac{1}{18}$.
 Choose $\alpha = 1/2$ then $q^* = 0.57$ which is less than 1. Therefore by existence theorem the given system has unique solution in the interval $[0, 1]$.

5 Remark

1. This method suggest not only the existence and uniqueness about the solution but it also suggest method to nd approximate solution of impulsive fractional dierentialequations (3.1) and (4.1).
2. This condition is not necessary condition this means equations (3.1) and (4.1) may have solution if one of the (H1) to (H7) not satised.

6 Conclusion

The system taken by Balachandran et. al. [19] is a special case of the system taken in this paper because of inclusion of the nonlinear Fredholm operator in the system which is more relevant in many physical situations.

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