International Journal of Pure and Applied Mathematics
Volume 120 No. 5 2018, 1547-1554
ISSN: 1314-3395 (on-line version)
url: http://www.acadpubl.eu/hub/
Special Issue
http://www.acadpubl.eu/hub/

A Note on Weighted Quasinormal and Hyponormal Composition Operators on the Fock Space over $\mathbb{C}$

C Santhoshkumar [1], Dr.T Veluchamy [2]

[1] Corporate and Industry Relation, Amrita University,
Coimbatore, Tamilnadu, India 641112.
Email: santhosh_csk@yahoo.com

[2] SNS College of Arts and Science, Saravanampatti,
Coimbatore, Tamilnadu, India 641035.
Email: veluchamy_t@yahoo.com

Abstract In this paper, we study quasinormal and Hyponormal composition operators on the Fock space and obtain simple characterization for the quasinormal and Hyponormal composition operators on the Fock space.

2010 AMS Subject Classification: 47B38 (Primary) 47B20 (Secondary)

Keywords: Composition Operators, Fock Space, Normal, Quasinormal and Hyponormal Composition Operator.

1 Introduction

Let $\mathscr{H}$ be a Hilbert space who elements are holomorphic functions on a domain Ω. For a holomorphic map $\phi : \Omega \to \Omega$ and a holomorphic function h on Ω, the weighted composition operator is defined as $W_{f,\phi}h = f.h \circ \phi$. The domain of $W_{f,\phi}$ consists of all $h \in \mathscr{H}$ for which $f.h \circ \phi$ belongs to $\mathscr{H}$. When the weighted function f is identically one, the operator $W_{f,\phi}$ reduced to the composition operator C_ϕ. Researchers are after interested in how the function theoretic properties of f and ϕ affect the operator theoretic properties of $W_{f,\phi}$. The books [1], [5] are excellent references.
Recall that an operator A is called normal if A*A = AA*, and that it is called quasinormal if A and A*A commute. Moreover, an operator A is called hyponormal if A*A $\geq$ AA*, where $\geq$ denotes the usual ordering on selfadjoint operators. The latter condition is easily seen to be equivalent to $||Ax|| \geq ||A^*x||$ for all vectors x. It is also easy to see that the hyponormality of A implies the hyponormality of the translates A - λI. Combining these observations, we see that if x is an eigenvector

for A, then x is also an eigenvector for A*. An excellent reference for background on these ideas is Conway's book *The Theory of Subnormal Operators* [4]
The Fock space $\mathscr{F}^2$, also known as Segal - Bargmann space, consists of all entire functions on the complex plane $\mathbb{C}$ that are square integrable with respect to the Gaussian measure $d\mu(z) = \pi^{-1}e^{-|z|^2}dA(z)$, where dA denotes the Lebesque measure on $\mathbb{C}$.
The inner product on $\mathscr{F}^2$ is given by $\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}dA(z)$. Here $||.||$ denotes the corresponding norm.
It is known that the set $\{e_m(z) = \frac{z^m}{\sqrt{m!}}, m \geq 0\}$ forms an orthonormal basis for $\mathscr{F}^2$. It is also known that $\mathscr{F}^2$ is a reproducing kernel Hilbert space with kernel $K_zw = K(z,w) = e^{\bar{z}w}$, that $f(z) = \langle f, K_z\rangle$, for all $f \in \mathscr{F}^2$ and $z \in \mathbb{C}$.
Also $||K_z||^2 = \langle K_z, K_z\rangle = K_zz = e^{|z|^2}$. The book [7] is an excellent reference.

In [3], Carswell, Maccluer and Schuster studied boundedness and compactness of C_ϕ on the Fock space over $\mathbb{C}^n$. In [8], Liankuo Zhao characterized unitary weighted composition operators and their spectrum on the Fock space over $\mathbb{C}^n$. In [9], Liankuo Zhao characterized the isometric weighted compostion operator on the Fock space over $\mathbb{C}^n$. In [11], Trieu Le investigated boundedness and compactness using much simpler characterization than in [3]. In [10] Liankuo Zhao investigated the bounded invertible weighted composition operators on the Fock space over $\mathbb{C}^n$.
In this paper we report simple characterization of quasinormal and hyponormal composition operators on the Fock space over $\mathbb{C}$.

2 Preliminary Results

In this section, we list the well - known lemmas and properties of weighted composition operators on Hilbert spaces of analytic functions with reproducing kernel functions.

Lemma 2.1 *Let $f_1, f_2, ...f_n$ be analytic functions on $\mathbb{C}$ and $\phi_1, \phi_2, ...\phi_n$ be an analytic self-map on $\mathbb{C}$. If $C_{f_1,\phi_1}, C_{f_2,\phi_2}, ...C_{f_n,\phi_n}$, are bounded operators on $\mathscr{F}^2$, then* $C_{f_1,\phi_1}C_{f_2,\phi_2}...C_{f_n,\phi_n} = C_{f_1(f_2\circ\phi_1)....(f_n\circ\phi_{n-1}\circ...\circ\phi_1),\phi_n\circ\phi_{n-1}\circ...\circ\phi_1}$.

Lemma 2.2 *Let f be an analytic function on $\mathbb{C}$ and ϕ be an analytic self-map of $\mathbb{C}$. If $C_{f,\phi}$ is a bounded operator on $\mathscr{F}^2$, then for $z \in \mathbb{C}$,*
$C^*_{f,\phi}K_z = \overline{f(z)}K_{\phi(z)}$

Theorem 2.3 ([11], Theorem 2.2) *Let f and ϕ be entire functions on $\mathbb{C}$ such that f is not identically zero. Then $W_{f,\phi}$ is bounded if and only if f belongs to $\mathscr{F}^2$, $\phi(z) = \phi(0) + \lambda z$ with $|\lambda| \leq 1$ and*
$M(f, \phi) := sup\{|f|^2 exp(|\phi(z)|^2 - |z|^2); z \in \mathbb{C}\} < \infty$.

Theorem 2.4 ([11], Theorem 3.2) *Let f, and ϕ be entire functions such that f is not identically zero. Then the operator $W_{f,\phi}$ is a normal bounded operator on $\mathscr{F}^2$ if and only if one of the following two cases occurs:*
a. $\phi(z) = \lambda z + b$ with $|\lambda| = 1$ and $f = f(0)K_{\overline{\lambda}b}$. In this case, $W_{f,\phi}$ is a constant multiple of a unitary operator.
b. $\phi(z) = \lambda z + b$ with $|\lambda| < 1$ and $f = f(0)K_c$, where $c = b(1-\lambda)^{-1}(1-\overline{\lambda})$. In this case, $W_{f,\phi}$ is unitarily equvalent to $f(0)C_{\lambda z}$.

3 Main Results

3.1 Quasinormality of C_ϕ

Proposition 3.1 *Suppose ϕ be an analytic self map on $\mathbb{C}$ and C_ϕ be bounded operator on $\mathscr{F}^2$. If C_ϕ is quasinormal then $\phi(0) = 0$.*
Proof.
Let C_ϕ be quasinormal on $\mathscr{F}^2$. Since $K_0 \equiv 1$,

$$\begin{aligned} \langle C_\phi C_\phi^* C_\phi K_0, K_w\rangle &= \langle C_\phi C_\phi^* K_0, K_w\rangle \\ &= \langle C_\phi^* K_0, C_\phi^* K_w\rangle \\ &= \langle K_{\phi(0)}, K_{\phi(w)}\rangle \\ &= K_{\phi(0)}\phi(w) \end{aligned} \tag{1}$$

$$\begin{aligned} \langle C_\phi^* C_\phi C_\phi K_0, K_w\rangle &= \langle C_\phi^* K_0, K_w\rangle \\ &= \langle K_{\phi(0)}, K_w\rangle \\ &= K_{\phi(0)} w \end{aligned} \tag{2}$$

Equating (1) and (2), we get

$$\begin{aligned} K_{\phi(0)}\phi(w) &= K_{\phi(0)} w \\ e^{\overline{\phi(0)}\phi(w)} &= e^{\phi(0)w} \end{aligned} \tag{3}$$

for all $w \in \mathbb{C}$.

Setting $w = 0$, we have

$$e^{\overline{\phi(0)}\phi(0)} = e^0$$
$$|\phi(0)|^2 = 0$$

Thus $\phi(0) = 0$.

Proposition 3.2 *Suppose* ϕ *be an analytic self map on* $\mathbb{C}$ *and* C_ϕ *be bounded operator on* $\mathscr{F}^2$*. Then* C_ϕ *is quasinormal implies* $\phi(z) = \lambda z$ *with* $|\lambda| \leq 1$.
Moreover in this case C_ϕ *is quasinormal if and only if* C_ϕ *is normal.*
Proof
Let C_ϕ *be quasinormal on* $\mathscr{F}^2$.
By [[11], Theorem 2.2], boundedness of C_ϕ *implies* $\phi(z) = \phi(0) + \lambda z$ *with* $|\lambda| \leq 1$.
Combining above result with [Proposition 3.1] we get $\phi(z) = \lambda z$ *with* $|\lambda| \leq 1$.
Moreover, by [[11], Theorem 3.2], C_ϕ *is normal.*
Converse is obvious, since every normal operator is quasinormal.

Lemma 3.3 *([6, Lemma 2.1])*
Let $f_1, f_2, ..., f_n$ *be analytical functions on* $\mathbb{C}$ *and* $\phi_1, \phi_2, ..., \phi_n$ *be self map of* $\mathbb{C}$.
If $W_{f_1,\phi_1}, W_{f_2,\phi_2}, ..., W_{f_n,\phi_n}$ *are bounded operators on* $\mathscr{F}^2$*, then* $W_{f_1,\phi_1}W_{f_2,\phi_2}....W_{f_n,\phi_n} = W_{f_1(f_2\circ\phi_1)...(f_n\circ\phi_{n-1}\circ...\phi_1),\phi_n\circ\phi_{n-1}\circ...\circ\phi_1}$.

Theorem 3.4 *Let* f, ϕ *be analytic functions on* $\mathbb{C}$*,* f *is not identically zero and* $W_{f,\phi}$ *is bounded on* $\mathscr{F}$ *. Then* $W_{f,\phi}$ *is quasinormal operators on* $\mathscr{F}^2$ *if and only if* $\phi(z) = \phi(0) + z$ *and* $f(z) = \overline{f(0)}e^{-|\phi(0)|^2}$.
Proof.
Let $W_{f,\phi}$ *be quasinormal on* $\mathscr{F}^2$*. Since* $K_0 \equiv 1$,

$$\begin{aligned} \langle W_{f,\phi}W_{f,\phi}^*W_{f,\phi}K_0, K_w\rangle &= \langle W_{f,\phi}W_{f,\phi}^*W_{f,\phi}K_0w\rangle \\ &= f(0)W_{f,\phi}W_{f,\phi}^*K_0w \\ &= f(0)\overline{f(w)}W_{f,\phi}K_{\phi(0)}w \\ &= f(0)\overline{f(w)}\langle W_{f,\phi}K_{\phi(0)}, K_w\rangle \\ &= f(0)\overline{f(w)}\langle K_{\phi(0)}, W_{f,\phi}^*K_w\rangle \\ &= f(0)\overline{f(w)}\langle K_{\phi(0)}, f(w)K_{\phi(w)}\rangle \\ &= f(0)\overline{f(w)}^2\langle K_{\phi(0)}, K_{\phi(w)}\rangle \\ &= f(0)\overline{f(w)}^2K_{\phi(0)}\phi(w) \end{aligned} \tag{4}$$

$$\begin{aligned}\langle W^*_{f,\phi}W_{f,\phi}W_{f,\phi}K_0, K_w\rangle &= f(0)^2\langle W^*_{f,\phi}K_0, K_w\rangle \\ &= f(0)^2\overline{f(w)}\langle K_{\phi(0)}, K_w\rangle \\ &= f(0)^2\overline{f(w)}K_{\phi(0)}w\end{aligned} \tag{5}$$

Equating (5) and (6), we get

$$\begin{aligned} f(0)\overline{f(w)}^2K_{\phi(0)}\phi(w) &= f(0)^2\overline{f(w)}K_{\phi(0)}w \\ f(0)\overline{f(w)}^2e^{\overline{\phi(0)}\phi(w)} &= f(0)^2\overline{f(w)}e^{\overline{\phi(0)}w} \\ \overline{f(w)}e^{\overline{\phi(0)}\phi(w)} &= f(0)e^{\overline{\phi(0)}w}\end{aligned} \tag{6}$$

for all $w \in \mathbb{C}$.

Since $W_{f,\phi}$ *is a bounded operator on* $\mathscr{F}^2$, *by (7, Proposition 2.1)* $\phi(z) = \phi(0)+\lambda z$ *for some* $\lambda \leqq 1$.

So equation (7) becomes

$$\begin{aligned}\overline{f(w)}e^{\overline{\phi(0)}(\phi(0)+\lambda w)} &= f(0)e^{\overline{\phi(0)}w} \\ \overline{f(w)}e^{|\phi(0)|^2}e^{w\lambda\overline{\phi(0)}} &= f(0)e^{\overline{\phi(0)}w}\end{aligned} \tag{7}$$

Setting $f(w) = \overline{f(0)}e^{-|\phi(0)|^2}$, *we get* $e^{(\lambda-1)w\overline{\phi(0)}} = 1$ *for all* $w \in \mathbb{C}$.
Thus $\lambda = 1$.
This completes the proof.
Converse of this theorem is easy to verify.

3.2 Quasinormality of C^*_ϕ

Lemma 3.5 *Let* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *and* $\phi(z) = \lambda z$, *then* $C_\phi = C_\psi$ *where* $\psi(z) = \overline{\lambda}z$.
Proof
Consider

$$\begin{aligned} C^*_\phi K_w z &= K_{\phi(w)}z \\ &= K(z, \phi(w)) \\ &= exp(\langle z, \phi(w)\rangle \\ &= exp(\langle z, \lambda w)\rangle \\ &= exp(\langle\overline{\lambda}z, w)\rangle \\ &= K_w(\overline{\lambda z})\end{aligned} \tag{8}$$

Letting, $\psi(z) = \overline{\lambda}z$*, we have* $C_\phi^* = C_\psi$

Proposition 3.6 *Let* ϕ *be analytic self map on* $\mathbb{C}$ *such that* $\phi(0) = 0$ *and* C_ϕ *be bounded operator on* $\mathscr{F}^2$*. Then* C_ϕ^* *is quasinormal.*
Proof.
By [[11], Theorem 2.2], with $\phi(0) = 0$ *we get,* $\phi(z) = \lambda z$.
By [Lemma 3.5], $\psi(z) = \overline{\lambda}z$ *with* $|\lambda| \leq 1$.
Consider

$$\begin{aligned} \langle C_\psi C_\psi^* C_\psi K_0, K_w\rangle &= \langle C_\psi C_\psi^* K_0, K_w\rangle \\ &= \langle C_\psi^* K_0, C_\psi^* K_w\rangle \\ &= \langle K_{\psi(0)}, K_{\psi(w)}\rangle \\ &= K_{\psi(0)}\psi(w) \end{aligned} \tag{9}$$

$$\begin{aligned} \langle C_\psi^* C_\psi C_\psi K_0, K_w\rangle &= \langle C_\psi^* K_0, K_w\rangle \\ &= \langle K_{\psi(0)}, K_w\rangle \\ &= K_{\psi(0)}w \end{aligned} \tag{10}$$

Thus C_ψ *is quasinormal. Hence* C_ϕ^* *is quasinormal.*

3.3 Hyponormality of C_ϕ

Proposition 3.7 *Let* ϕ *be an analytic self map on* $\mathbb{C}$ *and* C_ϕ *be bounded on* $\mathscr{F}^2$*. If* C_ϕ *is hyponormal then* $\phi(0) = 0$
Proof
Suppose C_ϕ *is hyponormal.*
Consider $C_\phi K_0 = K_0 \circ \phi = K_0$, K_0 *is an eigenvector of* C_ϕ*. The hyponormality of* C_ϕ *implies that* K_0 *is also an eigenvector of* C_ϕ^*.
Thus $C_\phi^* K_0 = K_{\phi(0)} = K_0$*, so we get* $\phi(0) = 0$.

Corollary 3.7.1 *Let* C_ϕ *be a bounded hyponormal composition operator on* $\mathscr{F}^2$ *and* ϕ *be an analytic self map on* $\mathbb{C}$*. Then* $\phi(z) = \lambda z$ *with* $|\lambda| \leq 1$.
Proof
By [[11], Theorem 2.2] and [Proposition 3.7], we get the desired result.

References

[1] Carl.C.Cowen and Barbara.D.MacCluer, Composition Operators on Spaces of Analytical Functions. Studies in Advanced Mathematics, CRC Press.

[2] Carlos S. Kubrusly, The Elements of Operator Theory, Birkhauser Publications, Boston.

[3] Carswell B.J, MacCluer B. D, Schuster.A, Composition Operators on the Fock Space, Acta.Sci.Math (Szeged) 69 (2003), 871-887.

[4] J.B. Conway, The theory of Subnormal Operators, American Mathematical Society, Providence, Rhode Island.

[5] Joel.H.Shapiro, Composition Operators and Classical Tunction Theory, Universitext: Tracts in Mathematics, Springer - Verlag, New York, 1993.

[6] John B Conway, The Theory of Subnormal Operators, Amer.Math.Soc. Providence, 1991.

[7] Kehe Zhu, Analysis on Fock Spaces, Graduate Text in Mathematics, Springer, New York.

[8] Liankuo Zhao, Unitary Weighted Composition Operators on the Fock Space of $\mathbb{C}^n$, Complex.Anal.Oper.Theory, 8 (2014), No.2, 581-590.

[9] Liankuo Zhao, Isometric Weighted Composition Operators on the Fock Space of $\mathbb{C}^n$, Bull.Korean.Math.Soc, 53 (2016), No.6, 1785-1794.

[10] Liankuo Zhao, Invertible Weighted Composition Operators on the Fock Space of $\mathbb{C}^n$, Hindawi Publicating Corporation, Journal of Function Spaces, Vol.2015, Article ID 250358.

[11] T.Le, Normal and Isometric Weighted Composition Operators on the Fock Space, Bull.Lond.Math.Soc, 46 (2014), No.4, 847-856.