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On compactness and connectedness in redefined soft topological spaces

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Abstract

The motivation of this paper is to develop different notions in redefined soft topological spaces. we introduce the notions of soft interior operator, soft closure operator, soft dense set, soft connectedness, locally soft compactness and locally soft connectedness in redefined soft topological spaces and discussed some of their properties.

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1 Introduction

Molodtsov felt the inadequacy of parameterization in fuzzy set theory and proposed the theory of soft sets for first time in 1999. Later in 2002 some operations are defined on soft sets. Since then much works have been done in developing mathematical structures in this setting ([1], [4], [7], [13], [6], [8]). This theory is very application friendly in different fields. In 2011 topological structure was introduced in this setting

by Shabir and Naz [15]. Later on Nazmul and Samanta [12] studied some properties on this topological spaces. Since then many authors defined soft topology in different manners. As a continuation of this process very recently in 2017, Chiney and Samanta [2] redefined soft topology using elementary union and elementary intersection and elementary complement, though these operations are not distributive and do not obey the excluded middle law.
In the present paper we introduce notions of interior and closure operators, subspace topology, dense set, locally compactness, connectedness, locally connectedness and related results are developed in Chiney and Samanta - type space. Different sections of this paper are as follows :
Section 2 contains the preliminary definitions and results. In section 3, we introduce soft interior and soft closure operators and soft topology induced by soft function. In section 4, we deal with a different type of subspace. In section 5, the notion of soft dense set is introduced. In subsequent sections some properties on soft compactness, locally soft compactness, soft connectedness and locally soft connectedness are discussed in details.

2 Preliminaries

Definition 2.1. *[11] Let X be a universal set and E be a set of parameters. Let $P(X)$ denotes the power set of X and A be a subset of E. A pair (F,A) is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $\alpha \in A$, $F(\alpha)$ may be considered as the set of α approximate elements of the soft set (F,A).*
In [8] the soft sets are redefined as follows:
Let E be the set of parameters and $A \subseteq E$. Then for each soft set (F,A) over X, a soft set (H,E) is constructed over X, $\forall \alpha \in E$,

$$H(\alpha) = \begin{cases} F(\alpha) & if\ \alpha \in A \\ \phi & if \alpha \in E \backslash A. \end{cases}$$

Thus the soft set (F,A) and (H,E) are equivalent to each other and the usual set operations of the soft sets (F_i,A), $i \in \Delta$ is the same as those of the soft sets (H_i,E), $i \in \Delta$. For this reason, in this paper, we have considered our soft sets over same parameter set A.

Following Molodtsov and Maji et al. ([9], [10], [11]) definition of soft subset, absolute soft set, null soft set, arbitrary union and arbitrary intersection of soft sets etc. are presented in [12] considering the same parameter set. Throughout this paper we use a fixed parameter set A and the soft set (F,A) of a universe X over the parameter set A will be denoted by F.

Definition 2.2. *[12] For two soft sets F and G over a common universe X.*

(a) F is said to be a soft subset of G if $F(\alpha)$ is a subset of $G(\alpha)\,\forall\alpha \in A$.

(b) Two soft sets F and G over a common universe X are said to be soft equal if F is a soft subset of G and G is a soft subset of F.

(c) The complement or relative complement of a soft set F is denoted by F^C and is defined by $F^C(\alpha) = X \setminus F(\alpha), \forall\alpha \in A$.

(d) (Null soft set) F over X is said to be a null soft set if $F(\alpha) = \phi, \forall\alpha \in A$ and it is denoted by $(\tilde{\Phi},A)$.

(e) (Absolute soft set) F over X is said to be an absolute soft set if $F(\alpha) = X, \forall\alpha \in A$.

(f) Union of two soft set F and G is denoted by $F\tilde{\cup}G$ and defined by $(F\tilde{\cup}G)(\alpha) = F(\alpha) \cup G(\alpha), \forall\alpha \in A$.

(g) Intersection of two soft sets F and G is denoted by $F\tilde{\cap}G$ and is defined by $(F\tilde{\cap}G)(\alpha) = F(\alpha) \cap G(\alpha), \forall\alpha \in A$.

Definition 2.3. *F over X is said to be a constant soft set if $F(\alpha) = Y, \forall\alpha \in A$, where, $Y \subset X$. We denote this by $(\tilde{Y},A)$.*

Definition 2.4. *[3] Let X be a non-empty set and A be a non-empty parameter set. Then a function, $\tilde{x} : A \to X$ is said to be a soft element of X. A soft element $\tilde{x}$ of X is said to belong to a soft set F over X, which is denoted by $\tilde{x}\tilde{\in}F$, if $\tilde{x}(\lambda) \in F(\lambda), \forall\lambda \in A$. Thus for a soft set F over X with respect to the index set A with $F(\lambda) \neq \phi, \forall\lambda \in A$, we have $F(\lambda) = \{\tilde{x}(\lambda) : \tilde{x}\tilde{\in}F\}$, for all $\lambda \in A$.*
In rest of the paper we shall call it just an element.

Let X be an initial universal set and A be a non-empty parameter set. Throughout the paper we consider the null soft set $(\tilde{\Phi},A)$ and those soft *sets F over X for which $F(\alpha) \neq \phi, \forall \alpha \in A$. We denote this collection by $S(\tilde{X})$. Thus for all $F[\neq (\tilde{\Phi}, A)] \in S(\tilde{X}), F(\alpha) \neq \phi, \forall \alpha \in A$. The soft set constructed from a collection $\mathscr{B}$ will be denoted by $SS(\mathscr{B})$. For any soft set $F \in S(\tilde{X})$, the collection of all soft element of F is denoted by $SE(F)$.*

Definition 2.5. *[3] Let $\mathbb{R}$ be the set of real numbers and $\mathfrak{B}(\mathbb{R})$ be the collection of all non-empty bounded subset of $\mathbb{R}$ and A be a set of parameters. Then a mapping $F : A \to \mathfrak{B}(\mathbb{R})$ is called a soft real set. If specifically F is a singleton set, then after identifying F with the corresponding soft element, it will be called a soft real number.*

$\tilde{r}, \tilde{s}, \tilde{t}$ denotes soft real numbers whereas $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(\lambda) = r, \forall \lambda \in A$.

Definition 2.6. *[5] For any two soft sets $F, G \in S(\tilde{X})$,*

(a) elementary union of F and G is denoted by $F \Cup G$ and is defined by $F \Cup G = SS(\mathscr{B})$, where, $\mathscr{B} = \{\tilde{x} \tilde{\in} (\tilde{X}, A) : \tilde{x} \tilde{\in} F$ or $\tilde{x} \tilde{\in} G\}$; i.e. $F \Cup G = SS(SE(F) \cup SE(G))$.

(b) elementary intersection of F and G is denoted by $F \Cap G$ and is defined by $F \Cap G = SS(\mathscr{B})$, where, $\mathscr{B} = \{\tilde{x} \tilde{\in} (\tilde{X}, A) : \tilde{x} \tilde{\in} F$ and $\tilde{x} \tilde{\in} G\}$i.e. $F \Cap G = SS(SE(F) \cap SE(G))$.

Definition 2.7. *[5] For any soft set $F \in S(\tilde{X})$, the elementary complement of F is denoted by $F^{\mathbb{C}}$ and is defined by $F^{\mathbb{C}} = SS(\mathscr{B})$, where, $\mathscr{B} = \{\tilde{x} \tilde{\in} (\tilde{X}, A) : \tilde{x} \tilde{\in} F^C\}$ and F^C is the complement of F.*

Proposition 2.8. *[5] For any two soft sets $F, G \in S(\tilde{X})$.*

(a) $F \Cup G = F \tilde{\cup} G$.

(b) $F \Cap G = F \tilde{\cap} G$ if $F \Cap G \neq (\tilde{\Phi}, A)$.

Proposition 2.9. *[5] For any soft set $F \in S(\tilde{X})$.*

(a) $F \Cap F^{\mathbb{C}}$=$(\tilde{\Phi}, A)$.

(b) In general $F \Cup F^{\mathbb{C}} \tilde{\subseteq} (\tilde{X}, A)$ but if $F^{\mathbb{C}} \neq (\tilde{\Phi}, A)$, then $F \Cup F^{\mathbb{C}} = (\tilde{X}, A)$.

(c) In general$(F^{\mathbb{C}})^{\mathbb{C}} \neq F$. If $F^{\mathbb{C}} \neq (\tilde{\Phi}, A)$, then $(F^{\mathbb{C}})^{\mathbb{C}} = F$.

Definition 2.10. *[2] Let τ be a collection of soft sets of $S(\tilde{X})$. Then τ is said to be a soft topology on $(\tilde{X},A)$ if*

(i) $(\tilde{\Phi},A)$ *and* $(\tilde{X},A)$ *belong to* τ.

(ii) the elementary intersection of any two soft sets of τ *belong to* τ.

(iii) the elementary union of any number of soft sets of τ *belong to* τ .

The triplet $(\tilde{X},\tau,A)$ is called a soft topological space.
In the rest of the paper by saying X it is understood that $(\tilde{X},\tau,A)$ is a soft topological space and a set F will denote a soft set (F,A) over X.

Definition 2.11. *[2] In X a set $F\in S(\tilde{X})$ is called a closed set in X if its relative complement $F^C \in S(\tilde{X})$ and $F^{\mathbb{C}} \in \tau$.*

Definition 2.12. *[2] Let $F\in S(\tilde{X})$ in X. Then the closure of F, is denoted by $\overline{F}$ is defined as the elementary intersection of all closed super sets of F.*

Definition 2.13. *[2] In X $F[\neq(\tilde{\Phi},A)]\in S(\tilde{X})$ is said to be a neighbourhood (nbd) of an element $\tilde{x}$ if there exists an open set G such that $\tilde{x}\tilde{\in}G\tilde{\subseteq}F$. The nbd system at a element $\tilde{x}$, denoted by $\aleph_\tau(\tilde{x})$, is the family of all its nbds.*

Proposition 2.14. *[14] Let X be a space and $Y\subset X$. Let $\tau_{\tilde{Y}} = \{(\tilde{Y},A)\Cap G; G\in\tau\}$. Then $(\tilde{Y},\tau_{\tilde{Y}},A)$ is a subspace of X.*

Definition 2.15. *[14] Let X be a space and $(\tilde{Y},\tau_{\tilde{Y}},A)$ be a subspace of X. A set $F\in S(\tilde{Y})$ is said to be closed in the subspace $(\tilde{Y},\tau_{\tilde{Y}},A)$, if $F^{\mathbb{C}}|_{\tilde{Y}}\in\tau_{\tilde{Y}}$ and $F^C|_{\tilde{Y}}\in S(\tilde{Y})$.*

Proposition 2.16. *[14] Let X be a space and $(\tilde{Y},\tau_{\tilde{Y}},A)$ be a subspace of X. Then for any closed set H in $(\tilde{Y},\tau_{\tilde{Y}},A)$ there exists a closed set F in X such that $H=(\tilde{Y},A)\Cap F$.*

Definition 2.17. *[14] Let X be a space. Let $F[\neq\tilde{\Phi},A)]\in S(\tilde{X})$. A collection of sets $\mathscr{B}=\{G_i\in S(\tilde{X}), i\in\Delta\}$ is said to be a cover of F if $F\tilde{\subset}\Cup_{i\in\Delta} G_i$. If G_i is open in a space X $\forall i\in\Delta$, then $\mathscr{B}$ is said to be a open cover of F.*

Definition 2.18. *[14] Let X be a space. A set $F\in S(\tilde{X})$ is said to be compact in X if any open cover of F has a finite subcover.*

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Proposition 2.19. *[14] Soft continuous image of a compact set is compact.*

Proposition 2.20. *[14] Let X be a T_2 space and F be a compact set such that $F^{\mathbb{C}} \neq (\tilde{\Phi},A)$, then F is closed in X.*

Some other preliminary results of redefined soft topological spaces that has been further used in this paper is taken from [2]

3 Soft interior and soft closure operators

Proposition 3.1. *Let X be a universal set and A be a non empty-set of parameters. Let $\tilde{\Omega} : S(\tilde{X}) \to S(\tilde{X})$ be a soft function such that*

(i) $\tilde{\Omega}[(\tilde{X},A)] = (\tilde{X},A)$.

(ii) $\tilde{\Omega}(F)\tilde{\subset}F$.

(iii) $\tilde{\Omega}(F \Cap G) = \tilde{\Omega}(F) \Cap \tilde{\Omega}(G)$.

(iv) $\tilde{\Omega}[\tilde{\Omega}(F)] = \tilde{\Omega}(F), \forall\, F, G \in S(\tilde{X})$.

Let $\tau = \{G \in S(\tilde{X}) : \tilde{\Omega}(G) = G\}$. Then τ is a soft topology on $(\tilde{X},A)$ and $\tilde{\Omega}(G) = Int G\ \forall\ G \in S(\tilde{X})$.

Proof. $\tilde{\Omega}[(\tilde{X},A)] = (\tilde{X},A)$. So $(\tilde{X},A) \in \tau$. $\tilde{\Omega}[(\tilde{\Phi},A)]\tilde{\subset}(\tilde{\Phi},A)$.
So $\tilde{\Omega}[(\tilde{\Phi},A)]$
$= (\tilde{\Phi},A)$. Hence $(\tilde{\Phi},A) \in \tau$.
Let $G_1, G_2 \in \tau$. Then $\tilde{\Omega}(G_1) = G_1$ and $\tilde{\Omega}(G_2) = G_2$. Now $\tilde{\Omega}(G_1) \Cap \tilde{\Omega}(G_2) = G_1 \Cap G_2 \Rightarrow \tilde{\Omega}(G_1 \Cap G_2) = G_1 \Cap G_2$. Therefore $G_1 \Cap G_2 \in \tau$.
Let $G_\alpha \in \tau\ \forall \alpha \in \Delta$. Then $\tilde{\Omega}(G_\alpha) = G_\alpha, \forall\ \alpha \in \Delta$. Now $G_\alpha \tilde{\subset} \underset{\alpha\in\Delta}{\Cup} G_\alpha \forall\ \alpha \in \Delta \Rightarrow \tilde{\Omega}\ (G_\alpha)\tilde{\subset}\ \tilde{\Omega}(\underset{\alpha\in\Delta}{\Cup} G_\alpha) \forall \alpha \in \Delta \Rightarrow G_\alpha \tilde{\subset}\ \tilde{\Omega}\ (\underset{\alpha\in\Delta}{\Cup} G_\alpha) \forall \alpha \in \Delta \Rightarrow \underset{\alpha\in\Delta}{\Cup} G_\alpha \tilde{\subset} \tilde{\Omega}(\underset{\alpha\in\Delta}{\Cup} G_\alpha)$. Again, $\tilde{\Omega}(\underset{\alpha\in\Delta}{\Cup} G_\alpha)\tilde{\subset} \underset{\alpha\in\Delta}{\Cup} G_\alpha$. Therefore, $\tilde{\Omega}(\underset{\alpha\in\Delta}{\Cup} G_\alpha) = \underset{\alpha\in\Delta}{\Cup} G_\alpha$. So $\underset{\alpha\in\Delta}{\Cup} G_\alpha \in \tau$. Hence τ is a soft topology on $(\tilde{X},A)$.

Again by definition of τ, $\tilde{\Omega}[\tilde{\Omega}(G)] = \tilde{\Omega}(G)$. So $\tilde{\Omega}(G) \in \tau$. Again by (ii) $\tilde{\Omega}(G)\tilde{\subset}\ G$. Then $\tilde{\Omega}(G)$ is a open set contained in G. Let F be another open set such that $F\ \tilde{\subset}G$. Then $\tilde{\Omega}(F)\ \tilde{\subset}\tilde{\Omega}(G) \Rightarrow F\ \tilde{\subset}\tilde{\Omega}(G)$. So $\tilde{\Omega}(G)$ is the largest open set contained in G. Therefore $\tilde{\Omega}(G) = Int\ G$.

The soft function defined as above is called the soft interior operator on $(\tilde{X},A)$. □

Proposition 3.2. *Let X be an universal set and A be a non-empty set of parameters. Let $\tilde{\Gamma}: S(\tilde{X}) \to S(\tilde{X})$ be a soft function such that*

(i) $\tilde{\Gamma}[(\tilde{\Phi},A)] = (\tilde{\Phi},A)$

(ii) $(F,A)\tilde{\subset}\tilde{\Gamma}[(F,A)]$

(iii) $\tilde{\Gamma}(F \Cup G) = \tilde{\Gamma}(F) \Cup \tilde{\Gamma}(G)$

(iv) $\tilde{\Gamma}[\tilde{\Gamma}(F)] = \tilde{\Gamma}(F), \forall F,G \in S(\tilde{X})$

Let $\tau = \{G \in S(\tilde{X}) : \tilde{\Gamma}(G^{\mathbb{C}}) = G^{\mathbb{C}}\}$. Then τ is a soft topology on $(\tilde{X},A)$.

Proof. $\tilde{\Gamma}[(\tilde{\Phi},A)] = (\tilde{\Phi},A)$ by $(i) \Rightarrow \tilde{\Gamma}[(\tilde{X},A)^{\mathbb{C}}] = (\tilde{X},A)^{\mathbb{C}}$. So $(\tilde{X},A) \in \tau$. Again $(\tilde{X},A))\tilde{\subset}$
$\tilde{\Gamma}[(\tilde{X},A)]$ by $(ii) \Rightarrow (\tilde{X},A) = \tilde{\Gamma}[(\tilde{X},A)] \Rightarrow (\tilde{\Phi},A)^{\mathbb{C}} = \tilde{\Gamma}[(\tilde{\Phi},A)^{\mathbb{C}}]$. Therefore $(\tilde{\Phi},A) \in \tau$.
Let $G_1,G_2 \in \tau$. Then $\tilde{\Gamma}(G_1^{\mathbb{C}}) = G_1^{\mathbb{C}}$ and $\tilde{\Gamma}(G_2^{\mathbb{C}}) = G_2^{\mathbb{C}}$. Then $\tilde{\Gamma}(G_1^{\mathbb{C}}) \Cup \tilde{\Gamma}(G_2^{\mathbb{C}}) = G_1^{\mathbb{C}} \Cup G_2^{\mathbb{C}} \Rightarrow \tilde{\Gamma}(G_1^{\mathbb{C}} \Cup G_2^{\mathbb{C}}) = G_1^{\mathbb{C}} \Cup G_2^{\mathbb{C}} \Rightarrow \tilde{\Gamma}(G_1 \Cap G_2)^{\mathbb{C}} = (G_1 \Cap G_2)^{\mathbb{C}}$. Therefore $G_1 \Cap G_2 \in \tau$.

Let $G_i \in \tau\, \forall i \in \Delta$. Then $\tilde{\Gamma}(G_i)^{\mathbb{C}} = G_i^{\mathbb{C}}, \forall i \in \Delta$. We have $\underset{i\in\Delta}{\Cap} G_i^{\mathbb{C}} \tilde{\subset} G_i^{\mathbb{C}}$, $\forall i \in \Delta \Rightarrow \tilde{\Gamma}(\underset{i\in\Delta}{\Cap} G_i^{\mathbb{C}})\tilde{\subset}\tilde{\Gamma}(G_i^{\mathbb{C}})$, $\forall i \in \Delta \Rightarrow \tilde{\Gamma}(\underset{i\in\Delta}{\Cap} G_i^{\mathbb{C}})\tilde{\subset} G_i^{\mathbb{C}}$, $\forall i \in \Delta \Rightarrow$
$\tilde{\Gamma}(\underset{i\in\Delta}{\Cap} G_i^{\mathbb{C}})\tilde{\subset} \underset{i\in\Delta}{\Cap} G_i^{\mathbb{C}} \Rightarrow \tilde{\Gamma}[(\underset{i\in\Delta}{\Cup} G_i)^{\mathbb{C}}]\tilde{\subset}(\underset{i\in\Delta}{\Cup} G_i)^{\mathbb{C}}$ Again by (ii) $(\underset{i\in\Delta}{\Cup} G_i)^{\mathbb{C}}\tilde{\subset}\tilde{\Gamma}$ $[(\underset{i\in\Delta}{\Cup} G_i)^{\mathbb{C}}]$ So $\tilde{\Gamma}[(\underset{i\in\Delta}{\Cup} G_i)^{\mathbb{C}}] = (\underset{i\in\Delta}{\Cup} G_i)^{\mathbb{C}}$ Therefore $\underset{i\in\Delta}{\Cup} G_i \in \tau$. Hence τ is a soft topology on $(\tilde{X},A)$. The soft function $\tilde{\Gamma}$ defined as above is called the soft closure operator on $(\tilde{X},A)$.

□

Remark 3.3. *In general $\tilde{\Gamma}(F)$ may not be equal to $\overline{F}$.*

Proposition 3.4. *Let $(\tilde{X},\tau,A)$ and $(\tilde{Y},\nu,A)$ be two spaces and $\mathscr{B}$ be a s-base for τ. Let $f : (\tilde{X},\tau,A) \to (\tilde{Y},\nu,A)$ be a soft function such that $\{f(B);B \in \mathscr{B}\}$ is a base for ν, then f is a open map.*

Proof. Let G be a open set in X. Since $\mathscr{B}$ is a base for τ , $G = \underset{i\in\Delta}{\Cup} B_i\, ;B_i \in \mathscr{B}, \forall\, i \in \Delta$. Now $f(G) = f(\underset{i\in\Delta}{\Cup} B_i) = \underset{i\in\Delta}{\Cup} f(B_i)$. Since $\{f(B);B \in$

$\mathscr{B}\}$ is a base for ν, then $f(B_i) \in \nu, \forall i \in \Delta$. Therefore $\Cup_{i\in\Delta} f(B_i) \in \nu$. Hence $f(G) \in \nu$. So f is a open map. $\square$

Proposition 3.5. *Let $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, A)$ be a soft function from a space $(\tilde{X}, \tau, A)$ to a soft set $(\tilde{Y}, A)$. Define $\nu = \{G \in S(\tilde{Y})\ ; f^{-1}(G) \in \tau\}$. Then*

(i) ν is a soft topology on $(\tilde{Y}, A)$.

(ii) $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, \nu, A)$ is a soft continuous function.

(iii) ν is the largest soft topology on $(\tilde{Y}, A)$ such that $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, \nu, A)$ is a soft continuous function.

Proof. $(i) f^{-1}[(\tilde{\Phi}, A)] = (\tilde{\Phi}, A) \in \tau$. $f^{-1}[(\tilde{Y}, A)] = (\tilde{X}, A) \in \tau$.
Let $G_1, G_2 \in \nu$. Then $f^{-1}(G_1) \in \tau$ and $f^{-1}(G_2) \in \tau$. Therefore $f^{-1}(G_1 \Cap G_2) = f^{-1}(G_1) \Cap f^{-1}(G_2) \in \tau$. So $G_1 \Cap G_2 \in \nu$.
Let $G_i \in \nu\ \ \forall i \in \Delta$. Then $f^{-1}(G_i) \in \tau \forall i \in \Delta \Rightarrow \Cup_{i\in\Delta} f^{-1}(G_i) \in \tau \Rightarrow f^{-1}(\Cup_{i\in\Delta} G_i) \in \tau$. Therefore $\Cup_{i\in\Delta} G_i \in \nu$. Therefore $(\tilde{Y}, \nu, A)$ is a space.

(ii) By construction of τ, f is soft continuous.

(iii) Let ν^* be a soft topology on $(\tilde{Y}, A)$, so that $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, \nu^*, A)$ is soft continuous. Let $G \in \nu^*$. Since f is soft continuous $f^{-1}(G) \in \tau$. So $G \in \nu$. Hence $\nu^* \subseteq \nu$. Therefore ν is largest soft topology on $(\tilde{Y}, A)$ so that $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, \nu, A)$ is soft continuous. $\square$

4 Soft subspace

Proposition 4.1. *Let X be a space and $Y[\neq (\tilde{\Phi}, A)] \in S(\tilde{X})$. Let $\tau_Y = \{Y \Cap G; G \in \tau\}$. Then (Y, τ_Y, A) is a soft subspace of $(\tilde{X}, \tau, A)$.*

Remark 4.2. *The soft subspace defined in 4.1 is called a subspace of second type which is different from the subspace defined in Definition 2.14. By saying subspace we shall mean a soft subspace defined in 2.14.*

Proposition 4.3. *Let X be a space and $\mathscr{B}$ be a base for τ. Let $(\tilde{Y}, \tau_{\tilde{Y}}, A)$ be a subspace of $(\tilde{X}, \tau, A)$. Then $\mathscr{B}^* = \{(\tilde{Y}, A) \Cap B; B \in \mathscr{B}\}$ is a base for $(\tilde{Y}, \tau_{\tilde{Y}}, A)$.*

Proof. Since $\mathscr{B}$ is a base for τ, $\mathscr{B} \subset \tau$. So $\mathscr{B}^* \subset \tau_{\tilde{Y}}$. Let $\tilde{x}\tilde{\in}(\tilde{Y},A)$ and $\tilde{x}\tilde{\in}G' \in \tau_{\tilde{Y}}$. Then $\exists\ G \in \tau$ such that $G' = (\tilde{Y},A) \Cap G$. Therefore $\tilde{x}\tilde{\in}(\tilde{Y},A) \Cap G$. So $\tilde{x}\tilde{\in}(\tilde{Y},A)$ and $\tilde{x}\tilde{\in}G$. Since $\mathscr{B}$ is a base for τ, $\exists\ B \in \mathscr{B}$ such that $\tilde{x}\tilde{\in}B\tilde{\subset}G$. Then $\tilde{x}\tilde{\in}(\tilde{Y},A) \Cap B$ and $(\tilde{Y},A) \Cap B \in \mathscr{B}^*$. Therefore $\tilde{x}\tilde{\in}(\tilde{Y},A) \Cap B \tilde{\subset}(\tilde{Y},A) \Cap G = G'$. Hence $\mathscr{B}^*$ is a base for $(\tilde{Y},\tau_{\tilde{Y}},A)$. □

Definition 4.4. *Let X be a space and $(Y,\tau_Y,A))$ is a subspace of X of second type. Then a set $F\tilde{\subset}Y$ is said to be closed in the subspace $(Y,\tau_Y,A))$ if $F^{\mathbb{C}}|_Y \in \tau_Y$ and $F^C|_Y \in S(\tilde{Y})$.*

Proposition 4.5. *Let X be a space and $Z \subset Y \subset X$. Then,*

(i) $\tau|_{\tilde{Z}} = (\tau|_{\tilde{Y}})|_{\tilde{Z}}$.

(ii) If $G\tilde{\subset}(\tilde{Y},A)\tilde{\subset}(\tilde{X},A)$ and $G \in \tau$, then $G \in \tau_{\tilde{Y}}$.

(iii)If $F\tilde{\subset}(\tilde{Y},A)\tilde{\subset}(\tilde{X},A)$, then $\overline{F}^{\tau_{\tilde{Y}}} = (\tilde{Y},A) \Cap \overline{F}^{\tau}$.

Proof. (i) Let $F[\neq (\tilde{\Phi},A)]\tilde{\in}(\tau|_{\tilde{Y}})|_{\tilde{Z}} \Leftrightarrow F = (\tilde{Z},A) \Cap G_1\,; G_1 \in \tau_{\tilde{Y}} \Leftrightarrow F = (\tilde{Z},A) \Cap [\,(\tilde{Y},A) \Cap G]$ where $G \in \tau \Leftrightarrow F = (\tilde{Z},A)\tilde{\cap}[(\tilde{Y},A)\tilde{\cap}G]$, $\Leftrightarrow F = [\,(\tilde{Z},A)\tilde{\cap}[(\tilde{Y},A)\,]\tilde{\cap}G \Leftrightarrow F = (\tilde{Z},A)\tilde{\cap}G \Leftrightarrow F = (\tilde{Z},A) \Cap G$. Therefore , $\tau|_{\tilde{Z}} = (\tau|_{\tilde{Y}})|_{\tilde{Z}}$.

(ii) Let $G\tilde{\subset}(\tilde{Y},A)\tilde{\subset}(\tilde{X},A)$ and $G \in \tau$. Then $(\tilde{Y},A) \Cap G\tilde{\in}\tau_{\tilde{Y}}$. So $G \in \tau_{\tilde{Y}}$.

(iii) Let $F\tilde{\subset}(\tilde{Y},A)\tilde{\subset}(\tilde{X},A)$. Now $\overline{F}^{\tau_{\tilde{Y}}}$ is the smallest closed set in $(\tilde{Y},\tau_{\tilde{Y}},A)$ which contains F. Again $\overline{F}^{\tau}$ is a closed set in $(\tilde{X},\tau,A)$. So $(\tilde{Y},A) \Cap \overline{F}^{\tau}$ is a closed set in $(\tilde{Y},\tau_{\tilde{Y}},A)$ containing F. Therefore, $\overline{F}^{\tau_{\tilde{Y}}}\tilde{\subset}(\tilde{Y},A) \Cap \overline{F}^{\tau}$........................(1).
Again , $\overline{F}^{\tau_{\tilde{Y}}}$ is a closed set in $(\tilde{Y},\tau_{\tilde{Y}},A)$ containing F. Then $\exists$ a s-closed set H in $(\tilde{X},\tau,A)$ such that $\overline{F}^{\tau_{\tilde{Y}}} = (\tilde{Y},A) \Cap H \tilde{\supset}(\tilde{Y},A) \Cap \overline{F}^{\tau}$................(2)
From (1) and (2), we have $\overline{F}^{\tau_{\tilde{Y}}} = (\tilde{Y},A) \Cap \overline{F}^{\tau}$. □

Remark 4.6. *The converse of (ii) in Proposition 4.5 may not be true. To show let $X = \{x,y,z,t\}$, $Y = \{x,y,z\}$ and $A = \{\alpha,\beta\}$. Let $\tau = \{(\tilde{\Phi},A),(\tilde{X},A),F_1,F_2,F_3,F_4,F_5,F_6,F_7\}$, where, $F_1(\alpha) = \{x\}, F_1(\beta) = \{t\}; F_2(\alpha) = \{y\}, F_2(\beta) = \{z\}; F_3(\alpha) = \{t\}, F_3(\beta) = \{x\}; F_4(\alpha) = \{x,y\}$, $F_4(\beta) = \{z,t\}; F_5(\alpha) = \{y,t\}, F_5(\beta) = \{x,z\}; F_6(\alpha) = \{x,t\}, F_6(\beta) = \{x,t\}; F_7(\alpha) = \{x,y,t\}, F_7(\beta) = \{x,z,t\}$. Then X is a space.*
Then, $\tau_{\tilde{Y}} = \{(\tilde{Y},A) \Cap F_i; F_i \in \tau, i = 1,2,.....,7\} \Rightarrow \tau_{\tilde{Y}} = \{(\tilde{\Phi},A),(\tilde{Y},A),G_2$, $G_4,G_5,G_6,G_7\}$, where, $G_2(\alpha) = \{y\}, G_2(\beta) = \{z\}; G_4(\alpha) = \{x,y\}$, $G_4(\beta) = \{z\}; G_5(\alpha) = \{y\}, G_5(\beta) = \{x,z\}; G_6(\alpha) = \{x\}, G_6(\beta) = \{x\}$

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$;G_7(\alpha) = \{x,y\}, G_7(\beta) = \{x,z\}$. *Then* $(\tilde{Y}, \tau_{\tilde{Y}}, A)$ *is a subspace of* X. *Now* $G_4 \tilde{\subset} (\tilde{Y}, A) \tilde{\subset} (\tilde{X}, A) \in \tau_{\tilde{Y}}$ *but* $G_4 \notin \tau$.

Proposition 4.7. *Let* X *be a space and* (Y, τ_Y, A) *is a subspace of* $(\tilde{X}, \tau, A)$ *of second type. Then*

(i) *For any closed set* F *in* (Y, τ_Y, A) , $\exists$ *a closed set* H *in* $(\tilde{X}, \tau, A)$*, such that* $F = Y \Cap H$.

(ii) *Let* $Z, Y[\neq (\tilde{\Phi}, A)] \in S(\tilde{X})$ *such that* $Z \tilde{\subset} Y$*, then* $\tau|_Z = (\tau|_Y)|_Z$.

Proof. (i) Let $F \neq (\tilde{\Phi}, A)$ be a closed set in (Y, τ_Y, A). Then $F^{\mathbb{C}}|_Y \in \tau_Y$. So $\exists\ G \in \tau$ such that $F^{\mathbb{C}}|_Y = Y \Cap G \Rightarrow (F^{\mathbb{C}}|_Y)^{\mathbb{C}}|_Y = (Y \Cap G)^{\mathbb{C}}|_Y$. Since F is closed in τ_Y so $F^C|_Y \in S(\tilde{Y})$, so $Y \Cap G \neq (\tilde{\Phi}, A)$. Therefore $F = G^{\mathbb{C}}|_Y = G^{\mathbb{C}}|_{\tilde{X}} \Cap Y$. Let $H = G^{\mathbb{C}}|_{\tilde{X}}$. Then H is closed in X and $F = Y \Cap H$.

(ii) Proof is similar as of (i) in Proposition 4.5.

□

5 Soft dense set

Definition 5.1. *Let* X *be a space. A set* $F[\neq (\tilde{\Phi}, A)] \in S(\tilde{X})$ *is said to be dense in* $(\tilde{X}, A)$ *if* $\overline{F} = (\tilde{X}, A)$.
For the rest of the paper a soft dense set will be called a dense set.

Proposition 5.2. *Let* X *be a space. A set* $F \in S(\tilde{X})$ *is dense in* $(\tilde{X}, A)$ *iff* $F \tilde{\cap} G \neq (\tilde{\Phi}, A) \forall G[\neq (\tilde{\Phi}, A)] \in \tau$.

Proof. Let $F \in S(\tilde{X})$ is dense in $(\tilde{X}, A)$. Then $\overline{F} = (\tilde{X}, A)$. Let $G \neq (\tilde{\Phi}, A)$ be any open set. If possible let $F \tilde{\cap} G = (\tilde{\Phi}, A)$. Then $F \tilde{\subset} G^{\mathbb{C}}$. Since $F \in S(\tilde{X})$ and $F \tilde{\subset} G^{\mathbb{C}}$, so $G^{\mathbb{C}} \neq (\tilde{\Phi}, A)$. Therefore $G^{\mathbb{C}}$ is a closed set containing F, which contradicts $\overline{F} = (\tilde{X}, A)$. Hence $F \tilde{\cap} G \neq (\tilde{\Phi}, A) \forall$ non-empty$G \in \tau$.

Conversely suppose $F \tilde{\cap} G \neq (\tilde{\Phi}, A) \forall G[\neq (\tilde{\Phi}, A)] \in \tau$. If possible let $\overline{F} \neq (\tilde{X}, A)$. Then $\overline{F}^{\mathbb{C}}[\neq (\tilde{\Phi}, A)] \in \tau$. Now we have $F \tilde{\subset} \overline{F}$ so $\overline{F}^{\mathbb{C}} \tilde{\subset} F^{\mathbb{C}}$. Since $F \tilde{\cap} F^{\mathbb{C}} = (\tilde{\Phi}, A)$ then $F \tilde{\cap} \overline{F}^{\mathbb{C}} = (\tilde{\Phi}, A)$, which contradicts $F \tilde{\cap} G \neq (\tilde{\Phi}, A) \forall G[\neq (\tilde{\Phi}, A)] \in \tau$. Therefore $\overline{F} = (\tilde{X}, A)$ □

Remark 5.3. *The following example shows that the of the Proposition 5.2. is not true for elementary intersection.*

Example 5.4. *Let $X = \{x,y,z\}$ and $A = \{\alpha,\beta\}$. Let $\tau = \{(\tilde{\Phi},A)$, $(\tilde{X},A),G\}$ where $G(\alpha) = \{x\}, G(\beta) = \{y,z\}$. Then X is a space. Let $F \in S(\tilde{X})$ be a set such that $F(\alpha) = \{x\}, F(\beta) = \{x\}$. Then F is dense in $(\tilde{X},A)$, but $F \Cap G = (\tilde{\Phi},A)$*

Proposition 5.5. *Let X be a space and $F \in S(\tilde{X})$. If $\overline{F}^w = (\tilde{X},A)$, then $F \tilde{\cap} G \neq (\tilde{\Phi},A)$, $\forall G[\neq (\tilde{\Phi},A)] \in \tau$.*

Proof. Let $G \neq (\tilde{\Phi},A)$ be any open set. Let $\tilde{x}\tilde{\in}G$, then $\tilde{x}\tilde{\in}(\tilde{X},A)$. Then $\tilde{x}\tilde{\in}\overline{F}^w = F \Cup F'$. Now if $\tilde{x}\tilde{\in}F$, then $F\tilde{\cap}G \neq (\tilde{\Phi},A)$, since $\tilde{x}\tilde{\in}G$. If $\tilde{x}$ is a limiting soft element of F , then $F(\alpha)\tilde{\cap}[G(\alpha)\backslash\tilde{x}(\alpha)]$
$\neq \phi$. Therefore $F\tilde{\cap}G \neq (\tilde{\Phi},A)$. If $\tilde{x}(\alpha) \in F(\alpha)$, for some $\alpha \in A$ and $\tilde{x}(\beta) \in F^C(\beta)$, for some $\beta \in A$, then $F(\alpha) \cap G(\alpha) \neq \phi$. Therefore $F\tilde{\cap}G \neq (\tilde{\Phi},A)$. □

Remark 5.6. *The converse of the Proposition 5.5 may not be true. This is shown in the following example.*

Example 5.7. *Let $X = \{x,y,z\}$ and $A = \{\alpha,\beta\}$. Let $\tau = \{(\tilde{\Phi},A)$, $(\tilde{X},A),F_1,F_2,F_3\}$. where, $F_1(\alpha) = \{y,z\}, F_1(\beta) = \{x\}; F_2(\alpha) = \{y\}$, $F_2(\beta) = \{y,z\}; F_3(\alpha) = \{y,z\}, F_3(\beta) = \{x,y,z\}$. Then X is a space. Let C be a set such that $C(\alpha) = \{x,z\}, C(\beta) = \{y,z\}$. Then $C\tilde{\cap}F_i \neq (\tilde{\Phi},A), i = 1,2,3$. Now $\tilde{\xi}_1, \tilde{\xi}_2$, where, $\tilde{\xi}_1(\alpha) = \{x\}, \tilde{\xi}_1(\beta) = \{y\}$ and $\tilde{\xi}_2(\alpha) = \{x\}, \tilde{\xi}_2(\beta) = \{z\}$, are only limiting element of C. Then C contains all its limiting element. Therefore $\overline{C}^w = C$.*

6 Soft compactness

In this section we study some properties on soft compactness and by saying a compact space we will mean a soft compact topological space.

Proposition 6.1. *A closed set in a compact space is compact.*

Proof. Let X be a compact space and F be any closed set in X. Let $\mathscr{C} = \{G_i \in \tau, \forall i \in \Delta\}$ be any open cover of F. Since F is closed $F^{\mathbb{C}} \in \tau$ and $F^C \in S(\tilde{X})$. So $F \Cup F^{\mathbb{C}} = (\tilde{X},A)$. Take $\mathscr{B} = \mathscr{C} \Cup F^{\mathbb{C}}$. Then $\mathscr{B}$ is a open cover of $(\tilde{X},A)$. Since X is compact $\exists$ a finite subcover $\mathscr{B}'$ of $\mathscr{B}$, such that $\mathscr{B}'$ also covers $(\tilde{X},A)$. Then $\mathscr{B}'$ also covers F. Now if $\mathscr{B}' = \{G_i,\ i = 1,2,......n\}$, then $\mathscr{B}'$ is a finite subcover of $\mathscr{C}$ which covers F. So in this case F is compact. If $\mathscr{B}' = \{F^{\mathbb{C}}, G_i,\ i = 1,2,......,n\}$, then we omit $F^{\mathbb{C}}$ from $\mathscr{B}'$ which results in a finite subcover of $\mathscr{C}$ covering F. So F is compact. □

Proposition 6.2. *Let X be a compact space . Then any family of closed set having finite intersection property (elementary intersection) has non-empty intersection.*

Proof. Let X be a compact space. Let $\{F_i, i \in \Delta\}$ be a family of closed set having finite intersection property. Since F_i is closed so, $F_i^{\mathbb{C}} \in \tau$ and $F_i^{C} \in S(\tilde{X}) \forall i \in \Delta$. If possible let $\underset{i\in\Delta}{\Cap} F_i = (\tilde{\Phi}, A) \Rightarrow [\underset{i\in\Delta}{\Cap} F_i]^{\mathbb{C}} = (\tilde{X}, A) \Rightarrow \underset{i\in\Delta}{\Cup} F_i^{\mathbb{C}} = (\tilde{X}, A)$.Then $\mathscr{B} = \{F_i^{\mathbb{C}} \in \tau, \forall i \in \Delta\}$ is a open cover of $(\tilde{X}, A)$. Since $(\tilde{X}, \tau, A)$ is compact , so $\mathscr{B}$ has a finite subcollection$\{F_i^{\mathbb{C}},\ i = 1, 2, ...n\}$ such that $(\tilde{X}, \tau, A \tilde{\subset} \overset{n}{\underset{i=1}{\Cup}} F_i^{\mathbb{C}} \Rightarrow [\overset{n}{\underset{i=1}{\Cup}} F_i^{\mathbb{C}}]^{\mathbb{C}} \tilde{\subset} (\tilde{\Phi}, A)$. Therefore $\overset{n}{\underset{i=1}{\Cap}} F_i = (\tilde{\Phi}, A)$, Which contradicts that $\{F_i, i \in \Delta\}$ has finite intersection property. Hence the result. □

The following example shows that the converse of the above Proposition is not true.

Example 6.3. *Consider $\mathbb{R}$, the set of all real number and A be a non-empty set of parameter. Define $F : A \to P(\mathbb{R})$ by $F(\alpha)_{(a,b)} = (a,b) \forall \alpha \in A, a, b \in \mathbb{R}$. Let $\mathscr{B} = \{F_{(a,b)} : a, b \in \mathbb{R}\}$. Let τ be the collection of all possible union of member of $\mathscr{B}$. Then $(\tilde{\mathbb{R}}, \tau, A)$ is a space. Now $\{[-n, n] : n = 1, 2, ...\}$ is a family of closed set having finite intersection property, though $(\tilde{\mathbb{R}}, \tau, A)$ is not compact as $\{F_{(-n,n)}, n = 1, 2, ...\}$ is a open cover of $(\tilde{\mathbb{R}}, A)$ having no finite subcover.*

Proposition 6.4. *Let (Y, τ_Y, A) be subspace of a space X of second type and $F \tilde{\subset} Y$. Then F is compact in X iff F is compact in (Y, τ_Y, A).*

Proof. Let $F \tilde{\subset} Y$ be compact in (Y, τ_Y, A). Let $\mathscr{B} = \{G_i \in \tau, i \in \Delta\}$ be any τ open cover of F. Then $\mathscr{C} = \{G_i \Cap Y, i \in \Delta\}$ is a s- τ_Y open cover of F. Since F is compact in (Y, τ_Y, A), so $F \tilde{\subset} \overset{n}{\underset{i=1}{\Cup}} [G_i \Cap Y]$ for some finite n. Therefore $F \tilde{\subset} \overset{n}{\underset{i=1}{\Cup}} G_i$. So F is τ compact.

Suppose F is τ compact. Let $\mathscr{C} = \{G_i \in \tau_Y, i \in \Delta\}$ be a τ_Y open cover of F. Then for each $i \in \Delta$, $\exists\ H_i \in \tau$ such that $G_i = Y \Cap H_i$. Therefore $F \tilde{\subset} \underset{i\in\Delta}{\Cup} H_i$. So $\{H_i \in \tau, i \in \Delta\}$ is a τ open cover of F. Since F is τ

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compact so $F \tilde{\subset} \overset{n}{\underset{i=1}{\Cup}} H_i$ for some finite n. So $F = F \Cap Y = \tilde{\subset} [\overset{n}{\underset{i=1}{\Cup}} H_i] \Cap Y = \overset{n}{\underset{i=1}{\Cup}} [H_i \Cap Y]$ [since $[\overset{n}{\underset{i=1}{\Cup}} H_i] \Cap Y \neq (\tilde{\Phi}, A)] = \overset{n}{\underset{i=1}{\Cup}} G_i \therefore F$ is τ_Y compact. □

Proposition 6.5. *Let $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, \nu, A)$ be a soft continuous function from a compact space $(\tilde{X}, \tau, A)$ to a T_2 space $(\tilde{Y}, \nu, A)$. Let $F \in S(\tilde{X})$ be a closed set in X. If $f(F)^{\mathbb{C}} \neq (\tilde{\Phi}, A)$, then $f(F)$ is closed in Y.*

Proof. Let X be a compact space and Y be a T_2 space and $f : (\tilde{X}, \tau, A) \to (\tilde{Y}, \nu, A)$ be a soft continuous map. Let F be a closed set in X. Since X is compact so F is compact in X. Since continuous image of a compact set is compact so $f(F)$ is compact in Y. Since Y is T_2 so $f(F)$ is closed in Y. Hence f is a closed map. □

Definition 6.6. *A space X is said to be a strong regular space if for all $\tilde{x} \in SE(\tilde{X})$ and for all open set U such that $\tilde{x} \tilde{\in} U, \exists V \in \tau$ such that $\tilde{x} \tilde{\in} V \tilde{\subseteq} \overline{V} \tilde{\subseteq} U$.*

Proposition 6.7. *A strong regular space is regular but not conversely.*

Example 6.8. *Let $X = \{a, b\}, A = \{\alpha, \beta\}$ and $\tau = \{(\tilde{\Phi}, A), (\tilde{X}, A), F_1, F_2, F_3, F_4\}$, where, $F_1(\alpha) = \{a, b\}, F_1(\beta) = \{a\}; F_2(\alpha) = \{b\}, F_2(\beta) = \{a, b\}; F_3(\alpha) = \{b\}, F_3(\beta) = \{a\}; F_4(\alpha) = \{a\}, F_4(\beta) = \{b\}$. Then X is a regular space. Consider the element $\tilde{x}$, where, $\tilde{x}(\alpha) = \{a\}, \tilde{x}(\beta) = \{a\}$. But F_1 is a open set containing the element $\tilde{x}$ for which the condition of strong regularity does not hold.*

Proposition 6.9. *Let $F \in S(\tilde{X})$ be a compact set in a strong regular space. Then for any open set G containing F, there is a closed set H such that $F \tilde{\subset} H \tilde{\subset} G$.*

Proof. Let X be a strong regular space and $F \in S(\tilde{X})$ be a compact set. If $G = (\tilde{X}, A)$, then we take $H = (\tilde{X}, A)$. Then $F \tilde{\subset} H \tilde{\subset} G$. Next suppose G be a open set other than $(\tilde{X}, A)$ such that $F \tilde{\subset} G$. Let $\tilde{x} \tilde{\in} F$, then $\tilde{x} \tilde{\in} G$. Since X is strong regular $\exists\ V \in \tau$ such that $\tilde{x} \tilde{\in} V \tilde{\subseteq} \overline{V} \tilde{\subset} G$. Let $\mathscr{B} = \{V_{\tilde{x}} : \tilde{x} \tilde{\in} F\}$. Then $\mathscr{B}$ is a open cover of F. Since F is compact there is a finite subcollection $\{V_i \in \tau, i = 1, 2, ..., n\}$ of $\mathscr{B}$ such that $F \tilde{\subset} \overset{n}{\underset{i=1}{\Cup}} V_i$. Now

$V_i \tilde{\subset} \overline{V_i} \tilde{\subset} G$, for $i = 1, 2, ..., n$. Therefore $\underset{i=1}{\overset{n}{\Cup}} V_i \tilde{\subset} \underset{i=1}{\overset{n}{\Cup}} \overline{V_i} \tilde{\subset} G$. Let $H = \underset{i=1}{\overset{n}{\Cup}} \overline{V_i}$. Then $F \tilde{\subset} H \tilde{\subset} G$. Now if $\underset{i=1}{\overset{n}{\Cap}} \overline{V_i}^{\mathbb{C}} = (\tilde{\Phi}, A)$, then $G = (\tilde{X}, A)$, which is a contradiction. So $\underset{i=1}{\overset{n}{\Cap}} \overline{V_i}^{\mathbb{C}} \neq (\tilde{\Phi}, A)$. Therefore H is closed. Hence the result. □

Proposition 6.10. *Closure of a compact set in a strong regular space is compact.*

Proof. Let X be a strong regular space and F be a compact set in X. Let $\mathscr{B} = \{G_i \in \tau : i \in \Delta\}$ be a open cover of $\overline{F}$. Since $F \tilde{\subset} \overline{F}$ so $\mathscr{B}$ is also a open cover of F. Since F is compact there is a finite subcollection $\{G_i, i = 1, 2, ..., n\}$ of $\mathscr{B}$ such that $F \tilde{\subset} \underset{i=1}{\overset{n}{\Cup}} G_i$. Let $G = \underset{i=1}{\overset{n}{\Cup}} G_i$. Then G is open and $F \tilde{\subset} G$. Then from Proposition 6.9 there is a closed set H such that $F \tilde{\subset} H \tilde{\subset} G$. Then $\overline{F} \tilde{\subset} H \tilde{\subset} G$. So $\overline{F} \tilde{\subset} \underset{i=1}{\overset{n}{\Cup}} G_i$. Hence $\overline{F}$ is compact. □

7 Locally soft compactness

Definition 7.1. *A space X is said to be locally soft compact if for each $\tilde{x} \in SE(\tilde{X})$ there exists a compact nbd of $\tilde{x}$.*
From here we will use the term locally compact space in place of locally soft compact topological space.

Remark 7.2. *A compact space is locally compact but not conversely which is shown in the following example.*

Example 7.3. *Let $\mathbb{N}$ be the set of natural numbers and A be a non-empty set of parameters. Let $\mathfrak{B}(\mathbb{N})$ denotes the collection of all bounded subset of $\mathbb{N}$. Choose $n \in \mathbb{N}$, define a mapping $F_n : A \to \mathfrak{B}(\mathbb{N})$ by $F_n(\alpha) = \{n, n+1, n+2, ...\}$, $\forall \alpha \in A$. Let $\tau = \{F_n^C : n \in \mathbb{N})\} \cup (\tilde{\mathbb{N}}, A)$. Then $(\tilde{\mathbb{N}}, \tau, A)$ is a space. Clearly $(\tilde{\mathbb{N}}, \tau, A)$ is not compact. Let $\tilde{x}$ be any element of $(\tilde{\mathbb{N}}, A)$, then $\tilde{x}(\alpha) \in \mathfrak{B}(\mathbb{N})$, $\forall \alpha \in A$. Let $\delta = Sup\{\tilde{x}(\alpha) : \alpha \in A\}$. Then $F_{\delta+1}^C$ is a compact nbd of $\tilde{x}$. Since $\tilde{x}$ is arbitrary $(\tilde{\mathbb{N}}, \tau, A)$ is locally compact space.*

Definition 7.4. *Let X be a space. Let $\tilde{x} \in SE(\tilde{X})$ and $\aleph_\tau(\tilde{x})$ be the nbd system at $\tilde{x}$. A subcollection $\mathscr{B}_\tau(\tilde{x})$ of $\aleph_\tau(\tilde{x})$ is said to be a nbd base at $\tilde{x}$ if for any $F \in \aleph_\tau(\tilde{x})$, $\exists\, G \in \mathscr{B}_\tau(\tilde{x})$ such that $\tilde{x} \tilde{\in} G \tilde{\subset} F$.*

Example 7.5. *Let* $X = \{x,y,z\}$ *and* $A = \{\alpha,\beta\}$. *Let* $\tau = \{(\tilde{\Phi},A)$ $,(\tilde{X},A),F_1,F_2,F_3,F_4\}$, *where,* $F_1(\alpha) = \{x,y,z\},F_1(\beta) = \{x,y\};F_2(\alpha)$ $= \{x,y\},F_2(\beta) = \{x,y\};F_3(\alpha) = \{x,z\},F_3(\beta) = \{x,y\};F_4(\alpha) = \{x\}$, $F_4(\beta) = \{x,y\}$. *Then* X *is a space. Consider the element* $\tilde{x}$, *where,* $\tilde{x}(\alpha) = \{x\},\tilde{x}(\beta) = \{x\}$. *Then* $\aleph_{\tilde{x}} = \{(\tilde{X},A),F_1,F_2,F_3,F_4,F_5,$ $F_6,F_7\}$, *where,* $F_5(\alpha) = \{x,y\},F_5(\beta) = \{x,y,z\};F_6(\alpha) = \{x,z\},F_6(\beta)$ $= \{x,y,z\};F_7(\alpha) = \{x\},F_7(\beta) = \{x,y,z\}$. *Consider* $\mathscr{B}_{\tilde{x}} = \{F_2,F_3$ $,F_4,F_5,F_6,F_7\}$. *Then* $\mathscr{B}_{\tilde{x}}$ *is a nbd base at* $\tilde{x}$.

Proposition 7.6. *Let* X *be a strong regular and locally compact space. Then each* $\tilde{x} \in SE(\tilde{X})$ *has a compact nbd base.*

Proof. Let X be a strong regular and locally compact space. Let $\tilde{x} \in SE(\tilde{X})$ and V be any nbd of $\tilde{x}$. Since X is locally compact $\exists$ a compact nbd C of $\tilde{x}$ in $(\tilde{X},\tau,A)$. Now $V \Cap C$ is a nbd of $\tilde{x}$. Since X is strong regular $\exists$ a closed nbd W of $\tilde{x}$ in X such that $W \tilde{\subset} V \Cap C$. Now (C,τ_C,A) is compact. Since W is closed in X so $W \Cap C = W$ is closed in (C,τ_C,A). Hence W is compact in (C,τ_C,A). So W is compact in X. So W is a compact nbd of $\tilde{x}$ such that $\tilde{x} \tilde{\in} W \tilde{\subset} V$. Hence $\tilde{x}$ has a compact nbd base □

Remark 7.7. *The following example shows that the Proposition 7.6 may not be true for a regular space.*

Example 7.8. *Let* $X = \{x,y\}$ *and* $A = \{\alpha,\beta\}$ *and* $\tau = \{(\tilde{\Phi},A),(\tilde{X},A),$ $F_1,F_2,F_3,F_4\}$, *where,* $F_1(\alpha) = \{x,y\},F_1(\beta) = \{x\};F_2(\alpha) = \{y\},F_2(\beta) =$ $\{x,y\};F_3(\alpha) = \{y\},F_3(\beta) = \{x\};F_4(\alpha) = \{x\},F_4(\beta) = \{y\}$. *Then* X *is locally compact and regular space. Consider the element* $\tilde{x}$, *where,* $\tilde{x}(\alpha) = \{x\},\tilde{x}(\beta) = \{y\}$. *Then there does not exist any compact nbd base at* $\tilde{x}$.

Proposition 7.9. *Every open set in a locally compact strong regular space is locally compact.*

Proof. Let X be a strong regular and locally compact space. Let Y be a open subset of $(\tilde{X},A)$. Let $\tilde{x} \tilde{\in} Y$. Since X is locally compact and strong regular space by Proposition 7.6, $\tilde{x}$ has a compact nbd base. So $\exists$ a compact nbd C of $\tilde{x}$ in X such that $C \tilde{\subset} Y$. Now $C \Cap Y = C$ is a compact nbd of $\tilde{x}$ in (Y,τ_Y,A). Hence Y is locally compact. □

Proposition 7.10. *Closed set in a locally compacts space is locally*

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compact.

Proof. Let X be a locally compact space and F be a closed subset of $(\tilde{X},A)$. Let $\tilde{x}\tilde{\in}F$. Since X is locally compact there is a compact nbd C of $\tilde{x}$ in X. Now (C,τ_C,A) is compact. Then $F\Cap C$ is a closed nbd of $\tilde{x}$ in (C,τ_C,A). Then $F\Cap C$ is compact in (C,τ_C,A). Hence $F\Cap C$ is a compact nbd of $\tilde{x}$ in (F,τ_F,A). So F is locally compact. □

8 Soft connectedness

Definition 8.1. *A space X is said to be soft disconnected if there exists $F,G[\neq(\tilde{\Phi},A)]\in S(\tilde{X})$ such that $(\tilde{X},A)=F\Cup G$ and $\overline{F}\tilde{\cap}\overline{G}=(\tilde{\Phi},A)$. A space X is said to be soft connected if it is not soft disconnected. Further by saying a connected space we will mean a soft connected topological space.*

Example 8.2. *Let $X=\{x,y,z\}$ and $A=\{\alpha,\beta\}$ and $\tau=\{(\tilde{\Phi},A),(\tilde{X},A),F,G\}$, where $F(\alpha)=\{x,y\},F(\beta)=\{x,z\};G(\alpha)=\{z\},G(\beta)=\{y\}$. Then X is a space. Now $(\tilde{X},A)=F\Cup G$. Here $\overline{F}=F$ and $\overline{G}=G$. Therefore $\overline{F}\tilde{\cap}\overline{G}=(\tilde{\Phi},A)$. So X is disconnected.*

Definition 8.3. *Let X be a space. A set $H\in S(\tilde{X})$ is said to be separated if there exists $F,G[\neq(\tilde{\Phi},A)]\in S(\tilde{X})$ such that $H=F\Cup G$ and $\overline{F}\tilde{\cap}G=(\tilde{\Phi},A)$ and $\overline{G}\tilde{\cap}F=(\tilde{\Phi},A)$.*

Remark 8.4. *Clearly a disconnected set in a space X has a separation but not conversely. This is shown in the following example.*

Example 8.5. *Let $X=\{x,y,z,t\}$ and $A=\{\alpha,\beta\}$ and $\tau=\{(\tilde{\Phi},A),(\tilde{X},A),G_1,G_2,G_3\}$, where, $G_1(\alpha)=\{x,y,z\},G_1(\beta)=\{x,t\};G_2(\alpha)=\{t\},G_2(\beta)=\{y,t\};G_3(\alpha)=\{x,y,z,t\},G_3(\beta)=\{x,y,t\}$. Then X is a space. Consider the set $H\in S(\tilde{X})$, where, $H(\alpha)=\{y,t\},H(\beta)=\{x,y\}$. Then $H=F\Cup G$, where, $F(\alpha)=\{t\},F(\beta)=\{y\};G(\alpha)=\{y\},G(\beta)=\{x\}$. Now $\overline{F}=(\{t\},\{y,z\})$ and $\overline{G}=(\{x,y,z\},\{x,z\})$. Then $\overline{F}\tilde{\cap}G=(\tilde{\Phi},A)$ and $\overline{G}\tilde{\cap}F=(\tilde{\Phi},A)$. But $\overline{F}\tilde{\cap}\overline{G}=(\phi,\{z\})$. Hence H is separated but not disconnected.*

Proposition 8.6. *Let X be a space. Then the following statements are equivalent*

(i) $(\tilde{X},\tau,A)$ *is connected.*

(ii) $(\tilde{X},A)$ *can not be expressed as union of two non-empty disjoint closed sets.*

(iii) $(\tilde{X},A)$ *can not be expressed as union of two non-empty disjoint open sets.*

(iv) There does not exists any proper clopen subset of $(\tilde{X},A)$.

Proof. $(i) \Rightarrow (ii)$:
Let X be a disconnected space. Then $\exists\, F,G[\neq (\tilde{\Phi},A)] \in S(\tilde{X})$ such that $(\tilde{X},A) = F \Cup G$ and $\overline{F} \tilde{\cap} \overline{G} = (\tilde{\Phi},A)$. Therefore $\overline{F} \tilde{\subset} \overline{G}^{\mathbb{C}}$. Now $G \tilde{\subset} \overline{G} \Rightarrow \overline{G}^{\mathbb{C}} \tilde{\subset} G^{\mathbb{C}}$. So $\overline{F} \tilde{\subset} G^{\mathbb{C}}$. Since $(\tilde{X},A) = F \Cup G$ and $\overline{F} \tilde{\cap} \overline{G} = (\tilde{\Phi},A)$. So $G^{\mathbb{C}} = F$ and $F^{\mathbb{C}} = G$. Hence $\overline{F} \tilde{\subset} F$. Therefore F is closed. Similarly G is closed.

$(ii) \Rightarrow (iii)$:
Let $(\tilde{X},A) = F \Cup G$, where, $F,G[\neq (\tilde{\Phi},A)] \in S(\tilde{X})$ are closed and $F \tilde{\cap} G = (\tilde{\Phi},A)$. Then $F^{\mathbb{C}} = G$ and $G^{\mathbb{C}} = F$. Again since F is closed so $F^{\mathbb{C}}[\neq (\tilde{\Phi},A)] \in \tau$. Therefore G is open. Similarly F is also open.

$(iii) \Rightarrow (iv)$:
Let $(\tilde{X},A) = F \Cup G$, where, $F,G[\neq (\tilde{\Phi},A)] \in S(\tilde{X})$ are open and $F \tilde{\cap} G = (\tilde{\Phi},A)$. Since $F \tilde{\cap} G = (\tilde{\Phi},A)$, F^{C} and $G^{C} \in S(\tilde{X})$. So $F^{\mathbb{C}} = G$. Therefore F is closed set. Hence F is a clopen subset of $(\tilde{X},A)$.

$(iv) \Rightarrow (i)$:
Let $F[\neq (\tilde{\Phi},A)] \in S(\tilde{X})$ be a proper clopen subset of $(\tilde{X},A)$. Since F is closed so $F^{\mathbb{C}} \neq (\tilde{\Phi},A)$. Therefore $(\tilde{X},A) = F \Cup F^{\mathbb{C}}$. Since F is closed $\overline{F} = F$. Again since F is open, $F^{\mathbb{C}}$ is closed. Therefore $\overline{F^{\mathbb{C}}} = F^{\mathbb{C}}$. Now $\overline{F} \tilde{\cap} \overline{F^{\mathbb{C}}} = F \tilde{\cap} F^{\mathbb{C}} = (\tilde{\Phi},A)$. Therefore X is disconnected. □

Proposition 8.7. *Soft continuous onto image of a connected space is connected.*

Proof. Let $f : (\tilde{X},\tau,A) \rightarrow (\tilde{Y},\nu,A)$ be a soft continuous onto function where, X is a connected space. If possible let $(\tilde{Y},\nu,A)$ is disconnected. Then $\exists\, F,G[\neq (\tilde{\Phi},A)] \in \nu$ such that $(\tilde{Y},A) = F \Cup G$ and $F \tilde{\cap} G = (\tilde{\Phi},A)$. Since f is onto $f[(\tilde{X},A)] = (\tilde{Y},A) \Rightarrow (\tilde{X},A) = f^{-1}[(\tilde{Y},A)] = f^{-1}(F \Cup G) = f^{-1}(F) \Cup f^{-1}(G)$. Now $f^{-1}(F)$ and $f^{-1}(G)$ are non-empty s-open sets in $(\tilde{X},\tau,A)$. Now $f^{-1}(F) \tilde{\cap} f^{-1}(G) = f^{-1}[(F) \tilde{\cap} G] =$

$f^{-1}[(\tilde{\Phi},A)] = (\tilde{\Phi},A)$. Therefore $(\tilde{X},\tau,A)$ is disconnected. A contradiction. Hence $(\tilde{Y},\nu,A)$ is connected. $\square$

Proposition 8.8. *Let $(\tilde{Y},\tau_{\tilde{Y}},A)$ be a subspace of the space X. Let $E\tilde{\subset}(\tilde{Y},A)\tilde{\subset}(\tilde{X},A)$. Then E is connected in $(\tilde{Y},\tau_{\tilde{Y}},A)$ iff E is connected in X.*

Proof. Let $F,G\tilde{\subset}(\tilde{Y},A)\tilde{\subset}(\tilde{X},A)$. Then
$[F\tilde{\cap}\overline{G}^{\tau}]\tilde{\cup}[G\tilde{\cap}\overline{F}^{\tau}]$
$= [\{F\tilde{\cap}(\tilde{Y},A)\}\tilde{\cap}\overline{G}^{\tau}]\tilde{\cup}[\{G\tilde{\cap}(\tilde{Y},A)\}\tilde{\cap}\overline{F}^{\tau}]$
$= [F\tilde{\cap}\{(\tilde{Y},A)\tilde{\cap}\overline{G}^{\tau}\}]\tilde{\cup}[G\tilde{\cap}\{(\tilde{Y},A)\tilde{\cap}\overline{F}^{\tau}\}]$
$= [F\tilde{\cap}\{(\tilde{Y},A)\Cap\overline{G}^{\tau}\}]\tilde{\cup}[G\tilde{\cap}\{(\tilde{Y},A)\Cap\overline{F}^{\tau}\}]$, Since $(\tilde{Y},A)\Cap\overline{G}^{\tau} \neq (\tilde{\Phi},A)$, $(\tilde{Y},A)\Cap\overline{F}^{\tau} \neq (\tilde{\Phi},A)$
$= [F\tilde{\cap}\overline{G}^{\tau_{\tilde{Y}}}]\tilde{\cup}[G\tilde{\cap}\overline{F}^{\tau_{\tilde{Y}}}]$
Thus for $F,G\tilde{\subset}(\tilde{Y},A)\tilde{\subset}(\tilde{X},A)$, $[F\tilde{\cap}\overline{G}^{\tau}]\tilde{\cup}[G\tilde{\cap}\overline{F}^{\tau}] = (\tilde{\Phi},A)$
$\Rightarrow [F\tilde{\cap}\overline{G}^{\tau_{\tilde{Y}}}]\tilde{\cup}[G\tilde{\cap}\overline{F}^{\tau_{\tilde{Y}}}] = (\tilde{\Phi},A)$.
Therefore E has a separation in $(\tilde{Y},\tau_{\tilde{Y}},A) \Leftrightarrow \exists\ F,G[\neq(\tilde{\Phi},A)] \in S(\tilde{Y})$ with $E = F\Cup G$ and $= [F\tilde{\cap}\overline{G}^{\tau_{\tilde{Y}}}]\tilde{\cup}[G\tilde{\cap}\overline{F}^{\tau_{\tilde{Y}}}] = (\tilde{\Phi},A) \Leftrightarrow \exists F,G[\neq(\tilde{\Phi},A)] \in S(\tilde{X})$ with $E = F\Cup G$ and $[F\tilde{\cap}\overline{G}^{\tau}]\tilde{\cup}[G\tilde{\cap}\overline{F}^{\tau}] = (\tilde{\Phi},A)$. Hence E is connected in $(\tilde{Y},\tau_{\tilde{Y}},A)$ iff E is connected in X. $\square$

Proposition 8.9. *Let $F,G \in S(\tilde{X})$ be two separated sets in a space X. Let $(\tilde{Y},\tau_{\tilde{Y}},A)$ be a connected subspace of X such that $(\tilde{Y},A)\tilde{\subset}F\Cup G$. Then either $(\tilde{Y},A)\tilde{\subset}F$ or $(\tilde{Y},A)\tilde{\subset}G$.*

Proof. Since F,G are separated so $F\tilde{\cap}\overline{G}^{\tau} = (\tilde{\Phi},A)$ and $G\tilde{\cap}\overline{F}^{\tau} = (\tilde{\Phi},A)$. Now $\overline{F}^{\tau}\Cap[F\Cup G] = \overline{F}^{\tau}\tilde{\cap}[F\tilde{\cup}G]$ $\left(\textit{Since } \overline{F}^{\tau}\Cap F \neq (\tilde{\Phi},A)\right) = [\overline{F}^{\tau}\tilde{\cap}F]\tilde{\cup}$ $[\overline{F}^{\tau}\tilde{\cap}G] = F\tilde{\cup}(\tilde{\Phi},A) = F$. So F is closed in $F\Cup G$. Similarly G is closed in $F\Cup G$. Therefore $(\tilde{Y},A)\Cap F$ and $(\tilde{Y},A)\Cap G$ are closed in $(\tilde{Y},\tau_{\tilde{Y}},A)$. Now $(\tilde{Y},A) = [(\tilde{Y},A)\tilde{\cap}F]\tilde{\cup}[(\tilde{Y},A)\tilde{\cap}G]$. If $(\tilde{Y},A)\Cap F = (\tilde{\Phi},A)$, then $[(\tilde{Y},A)\Cap F]^{\mathbb{C}}|_{\tilde{Y}} = (\tilde{Y},A) \Rightarrow (\tilde{Y},A)^{\mathbb{C}}|_{\tilde{Y}}\ \Cup\ (F^{\mathbb{C}}|_{\tilde{Y}}) = (\tilde{Y},A) \Rightarrow F^{\mathbb{C}}|_{\tilde{Y}} = (\tilde{Y},A)$. Since $F\tilde{\cap}\overline{G}^{\tau} = (\tilde{\Phi},A) \Rightarrow F\tilde{\cap}G = (\tilde{\Phi},A) \Rightarrow F^{\mathbb{C}}|_{\tilde{Y}}\tilde{\subset}G \Rightarrow (\tilde{Y},A)\tilde{\subset}G$. Similarly if $(\tilde{Y},A)\Cap G = (\tilde{\Phi},A)$, then $(\tilde{Y},A)\tilde{\subset}F$. Suppose $(\tilde{Y},A)\Cap F \neq (\tilde{\Phi},A)$ and $(\tilde{Y},A)\Cap G \neq (\tilde{\Phi},A)$, then $(\tilde{Y},A) = [(\tilde{Y},A)\Cap(F,A)]\Cup[(\tilde{Y},A)\Cap G]$. Since $(\tilde{Y},\tau_{\tilde{Y}},A)$ is connected so either $(\tilde{Y},A)\Cap F = (\tilde{\Phi},A)$ or $(\tilde{Y},A)\Cap G = (\tilde{\Phi},A)$. So in either case $(\tilde{Y},A)\tilde{\subset}G$ or $(\tilde{Y},A)\tilde{\subset}F$. $\square$

Proposition 8.10. *Let C be a connected set in a space X. Let $(\tilde{Y},A)$ be a set such that $C\tilde{\subset}(\tilde{Y},A)\tilde{\subset}\overline{C}$. Then $(\tilde{Y},A)$ is connected.*

Proof. If possible let $(\tilde{Y},A)$ is not connected. Then $(\tilde{Y},A)$ has a separation. So $\exists\, F,G \in S(\tilde{X})$ such that $(\tilde{Y},A) = F \Cup G$ and $\overline{F} \tilde{\cap} G = (\tilde{\Phi},A)$ and $\overline{G} \tilde{\cap} F = (\tilde{\Phi},A)$. Since C is connected in $(\tilde{X},\tau,A)$ and $C \tilde{\subset} (\tilde{Y},A) \tilde{\subset} (\tilde{X},A)$, so C is connected in $(\tilde{Y},\tau_{\tilde{Y}},A)$. Then by Proposition 8.9, either $C \tilde{\subset} F$ or $C \tilde{\subset} G$. Without loss of generality let $C \tilde{\subset} F$. Now $\overline{C} \tilde{\subset} \overline{(Y,A)}$. Therefore $\overline{C} \,\tilde{\cap}\, G \,\tilde{\subset}\, \overline{F} \,\tilde{\cap}\, G = (\tilde{\Phi},A)$. Therefore $\overline{C} \tilde{\cap} G = (\tilde{\Phi},A)$. Again $G \tilde{\subset} \overline{C}$.Therefore $\overline{C} \tilde{\cap} G = G$, which implies $G = (\tilde{\Phi},A)$. A contradiction. Hence $(\tilde{Y},A)$ is connected. $\square$

Remark 8.11. *Closure of a connected set $F \in S(\tilde{X})$ in a space X is connected if $\overline{F}$ is a constant soft set.*

Remark 8.12. *In general the closure of a connected set is not connected which is shown in the following example.*

Example 8.13. *Let $X = \{x,y,z,t\}$ and $A = \{\alpha,\beta\}$. Let $\tau = \{(\tilde{\Phi},A), (\tilde{X},A), G_1, G_2, G_3\}$, where, $G_1(\alpha) = \{z\}, G_1(\beta) = \{x\}; G_2(\alpha) = \{z,t\}, G_2(\beta) = \{x,y,z\}; G_3(\alpha) = \{x,y,z\}, G_3(\beta) = \{x,t\}$. Then X is a space. Consider the connected set C, where, $C(\alpha) = \{x,y,t\}, C(\beta) = \{y,z\}$. Then $\overline{C} = \Big(\{x,y,t\}, \{y,z,t\}\Big)$. Then $\overline{C}$ is not connected, since for $F,G \in S(\tilde{X})$, where, $F(\alpha) = \{x,y\}, F(\beta) = \{t\}; G(\alpha) = \{t\}, G(\beta) = \{y,z\}$, $\overline{C} = F \Cup G$ and $\overline{F} \tilde{\cap} \overline{G} = (\tilde{\Phi},A)$.*

Proposition 8.14. *Let $C_1, C_2 \in S(\tilde{X})$ be two connected set in a space$(\tilde{X},\tau,A)$. Then $C_1 \Cup C_2$ is connected if $C_1 \Cap C_2 \neq (\tilde{\Phi},A)$.*

Proof. Let $C = C_1 \Cup C_2$. If possible let C is not connected. Then C has a separation. So there exists $F, G[\neq (\tilde{\Phi},A)] \in S(\tilde{X})$ such that $C = F \Cup G$ and $\overline{F} \tilde{\cap} G = (\tilde{\Phi},A)$ and $\overline{G} \tilde{\cap} F = (\tilde{\Phi},A)$. Now $C_1 \tilde{\subset} F \Cup G$. Since $F \tilde{\cap} G = (\tilde{\Phi},A)$ so either $C_1 \tilde{\subset} F$ or $C_1 \tilde{\subset} G$. Without loss of generality let $C_1 \tilde{\subset} F$. Since $C_1 \Cap C_2 \neq (\tilde{\Phi},A)$ so $C_2 \tilde{\subset} F$. Then $C \tilde{\subset} F$. So $G = (\tilde{\Phi},A)$ which is a contradiction. Hence C is connected. $\square$

Remark 8.15. *The following example shows that for two connected setS $C_1, C_2 \in S(\tilde{X})$ if $C_1 \tilde{\cap} C_2 \neq (\tilde{\Phi},A)$ then $C_1 \Cup C_2$ may not be connected.*

Example 8.16. *Consider a space X as per as in Example 8.2. Consider two connected sets $C_1, C_2 \in S(\tilde{X})$, where, $C_1(\alpha) = \{x,y\}, C_1(\beta) = \{z\}; C_2(\alpha) = \{x\}, C_2(\beta) = \{x,y\}$. Then $C_1 \Cup C_2 = (\tilde{X},A)$, which is not*

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connected.

Definition 8.17. *In a space X a connected set $F \in S(\tilde{X})$ is said to be a soft component of $(\tilde{X}, A)$ if for any connected set $G \in S(\tilde{X})$, $F \tilde{\subset} G \Rightarrow F = G$.*

Proposition 8.18. *In a space X a component of $(\tilde{X}, A)$ is closed if its closure is a constant soft set.*

Proof. Let $F \in S(\tilde{X})$ be a component of $(\tilde{X}, A)$. Then$F$ is a connected set. Then by Remark 8.11, $\overline{F}$ is connected. Now $F \tilde{\subset} \overline{F}$. Since F is a component and $F \tilde{\subseteq} \overline{F}$. So $F = \overline{F}$. Hence $\overline{F}$ is closed. □

Proposition 8.19. *Let X be a space. For any two distinct components $F, G \in S(\tilde{X})$, $F \Cap G = (\tilde{\Phi}, A)$.*

Proof. Let $F, G \in S(\tilde{X})$ be two distinct component of $(\tilde{X}, A)$. If possible let $F \Cap G \neq (\tilde{\Phi}, A)$. Then by Proposition 8.14, $F \Cup G$ is connected. Now $F \tilde{\subset} F \Cup G$ and $G \tilde{\subset} F \Cup G$. Since F, G are components so $F = F \Cup G$ and $G = F \Cup G$. Therefore $F = G$ which is a contradiction. Hence $F \Cap G = (\tilde{\Phi}, A)$. □

Proposition 8.20. *Let X be a space. Then $(\tilde{X}, A)$ can be expressed as union of its component.*

Proof. Let $x \in X$. Let $\bar{x} \in S(\tilde{X})$ be a element such that $\bar{x}(\lambda) = x, \forall \lambda \in A$. Consider the set $(\bar{x}, A)$. Then $(\bar{x}, A)$ is connected. Then the elementary union of all connected set containing the element $\bar{x}$ is connected and this will be a component containing $\bar{x}$. Similarly for each $x \in X$ we get a component containing the element $\bar{x}$. Hence the result. □

9 Locally soft connectedness

Definition 9.1. *A space X is said to be locally soft connected at $\tilde{x} \in SE(\tilde{X})$ if for any nbd F of $\tilde{x}$ there exists a connected nbd G of $\tilde{x}$ such that $\tilde{x} \tilde{\in} G \tilde{\subset} F$. A space $(\tilde{X}, \tau, A)$ is said to be locally soft connected if it is locally soft connected at each $\tilde{x} \in SE(\tilde{X})$.*
From here by saying locally connected space, we will mean a locally soft connected topological space.

Remark 9.2. *The following two examples shows that connectedness and locally connectedness are independent of each other.*

Example 9.3. *Let* $X = \{x,y\}$ *and* $A = \{\alpha,\beta\}$. *Let* $\tau = \{(\tilde{\Phi},A),(\tilde{X},A), F_1,F_2,F_3,F_4,F_5,F_6,F_7,F_8\}$, *where,* $F_1(\alpha) = \{x\}, F_1(\beta) = \{y\}; F_2(\alpha) = \{y\}, F_2(\beta) = \{x\}; F_3(\alpha) = \{x\}, F_3(\beta) = \{x\}; F_4(\alpha) = \{y\}, F_4(\beta) = \{y\}; F_5(\alpha) = \{x,y\}, F_5(\beta) = \{x\}; F_6(\alpha) = \{x,y\}, F_6(\beta) = \{y\}; F_7(\alpha) = \{x\}, F_7(\beta) = \{x,y\}; F_8(\alpha) = \{y\}, F_8(\beta) = \{x,y\}$. *Then* X *is a locally connected space. But* X *is not connected.*

Example 9.4. *Let* $X = \{x,y,z\}$ *and* $A = \{\alpha,\beta\}$. *Let* $\tau = \{(\tilde{\Phi},A), (\tilde{X},A),F,G,H\}$, *where,* $F(\alpha) = \{x,z\}, F(\beta) = \{y,z\}; G(\alpha) = \{x\}, G(\beta) = \{y\}; H(\alpha) = \{z\}, H(\beta) = \{z\}$. *Then* X *is a connected space. Consider the element* $\tilde{\xi}$, *where,* $\tilde{\xi}(\alpha) = \{x\}, \tilde{\xi}(\beta) = \{z\}$. *Then* F *is a nbd of* $\tilde{\xi}$ *but there does not exist any connected nbd of* $\tilde{\xi}$. *So* $(\tilde{X},\tau,A)$ *is not locally connected.*

Proposition 9.5. *A space is locally connected iff the component of any nbd of a element is a nbd of that element.*

Proof. Let X be a locally connected space. Let $\tilde{x} \in SE(\tilde{X})$. Let F be any nbd of of $\tilde{x}$ and V be a component of F containing $\tilde{x}$. Since $(\tilde{X},\tau,A)$ is locally connected $\exists$ a connected nbd W of $\tilde{x}$ such that $\tilde{x}\tilde{\in}W\tilde{\subset}F$. Since V is a component containing $\tilde{x}$ and W is a connected nbd of $\tilde{x}$, $W\tilde{\subset}V$. Therefore $\tilde{x}\tilde{\in}W\tilde{\subset}V$. Hence V is a nbd of $\tilde{x}$.
The converse is straightforward. □

Proposition 9.6. *In a locally connected space component of every open set is open.*

Proof. Let X be a locally connected space. Let G be a open set and V be its component. Let $\tilde{x}\tilde{\in}V$. Then $\tilde{x}\tilde{\in}G$. Since G is open G is a nbd of $\tilde{x}$. By locally connectedness there exists a connected nbd F of $\tilde{x}$ such that $\tilde{x}\tilde{\in}F\tilde{\subset}G$. Since V is a component containing $\tilde{x}$, $\tilde{x}\tilde{\in}F\tilde{\subset}V$. Hence V is a nbd of $\tilde{x}$. Therefore V is open. □

10 Conclusion

In this paper, we have studied some topological properties of subspace, compactness, connectedness, locally compactness and locally connectedness on soft topology of a new structure. Still there are many scopes to study some important properties like product space, Tychonoff space, Urysohn's lemma, Tietze's extension theorem, metrizability, uniformity in this context. It will be necessary to carry out more theoretical research to establish a general framework for the topological applications.

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