International Journal of Pure and Applied Mathematics
Volume 120 No. 5 2018, 1423-1439
ISSN: 1314-3395 (on-line version)
url: http://www.acadpubl.eu/hub/
Special Issue
http://www.acadpubl.eu/hub/

A numerical method for two point singularly perturbed coupled system of diffusion-convection-reaction problems with discontinuous source terms

T. Prabha [1] , V. Shanthi[2]

[1] Department of Mathematics, National Institute of Technology,
Tiruchirappalli-620 015, Tamilnadu, INDIA
e-mail: prabha.thevaraj@gmail.com

[2]Department of Mathematics, National Institute of Technology,
Tiruchirappalli-620 015, Tamilnadu, INDIA
e-mail: vshanthi@nitt.edu

August 23, 2018

Abstract

In this article we have presented a weakly coupled system of Singularly Perturbed Diffusion-Convection-Reaction Problems (SPDCRPs) having two small parameters (ε, μ) multiplying the diffusion and convection term with discontinuous source term. The solution of the problem exhibits boundary and interior layers for sufficiently small values of the parameters. The behavior of the exact solutions with respect to both the parameters are examined involving two cases. Theoretical bounds for the solution and its derivatives are estimated. The numerical experiments of the algorithm on piecewise uniform Shishkin mesh are constructed, using finite difference scheme. The analysis and the estimation of the error are carried out and the algorithm is shown to have uniform convergence of almost first order. Numerical experiments are discussed to illustrate the parameter-uniform convergence of the theoretical results.

Math Subject Classification:34B05,65L10,65L12

Key Words :Singularly perturbed problem ,two point differential equations, interior layers, weakly coupled system, Two parameters, Shishkin mesh, Difference scheme

1 Introduction

The Singular Perturbation Problems(SPPs) commonly occur in many branches of applied mathematics, e.g., as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in solid mechanics, transition points in quantum mechanics and Stokes lines and surfaces in mathematics. In these kind of problems perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid change. These narrow regions frequently adjoin the boundaries or some interior point of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Therefore, these kind of problems exhibit boundary and/or interior layers, i.e., there are thin regions where the solution changes rapidly. Despite large amount of work being done in the research field of SPPs for smooth [5, 7, 14, 11, 10]and non-smooth data [2, 4, 8, 9, 16].still there is a platform for more relevant research to explore. All the study discussed so far relate to SPPs in which a small parameter affects the highest derivative term. Authors in [15, 3, 13] considered convection-reaction-diffusion problem with non-smooth data. The solution of linear system of SPPs for convection-reaction-diffusion type problems is a broad area to work. This type of problems can be modeled as turbulence flow owing to the interactions of waves with the steady current[10]. These type of equations also have applications in electroanalytical chemistry when investigating diffusion processes complicated by chemical reactions. The parameters multiplying the highest derivatives characterize the diffusion coefficient of the substances. Motivated by the works of [16, 1] in this study a numerical technique for solving a weakly coupled system of SPCRDPs with discontinuous source terms are considered on the unit interval $\Omega = (0,1)$.. A single discontinuity in the source terms is assumed to occur at a point $d \in \Omega$. The corresponding problem is defined as follows: Find $u_1, u_2 \in \mathscr{C}^0(\bar{\Omega}) \cap \mathscr{C}^1(\Omega) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$ such that

$$L_1\bar{u}(x) \equiv \varepsilon u_1''(x) + \mu a_1(x)u_1'(x) - b_{11}(x)u_1(x) - b_{12}(x)u_2(x) = f_1(x) \quad (1)$$

$$L_2\bar{u}(x) \equiv \varepsilon u_2''(x) + \mu a_2(x)u_2'(x) - b_{21}(x)u_1(x) - b_{22}(x)u_2(x) = f_2(x) \quad (2)$$

$$\forall\ x \in (\Omega^- \cup \Omega^+)$$

$$u_1(0) = p, u_2(0) = q, u_1(1) = r, u_2(1) = s, \quad (3)$$

$$\mid [f](d) \mid \le C.$$

where $0 < \varepsilon << 1$, $0 \le \mu \le 1$, are small parameters with

$$a_1(x) > \alpha_1 > 0, a_2(x) > \alpha_2 > 0,$$

$$\begin{cases} b_{11}(x), b_{12}(x) \ge 0, b_{11}(x) \ge |b_{12}(x)|, \\ b_{11}(x) + b_{12}(x) \ge \beta_1(x) > 0, \end{cases} \quad \begin{cases} b_{21}(x), b_{22}(x) \ge 0, b_{22}(x) \ge |b_{21}(x)|, \\ b_{21}(x) + b_{22}(x) \ge \beta_2(x) > 0. \end{cases}$$

The coefficients $a_i(x) and b_{i,j}(x)$ for $(i, j = 1, 2)$ are sufficiently smooth functions in $\bar{\Omega}$. The source term functions $f_1(x) and f_2(x)$ are assumed to be sufficiently smooth on $(\Omega^- \cup \Omega^+)$. Their derivatives have a single jump discontinuity at $d \in \Omega$, denoted by $[w](d) = w(d+) - w(d-)$. Eventually this discontinuity would rise to interior layers in the solution of $\bar{u}(x)$ of the continuous problem (1)-(3). It is convenient to introduce the notations $\bar{\Omega} = [0, 1]$, $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1), d \in \Omega$.When $\mu = 1$, the problem behaves like the well known convection-diffusion problem [8] and when $\mu = 0$, it behaves like the reaction-diffusion problem [2]. In the present study, two cases are assumed for the analysis of the problem (1)-(3) given as,

Case (i): $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ and **C**ase (ii): $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$,
where $\alpha = \min \alpha_1, \alpha_2$ and $\rho = \min\{\rho_1, \rho_2\}$.It is assumed that

$$\begin{cases} \rho_1 = \min_{\bar{\Omega}\setminus\{d\}} \left\{ \dfrac{b_{11}(x) + b_{12}(x)}{a_1(x)} \right\}, \\ \rho_2 = \min_{\bar{\Omega}\setminus\{d\}} \left\{ \dfrac{b_{21}(x) + b_{22}(x)}{a_2(x)} \right\} \end{cases}$$

The results in forthcoming section show that the considered convection-reaction-diffusion problem behaves more like the reaction type problem for $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ with a layer width of $\mathcal{O}(\sqrt{\varepsilon})$ appearing in the neighborhood of $x = 0$, $x = d$ and $x = 1$. When $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$, a layer width of $\mathscr{O}(\frac{\varepsilon}{\mu})$in the neighborhood of $x = 0, x = d$ and a layer width of $\mathcal{O}(\mu)$in the neighborhood of $x = d$ and $x = 1$ can be predicted. The matrix-vector form of the considered problem (1)-(3) can be represented as

$$\bar{L}\bar{u} = \varepsilon^*\bar{u}'' + \mu\mathbf{A}(x)\bar{u}' - \mathbf{B}(x)\bar{u} = \bar{f}(x),\ x \in (\Omega^- \cup \Omega^+), \quad (4)$$

with the boundary conditions

$$\bar{u}(0) = \begin{pmatrix} p \\ q \end{pmatrix}, \qquad \bar{u}(1) = \begin{pmatrix} r \\ s \end{pmatrix} \quad (5)$$

where

$$\bar{L} = \begin{pmatrix} L_1(x) \\ L_2(x) \end{pmatrix}, \bar{u} = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \varepsilon^* = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \mathbf{A}(x) = \begin{pmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{pmatrix},$$

$$\mathbf{B}(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix},\ \bar{f}(x) = \begin{pmatrix} f_1(x) = \begin{cases} f_{1l}(x), & \text{for } x \le d \\ f_{1r}(x), & \text{for } x \ge d \end{cases} \\ f_2(x) = \begin{cases} f_{2l}(x), & \text{for } x \le d \\ f_{2r}(x), & \text{for } x \ge d \end{cases} \end{pmatrix}.$$

The rest of the paper is organized as below. In Section 2 some a *priori* results and decomposition of the continuous problem are described . Discretization of the continuous problem and the methods to be applied with the discrete bounds are described in Section 3. Decomposition and bounds for the discrete solution are presented in Section 4. In Section 5, convergence of the numerical method to the exact solution is derived. An almost first order $\varepsilon-\mu$ uniform accuracy is proved and estimated with suitable analysis. A numerical illustration is shown in Section 6 to find the applicability of the proposed method. The paper ends with conclusion.

Throughout this paper C represents a positive constant independent of the mesh size (N) and the perturbation parameters ε, μ. All the functions are measured in the supremum norm, denoted by

$$||w||_{\bar{\Omega}} = \sup_{x\in\bar{\Omega}} |w(x)|.$$

and the corresponding discrete norm is denoted by

$$||w||_{\bar{\Omega}^N} = \sup_{x\in\bar{\Omega}^N} |w(x_i)|.$$

2 A Priori Bounds and Decomposition of the Solution and its Derivatives

In this section, we study certain analytical properties, the existence of the solution minimum principle, uniform stability and bounds for the derivatives of the solution (1)-(2) as well as for its regular and singular components.

Lemma 1. *The SPCRDP (1)-(2) has a solution such that* $u_1(x), u_2(x) \in \mathscr{C}^0(\bar{\Omega}) \cap \mathscr{C}^1(\Omega) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$.

The differential operator $\bar{L}$ of the continuous problem (1)-(3)satisfies the following minimum principle.

Lemma 2. *Let us suppose that a function* $\bar{u}(x) \in \mathscr{C}^0(\bar{\Omega}) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$ *satisfies* $\bar{u}(0) \geq \bar{0}$, $\bar{u}(1) \geq \bar{0}$, *and* $L_1\bar{u}(x) \leq \bar{0}, L_2\bar{u}(x) \leq \bar{0}\ \forall x \in (\Omega^- \cup \Omega^+)$ *and* $[\bar{u}'](d) \leq \bar{0}$. *Then* $\bar{u}(x) \geq \bar{0},\ \forall\ x \in \bar{\Omega}$. *Then if there exists a function* $\bar{t} = (t_1, t_2) \in \mathscr{C}^0(\bar{\Omega}) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$, *such that* $\bar{t}(0) > \bar{0}, \bar{t}(1) > 0$, $L_1\bar{t}(x) \leq \bar{0}, L_2\bar{t}(x) \leq \bar{0}0\ \forall\ x\ \in\ (\Omega^- \cup \Omega^+)$ *and* $[\bar{t}'](d) \leq \bar{0}$,*then* $\bar{t}(x) \geq \bar{0},\ \forall\ x\ \in\ \bar{\Omega}$.

Proof. Proof Define

$$\psi = max\left\{\psi_1 = max_{x\in\bar{\Omega}}\left(-\frac{u_1}{t_1}\right)(x), \psi_2 = max_{x\in\bar{\Omega}}\left(-\frac{u_2}{t_2}\right)(x)\right\}.$$

Let $x_* \in \Omega$ such that $\bar{u}(x_*)$ attains its minimum value in $\bar{\Omega}$.
With the assumptions on the boundary values it is clear that $x_* \in (\Omega^- \cup \Omega^+)$ or $x_* = d$.
Assume that the lemma is not true. The proof is completed by showing that this leads to a contradiction.
Let $\bar{u}(x) < 0$,then $\psi > 0$ and there exists a point $x_* \in \Omega$,such that either $\psi_1 = \psi$ or $\psi_2 = \psi$ or $\psi_1 = \psi_2 = \psi$ and $(\bar{u} + \psi\bar{t})(x) \geq 0), \forall\ x\ \in \bar{\Omega}$.
Case (i): Consider $x_* \in (\Omega^- \cup \Omega^+)and\psi_1 = \left(-\frac{u_1}{t_1}\right)(x_*) = \psi$.and $(u_1 + \psi t_1)(x_*) = 0$. This shows that $(u_1 + \psi t_1)$ attains its minimum value at $x = x_*$. Hence

$$\begin{aligned} L_1(\bar{u} + \psi\bar{t})(x_*) &= \varepsilon(u_1 + \psi t_1)''(x_*) + \mu a_1(x_*)(u_1 + \psi t_1)'(x_*) \\ &-b_{11}(x_*)(u_1 + \psi t_1)(x_*) + b_{12}(x_*)(u_2 + \psi t_2)(x_*) \geq 0, \end{aligned}$$

which is a contradiction.
Similarly a contradiction would be arrived if we consider $x_* \in (\Omega^- \cup \Omega^+)$and $\psi_2 = \psi \left(-\frac{u_2}{t_2}\right)(x_*) = \psi$.
Case (ii). Consider Consider $x_* = d$ and $\left(-\frac{u_1}{t_1}\right)(x_*) = \psi$. Here again $(u_1 + \psi t_1)(x_*) = 0, (u_1 + \psi t_1)$ attains its minimum value at $x = x_*$. Hence,

$$0 \leq [(u_1 + \psi t_1)'](x_*) = [u_1'](d) + \psi[t_1'](d) \leq 0,$$

which is a contradiction.
Similarly a contradiction is arrived if we choose $\left(-\frac{u_2}{s_2}\right)(x_*) = \psi$and $x_* = d$.
Combining all the above results we prove that our assumption is wrong. Hence $\bar{u}(x) \geq \bar{0}\ \forall\ x\ \in\ \bar{\Omega}$.

An immediate consequence of the minimum principle is the stability result. The Lemmas 3 and 4 can be proved following the the steps and techniques adopted in [12, 7].

Lemma 3. *Let $u_1(x), u_2(x) \in \mathscr{C}^0(\bar{\Omega}) \cap \mathscr{C}^1(\Omega) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$ then*

$$||u_i(x)||_{\bar{\Omega}} \leq C \max\{|u_1(0)|, |u_2(0)|, |u_1(1)|, |u_2(1)|\} + \Sigma_i \left\{\frac{1}{\rho_i}\{||f_i||_{(\Omega^- \cup \Omega^+)}\}\right\},\ x \in \bar{\Omega}, i = 1, 2.$$

Lemma 4. *Let $\bar{u}(x)$ be the solution of the continuous problem (1)-(3), where $|\bar{u}(0)| \leq C,\ |\bar{u}(1)| \leq C$. Then, for all $0 \leq k \leq 3$ satisfying the following bounds*

$$||u_j^{(k)}||_{\bar{\Omega}} \leq \frac{C}{(\sqrt{\varepsilon})^k}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^k\right)\max\{\|y\|, \|f\|\},\ k = 1, 2,$$
$$||u_j^{(3)}||_{\bar{\Omega}} \leq \frac{C}{(\sqrt{\varepsilon})^3}\left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right)\max\{\|y\|, \|f\|, \|f'\|, \}\ j = 1, 2$$

To obtain sharper bounds in the error analysis the solution $\bar{u}(x)$ is decomposed into regular $\bar{v}(x)$ and singular $\bar{w}(x)$ components. It is inevitable to split the analysis into two cases depending on the ratio of μ to $\sqrt{\varepsilon}$ given by $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$ and $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$. For both the cases the solution $\bar{u}(x)$ is decomposed as $\bar{u}(x) = \bar{v}(x) + \bar{w}_l(x) + \bar{w}_r(x)$, where $\bar{v}(x) = (v_1, v_2)^T$ and $\bar{w}(x) = (w_1(x), w_2(x))^T$. The regular component $\bar{v}(x)$ is the solution of

$$L\bar{v}(x) = \bar{f}(x), \quad x \in (\Omega^- \cup \Omega^+), \tag{6}$$
$$\bar{v}(0) = y(0),\ \bar{v}(1) = y(1),\ \bar{v}(d-) \text{ and } \bar{v}(d+) \text{ are chosen}, \tag{7}$$

where

$$\bar{v}(x) = \begin{cases} \bar{v}^-(x), & x \in \Omega^-, \\ \bar{v}^+(x), & x \in \Omega^+. \end{cases}$$

The singular component $\bar{w}_l(x)$ and $\bar{w}_r(x)$ are the solutions of

$$L\bar{w}_l(x) = 0, \quad x \in (\Omega^- \cup \Omega^+), \tag{8}$$
$$\bar{w}_l(0) = \bar{u}(0) - \bar{v}(0),\ \bar{w}_l(1) = 0,$$
$$L\bar{w}_r(x) = 0, \quad x \in (\Omega^- \cup \Omega^+), \tag{9}$$
$$\bar{w}_r(0) = 0,\ \bar{w}_r(1) = \bar{u}(1) - \bar{v}(1),$$
$$[\bar{w}_r](d) = -[\bar{v}](d) - [\bar{w}_l](d) \text{ and } [\bar{w}_r'](d) = -[\bar{v}'](d) - [\bar{w}_l'](d). \tag{10}$$

where

$$\bar{w}_l(x) = \begin{cases} \bar{w}_l^-(x), & x \in \Omega^-, \\ \bar{w}_l^+(x), & x \in \Omega^+, \end{cases} \qquad \bar{w}_r(x) = \begin{cases} \bar{w}_r^-(x), & x \in \Omega^-, \\ \bar{w}_r^+(x), & x \in \Omega^+. \end{cases}$$

Note that $\bar{v}(x)$, $\bar{w}_l(x)$ and $\bar{w}_r(x)$ are discontinuous at $x = d$, but by (10) their sum is in $\mathscr{C}^1(\Omega)$.

Consider the case(i): $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$.
The regular component $\bar{v}(x)$ is represented in the matrix form:

$$\bar{v}(x) = \bar{v}_0 + \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \bar{v}_1 + \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \bar{v}_2.$$

$$\begin{aligned}
&\mathbf{B}\bar{v}_0(x) = f(x), \quad x \in (\Omega^- \cup \Omega^+),\\
&\mathbf{B}(x)\bar{v}_1(x) = \frac{-\mu}{\sqrt{\varepsilon}}\mathbf{A}(x)\bar{v}_0'(x) - \sqrt{\varepsilon}\bar{v}_0''(x), \quad x \in (\Omega^- \cup \Omega^+),\\
&L\bar{v}_2(x) = \frac{-\mu}{\sqrt{\varepsilon}}\mathbf{A}(x)\bar{v}_1'(x) - \sqrt{\varepsilon}\bar{v}_1''(x), \quad x \in (\Omega^- \cup \Omega^+),\\
&\bar{v}_2(0) = \bar{v}_2(1) = 0, \ \bar{v}_2(d-), \ \bar{v}_2(d+) \text{ are chosen,}
\end{aligned}$$

where $\bar{v}_2 \in \mathscr{C}^0(\bar{\Omega}) \cap \mathscr{C}^1(\Omega) \cap \mathscr{C}^2(\Omega^- \cup \Omega^+)$. The Lemmas 5 and 6 can be proved using the principles and methods adopted in [7, 12]

Lemma 5. *The regular component $\bar{v}(x)$ satisfies the following bound*

$$||\bar{v}^{(k)}||_{\Omega\setminus\{d\}} \ \leq \ C\left(1 + \frac{1}{(\sqrt{\varepsilon})^{k-2}}\right), \quad 0 \leq k \leq 3. \qquad \square$$

Lemma 6. *The singular components $\bar{w}_l(x)$ and $\bar{w}_r(x)$ satisfy the following bounds*

$$||\bar{w}^{(k)}||_{\Omega\setminus\{d\}} \ \leq \ \frac{C}{(\sqrt{\varepsilon})^k}\begin{cases} Ce^{-\theta_1 x}, & x \in \Omega^-, \\ Ce^{-\theta_1(x-d)}, & x \in \Omega^+, \end{cases} \quad 0 \leq k \leq 3,$$

$$||\bar{w}^{(k)}||_{\Omega\setminus\{d\}} \ \leq \ \frac{C}{(\sqrt{\varepsilon})^k}\begin{cases} Ce^{-\theta_2(d-x)}, & x \in \Omega^-, \\ Ce^{-\theta_2(1-x)}, & x \in \Omega^+, \end{cases} \quad 0 \leq k \leq 3.$$

with

$$\theta_1 = \frac{\sqrt{\rho\alpha}}{\sqrt{\varepsilon}}, \theta_2 = \frac{\sqrt{\rho\alpha}}{\sqrt{\varepsilon}}. \qquad \square \tag{11}$$

Consider the case(ii): $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$.
Let $\bar{v}(x) = \bar{v}_0(x) + \varepsilon\bar{v}_1(x) + \varepsilon^2\bar{v}_2(x)$, where $\bar{v}_0(x)$, $\bar{v}_1(x)$ and $\bar{v}_2(x)$ be the solution of the problems.

$$\begin{aligned}
&\mu\mathbf{A}(x)\bar{v}_0'(x) - \mathbf{B}(x)\bar{v}_0(x) = f(x), \ x \in (\Omega^- \cup \Omega^+), \ \bar{v}_0(0) = \bar{u}(0), \bar{v}_0(1) = \bar{u}(1)\\
&\mu\mathbf{A}(x)\bar{v}_1'(x) - \mathbf{B}(x)\bar{v}_1(x) = -\bar{v}_0''(x), \ x \in (\Omega^- \cup \Omega^+), \ \bar{v}_1(0) = 0 = \bar{v}_1(1) = 0,\\
&L\bar{v}_2(x) = -\bar{v}_1''(x), \ x \in (\Omega^- \cup \Omega^+),\\
&\bar{v}_2(0) = \bar{v}_2(1) = 0, \ \bar{v}_2(d-), \ \bar{v}_2(d+) \text{ are chosen,}
\end{aligned}$$

where $\bar{v}_2(x) \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.

Lemma 7. *The regular component $\bar{v}(x)$ satisfies the following bounds*

$$||\bar{v}^{(k)}||_{\Omega\setminus\{d\}} \ \leq \ C\left(1 + \left(\frac{\varepsilon}{\mu}\right)^{2-k}\right), \quad 0 \leq k \leq 3. \qquad \square \tag{12}$$

Lemma 8. *The singular components $\bar{w}_l(x)$ and $\bar{w}_r(x)$ satisfy the following bounds*

$$||\bar{w}^{(k)}||_{\Omega\setminus\{d\}} \ \leq \ C\left(\frac{\mu}{\varepsilon}\right)^k\begin{cases} Ce^{-\theta_1 x}, & x \in \Omega^-, \\ Ce^{-\theta_1(x-d)}, & x \in \Omega^+, \end{cases} \quad 0 \leq k \leq 3,$$

$$||\bar{w}^{(k)}(x)||_{\Omega\setminus\{d\}} \ \leq \ C\left(\frac{1}{\mu}\right)^k\begin{cases} Ce^{-\theta_2(d-x)}, & x \in \Omega^-, \\ Ce^{-\theta_2(1-x)}, & x \in \Omega^+, \quad 0 \leq k \leq 3. \end{cases} \qquad \square$$

with

$$\theta_1 = \frac{\alpha\mu}{2\varepsilon}, \quad \theta_2 = \frac{\rho}{2\mu} \tag{13}$$

The unique solution $\bar{u}(x)$ of the continuous problem (1)-(3) is now given by

$$\bar{u}(x) = \begin{cases} \bar{v}^-(x) + \bar{w}_l^-(x) + \bar{w}_r^-(x), & x \in \Omega^-, \\ \bar{v}^-(d-) + \bar{w}_l^-(d-) + \bar{w}_r^-(d-) = \bar{v}^+(d+) + \bar{w}_l^+(d+) + \bar{w}_r^+(d+) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{at } x = d, & \\ \bar{v}^+(x) + \bar{w}_l^+(x) + \bar{w}_r^+(x), & x \in \Omega^+. \end{cases}$$

3 Discrete Problem

The continuous problem is discretized using finite difference methods with suitable Shishkin mesh. On $\bar{\Omega}$ a piecewise uniform mesh size N (let N be even and $N \geq 16$) is constructed as follows. The domain $\bar{\Omega}$ is subdivided into six subintervals as $\bar{\Omega} = [0, \tau_1] \cup [\tau_1, d-\tau_2] \cup [d-\tau_2, d] \cup [d, d+\tau_3] \cup [d+\tau_3, 1-\tau_4] \cup [1-\tau_4, 1]$. The subintervals $[0, \tau_1]$, $[d - \tau_2, d]$, $[d, d + \tau_3]$ and $[1 - \tau_4, 1]$ are scaled with a uniform mesh of $N/8$ mesh intervals, while $[\tau_1, d - \tau_2]$ and $[d + \tau_3, 1 - \tau_4]$ have a uniform mesh with $N/4$ mesh intervals. The step sizes in each subinterval is defined by $H_1 = 8\tau_1/N$, $H_2 = 4(d - \tau_1 - \tau_2)/N$, $H_3 = 8\tau_2/N$, $H_4 = 8\tau_3/N$, $H_5 = 4(1 - d - \tau_3 - \tau_4)/N$ and $H_6 = 8\tau_4/N$. If the discontinuity is assumed at the point $i = d$ then the interior points of the mesh are denoted by

$$\Omega^N = \{x_i : 1 \leq i \leq d - 1\} \cup \{x_i : d + 1 \leq i \leq N - 1\},$$

and the mesh points of the discrete domain are denoted by $\bar{\Omega}^N = \{x_i\}_0^N \cup \backslash\{d\}$. It is obvious that the mesh is uniform when $\tau_1 = \tau_2 = d/4$ and $\tau_3 = \tau_4 = (1 - d)/4$ and $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1/8$ (it is a special case of the discontinuous point at $d = 1/2$). The transition values in $\bar{\Omega}$ are chosen as

$$\begin{cases} \tau_1 = \min\left\{\dfrac{d}{4}, \dfrac{1}{\theta_1}\ln N\right\}, \ \tau_2 = \min\left\{\dfrac{d}{4}, \dfrac{2}{\theta_2}\ln N\right\}, \\ \tau_3 = \min\left\{\dfrac{1-d}{4}, \dfrac{1}{\theta_1}\ln N\right\}, \ \tau_4 = \min\left\{\dfrac{1-d}{4}, \dfrac{2}{\theta_2}\ln N\right\}, \end{cases} \tag{14}$$

where θ_1, θ_2, are defined in the previous section. On the piecewise uniform mesh $\bar{\Omega}^N$ the BVP (1)-(3) is discretized using standard upwind finite difference scheme as follows:
Find a mesh function $\bar{U}(x_i)$, $\forall\ x_i \in \Omega^N$ such that

$$\bar{L}^N\bar{U}(x_i) \equiv \varepsilon\delta^2\bar{U}(x_i) + \mathbf{A}(x_i)D^+\bar{U}(x_i) + \mathbf{B}(x_i)\bar{U}(x_i) = \bar{f}(x_i) \tag{15}$$

$$\bar{U}(x_0) = \bar{u}(0), \bar{U}(x_N) = \bar{u}(1), \tag{16}$$

$$D^-\bar{U}(x_{N/2}) = D^+\bar{U}(x_{N/2}), \tag{17}$$

where the matrix $\mathbf{A}$ and $\mathbf{B}$ are defined in section 2. More precisely, the stiffness matrix is obtained from the above discrete problem with,

$$D^+U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i}, \quad D^-U(x_i) = \frac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}}.$$

$$\delta^2U(x_i) = \frac{2(D^+U(x_i) - D^-U(x_i))}{x_{i+1} - x_{i-1}},$$

The following lemma shows that the finite difference operator L^N has properties equivalent to those of the differential operator L defined in Section2

Lemma 9. *(Discrete minimum principle). Suppose that a mesh function $\bar{U}(x_i)$ satisfies $\bar{U}(x_0) \geq \bar{0}$, $\bar{U}(x_N) \geq \bar{0}$, and $L_1^N\bar{U}_i(x) \leq 0, L_2^N\bar{U}(x) \leq 0 \forall x_i \in (\Omega^- \cup \Omega^+)$ and $D^+\bar{U}(x_{N/2}) - D^-\bar{U}(x_{N/2}) \leq 0$. Then $U(x_i) \geq 0$, $\forall\ x_i \in \bar{\Omega}$.*

Lemma 10. *If $\bar{U}(x_i)$ is any mesh function, then $|\bar{U}(x_i)| \leq \begin{pmatrix} C \\ C \end{pmatrix}$, for all $x_i \in \bar{\Omega}^N$.*

4 Decomposition and Bounds for the Discrete Solution

The error at each mesh point $x_i \in \bar{\Omega}^N$ is denoted by $|\bar{e}(x_i)| = |\bar{U}(x_i) - \bar{u}(x_i)|$. To bound the nodal error $|\bar{e}(x_i)|$, we decompose the solution of the discrete problem (15)- (17)as $\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}_l(x_i) + \bar{W}_r(x_i)$ in a way similar to the decomposition of continuous solutions. To obtain sharper bounds the discrete regular component $\bar{V}(x_i)$ and singular components $\bar{W}_l(x_i)$, $\bar{W}_r(x_i)$ are further decomposed as $\bar{V}^-(x_i)$ and $\bar{V}^+(x_i)$, which approximate $\bar{V}(x_i)$ respectively to the left and right sides of the point of discontinuity $x_i = x_d$ while the mesh functions $\bar{W}_l^-(x_i)$, $\bar{W}_l^+(x_i)$ and $\bar{W}_r^-(x_i)$, $\bar{W}_r^+(x_i)$ approximate respectively $\bar{W}_l(x_i)$ and $\bar{W}_r(x_i)$ on either side of the point of discontinuity $x_i = x_d$. These mesh functions help in deriving the convergence of the nodal error $|\bar{e}(x_i)|$ in the boundary and interior layers.

The regular discrete component $\bar{V}(x_i)$ is defined as

$$\bar{V}(x_i) = \begin{cases} \bar{V}^-(x_i), & \text{for } 1 \le i \le d-1, \\ \bar{V}^+(x_i), & \text{for } d+1 \le i \le N-1, \end{cases}$$

where, $\bar{V}^-(x_i)$ and $\bar{V}^+(x_i)$ are respectively, the solutions of the following discrete problems:

$$L^N \bar{V}^-(x_i) = f(x_i), \text{ for } 1 \le i \le d-1,$$
$$\bar{V}^-(0) = v(0), \ \bar{V}^-(d) = v(d-),$$

and

$$L^N \bar{V}^+(x_i) = \bar{f}(x_i), \text{ for } d+1 \le i \le N-1,$$
$$\bar{V}^+(d) = v(d+), \ \bar{V}^+(1) = v(1).$$

Further the discrete singular components $\bar{W}_l^-(x_i)$, $\bar{W}_l^+(x_i)$, $\bar{W}_r^-(x_i)$ and $\bar{W}_r^+(x_i)$ are defined as

$$\begin{aligned} \bar{W}(x_i) &= \bar{W}_l(x_i) + \bar{W}_r(x_i) \\ &= \begin{cases} (\bar{W}_l^- + \bar{W}_r^-)(x_i), & \text{for } 1 \le i \le d-1, \\ (\bar{W}_l^+ + \bar{W}_r^+)(x_i), & \text{for } N/2+1 \le i \le N-1, \end{cases} \end{aligned}$$

where, $\bar{W}_l^-(x_i)$, $\bar{W}_l^+(x_i)$, $\bar{W}_r^-(x_i)$ and $\bar{W}_r^+(x_i)$ are respectively the solutions of the following discrete problems:

$$\bar{L}^N \bar{W}_l^-(x_i) = 0, \text{ for } 1 \le i \le d-1,$$
$$\bar{W}_l^-(0) = \bar{w}_l^-(0), \ \bar{W}_l^-(d) = \bar{w}_l^-(d),$$
$$\bar{L}^N \bar{W}_l^+(x_i) = 0, \text{ for } d/2+1 \le i \le N-1,$$
$$\bar{W}_l^+(d) = \bar{w}_l^+(d), \ \bar{W}_l^+(1) = \bar{w}_l^+(1),$$
$$\bar{L}^N \bar{W}_r^-(x_i) = 0, \text{ for } 1 \le i \le d-1,$$
$$\bar{W}_r^-(0) = 0, \ \bar{W}_r^-(d) = \bar{w}_r^-(d),$$
$$\bar{L}^N \bar{W}_r^+(x_i) = 0, \text{ for } d/2+1 \le i \le N-1,$$
$$\bar{W}_r^+(d) = 0, \ \bar{W}_r^+(1) = \bar{w}_r^+(1).$$

The solution $\bar{Y}(x_i)$ of the discrete problem (15)- (17) can be now defined as

$$\bar{U}(x_i) = \begin{cases} \left(\bar{V}^- + \bar{W}_l^- + \bar{W}_r^-\right)(x_i), & \text{for } 1 \le i \le d-1, \\ (\bar{V}^- + \bar{W}_l^- + \bar{W}_r^-)(x_i) = \left(V^+ + \bar{W}_l^+ + \bar{W}_r^+\right)(x_i), & \text{for } i = d, \\ \left(\bar{V}^+ + \bar{W}_l^+ + \bar{W}_r^+\right)(x_i), & \text{for } d/2+1 \le i \le N-1. \end{cases}$$

Lemma 11. *The following bounds on $\bar{W}_l^-(x_i), \bar{W}_l^+(x_i), \bar{W}_r^-(x_i)$ and $\bar{W}_r^+(x_i)$ are given by*

$$|\ \bar{W}_l^-(x_i)\ | \le C \prod_{j=1}^{i} (1+\theta_1 h_j)^{-1} = \psi_{li}^-, \ \psi_{l,0}^- = C,$$

$$|\ \bar{W}_l^+(x_i)\ | \le C \prod_{j=N/2+1}^{i} (1+\theta_1 h_j)^{-1} = \psi_{li}^+, \ \psi_{l,d}^+ = C,$$

$$|\ \bar{W}_r^-(x_i)\ | \le C \prod_{j=i+1}^{d} (1+\theta_2 h_j)^{-1} = \psi_{ri}^-, \ \psi_{r,d}^- = C,$$

$$|\ W_r^+(x_i)\ | \le C \prod_{j=i+1}^{N} (1+\theta_2 h_j)^{-1} = \psi_{ri}^+, \ \psi_{r,N}^+ = C,$$

where $h_i = x_i - x_{i-1}$.

$$\bar{\phi}_{li}^{-} = \bar{\psi}_{li}^{-} \pm \bar{W}_l^{-}(x_i) \quad \text{and} \quad \bar{\phi}_{ri}^{-} = \bar{\psi}_{ri}^{-} \pm \bar{W}_r^{-}(x_i),$$

where

$$\bar{\psi}_{li}^{-} = \begin{cases} \prod_{j=1}^{i} (1+\theta_1 h_j)^{-1}, & 1 \le i \le d, \\ 1, & i = 0, \end{cases}$$

$$\bar{\psi}_{ri}^{-} = \begin{cases} \prod_{j=i+1}^{d} (1+\theta_2 h_j)^{-1}, & 0 \le i < d, \\ 1, & i = d, \end{cases}$$

θ_1, θ_2 *are defined in Section 3*

5 Truncation Error Analysis

Lemma 12. *At each mesh point $x_i \in \Omega^N$ the regular component of the truncation error satisfies the following estimate*

$$\|\bar{V} - \bar{v}\| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}. \tag{18}$$

where $\bar{V}$ and $\bar{v}$ are the solutions of the discrete and continuous decompositions defined in Section 4 and 2 respectively.

Proof. Proof Applying the regular arguments on the truncation error and the bounds on $\bar{v}(x)$ for both the cases $\sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}$ and $\sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}$ we get

$$\begin{aligned} |\bar{L}^N(\bar{V}^- - \bar{v}^-)(x_i)| &= |\bar{L}^N(\bar{V}^- - f(x_i)| \\ &\le \left|\varepsilon\left(\delta^2 - \frac{d^2}{dx^2}\right)\right| + \left|\mu a(x_i)\left(D^+ - \frac{d}{dx}\right)\right| \\ &\le C\varepsilon\,(x_{i+1} - x_i)^2\,|\bar{v}^{-(3)}| + \mu a(x_i)\,(x_{i+1} - x_i)\,|\bar{v}^{-(2)}| \\ &\le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}. \end{aligned}$$

Similarly, we could prove that

$$|\bar{L}^N(\bar{V}^- - \bar{v}^-)(x_i)| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}, \text{ for } d+1 \le i \le N-1.$$

Define the barrier function

$$\bar{\psi}_i^{\pm} = CN^{-1} \pm (\bar{V}^- - \bar{v}^-)(x_i), \ \forall\ x_i \in \bar{\Omega}^N.$$

It is clear that $\bar{\psi}^{\pm}(0) \ge 0$ and $\psi^{\pm}(1) \ge 0$. For greater values of C, $\bar{L}^N\bar{\psi}_i^{\pm} \le 0$. Applying Lemma 9 on $\bar{\psi}_i^{\pm}$ we get $\bar{\psi}_i^{\pm} \ge 0$ for $0 \le i \le N$. Combining the above results, we obtain

$$\|\bar{V}^- - \bar{v}^-)\| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}. \ \forall\ x_i \in \Omega^N. \qquad \square$$

Lemma 13. *At each mesh point $x_i \in \Omega^N$ the right singular component of the truncation error satisfies the following estimate*

$$\|\bar{W}_r - \bar{w}_r\| \le \begin{cases} \begin{pmatrix} CN^{-1}(\ln N) \\ CN^{-1}(\ln N) \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \le \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \ge \sqrt{\rho\varepsilon}. \end{cases}$$

Proof. Proof Using the classical argument for the truncation error and the Lemma 4, we obtain

$$\begin{aligned}
|\bar{L}^N(\bar{W}_r^- - \bar{w}_r^-)(x_i)| &= |(\bar{L}^N - \bar{L})\bar{w}_r^-(x_i)| \\
&\leq \begin{pmatrix} C(h_{i+1}+h_i)\varepsilon \\ C(h_{i+1}+h_i)\varepsilon \end{pmatrix} \|(\bar{w}_r^-)^3\| + \mu\|(\bar{w}_r^-)^2\| \\
&\leq \begin{pmatrix} C(h_{i+1}+h_i) \\ C(h_{i+1}+h_i) \end{pmatrix} \left\{ \frac{1}{\sqrt{\varepsilon}}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right) + \frac{\mu}{\varepsilon}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^2\right)\right\} \\
&\leq \begin{pmatrix} C(h_{i+1}+h_i) \\ C(h_{i+1}+h_i) \end{pmatrix} \frac{1}{\sqrt{\varepsilon}}\left(1+\left(\frac{\mu}{\sqrt{\varepsilon}}\right)^3\right). \qquad (19)
\end{aligned}$$

If $\tau_2 = d/4$, the mesh is uniform and $\theta_1 \leq 16\ln N$. Hence (19) reduces to

$$|\bar{L}^N(\bar{W}_r^- - \bar{w}_r^-)(x_i)| \leq \begin{cases} \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix} \left(\frac{1}{\sqrt{\varepsilon}}\right), & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix} \left(\frac{\mu^3}{\varepsilon^2}\right), & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases} \qquad (20)$$

The error in the fine mesh region $(d-\tau_2, d)$ and the coarse mesh region $(0, d-\tau_2]$ are analysed using the bounds defined in lemmas 6 and 8. On simplifying (19) we obtain

$$|\bar{L}^N(\bar{W}_r^- - \bar{w}_r^-)(x_i)| \leq \begin{pmatrix} C_1N^{-1}\ln N \\ C_1N^{-1}\ln N \end{pmatrix} + \begin{pmatrix} C_2N^{-1}\ln N \\ C_2N^{-1}\ln N \end{pmatrix} \left(\frac{\mu}{\varepsilon}\right)\ln N. \qquad (21)$$

Using the values of θ_1 and $\tau_2 \leq \dfrac{2}{\theta_1}\ln N$, we get

$$|\bar{L}^N(\bar{W}_r^- - \bar{w}_r^-)(x_i)| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}\mu(\ln N)^2 \\ CN^{-1}\mu(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases} \qquad (22)$$

Using the barrier function technique and discrete minimum principle we derive

$$\|(\bar{W}_r^- - \bar{w}_r^-)\| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases}$$

Following similar arguments in the domain $(d, 1-\tau_4]$ and $(1-\tau_4, 1)$ we obtain

$$\|(\bar{W}_r^- - \bar{w}_r^-)\| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}, \end{cases} \qquad (23)$$

Combining the results (5) and (23) the desired result is obtained. □

Lemma 14. *At each mesh point $x_i \in \Omega^N$ the left singular component of the truncation error satisfies the following estimate*

$$\|(\bar{W}_l^- - \bar{w}_l^-)\| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases}$$

Proof. Proof Using the derivative estimate, we have

$$|\bar{L}^N(\bar{W}_l^- - \bar{w}_l^-)(x_i)| \leq \begin{pmatrix} C(h_{i+1}+h_i)\varepsilon \\ C(h_{i+1}+h_i)\varepsilon \end{pmatrix} \left\{ \frac{1}{\sqrt{\varepsilon}} \left(1 + \left(\frac{\mu}{\sqrt{\varepsilon}} \right)^3 \right) \right\}. \tag{24}$$

We begin with the case $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$. If $\tau_1 = \dfrac{d}{4}$, with $\sqrt{\alpha}\mu/\sqrt{\varepsilon} \leq \theta_2 \leq 16\ln N$ we obtain the following bounds on the left singular component given by

$$|\bar{L}^N(\bar{W}_l^- - \bar{w}_l^-)(x_i)| \leq \frac{C}{\sqrt{\varepsilon}}(h_{i+1}+h_i) \leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

When $\tau_1 < \dfrac{d}{4}$, we have to analyse the error in the fine and coarse mesh regions separately. Following the methods applied in [12] we could prove the bound for the coarse mesh region $[\tau_1, d)$ given by

$$|\bar{L}^N(\bar{W}_l^- - \bar{w}_l^-)(x_i)| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix} \frac{\sqrt{\rho\alpha}}{\sqrt{\varepsilon}} \leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

In the fine mesh region $(0, \tau_1)$ the inequality (24) still holds and we have $h_i = h_{i+1} = \dfrac{16N^{-1}\ln N}{\theta_2}$. Using the value of θ_2 we have $\dfrac{\sqrt{\rho\alpha}(h_{i+1}-h_i)}{\sqrt{\varepsilon}} \leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}$. Hence

$$|\bar{L}^N(\bar{W}_l^- - \bar{w}_l^-)(x_i)| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix} \frac{\sqrt{\rho\alpha}}{\sqrt{\varepsilon}} \leq \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}.$$

Consider the case $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$.
Following the methods constructed for the error bounds in [12] when the transition point $\tau_2 = d/4$, and $\dfrac{\rho}{\mu} \leq \theta_2 \leq 16\ln N$ we obtain

$$|\bar{L}^N(\bar{W}_l^- - \bar{w}_l^-)(x_i)| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix} \left(\frac{\rho}{\mu^2} \right) \leq \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}.$$

For the coarse mesh region $[\tau_1, d)$, if the transition point $\tau_1 < \dfrac{d}{4}$ the error bound in (24) still holds. In the fine mesh region $(0, \tau_1)$ the inequality (24) reduces to

$$|\bar{L}^N(\bar{W}_l^- - \bar{w}_l^-)(x_i)| \leq \begin{pmatrix} C \\ C \end{pmatrix} \frac{h_{i+1}+h_i}{\mu}^2.$$

Since $h_{i+1} = h_i = \left(\dfrac{16}{\theta_2} \right) \ln N$ and using the value of θ_2 we arrive at the following error bound for the left singular component

$$|\bar{L}^N(\bar{W}_l^- - \bar{w}_l^-)(x_i)| \leq \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}.$$

Using the barrier function technique and discrete minimum principle we can prove that

$$\|(\bar{W}_l^- - \bar{w}_l^-)\| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases}$$

Similarly, we can prove the result for $d+1 \leq i < N$ by taking appropriate transition parameter τ_3 to obtain

$$\|\bar{L}^N(\bar{W}_l^+ - \bar{w}_l^+)\| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases}$$

In particular $d = 1/2$ could be seen as a special case here since, $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1/8$. If $\tau_1 < d/4$ and , $\tau_2 < d/4$, then the mesh is piecewise uniform. Combining all the results analysed above we can say that

$$\|\bar{W} - \bar{w}\| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases} \qquad \square$$

Lemma 15. *At the point of discontinuity $x_d = d$, the error $\bar{e}(x_d)$ satisfies the following estimate*

$$|(D^+ - D^-)\bar{e}(x_d)| \leq \begin{cases} \left(\frac{\rho\alpha\tau}{N\varepsilon}, \frac{\rho\alpha\tau}{N\varepsilon}\right)^T, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \left(\frac{\alpha\tau\mu^2}{N\varepsilon^2}, \frac{\alpha\tau\rho^2}{N\mu^2}\right)^T, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}, \end{cases}$$

where $\tau = min\{\tau_2, \tau_3\}$.

The next theorem presents the main contribution of the article, which conveys the ε-μ-uniform convergence error estimate.

Theorem 16. *Let $\bar{u}(x)$ and $\bar{U}(x_i)$ be respectively the solutions of the problems (1), and (15). Then, for sufficiently large N, we have*

$$\|\bar{U} - \bar{u}\| \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \text{if } \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}. \end{cases}$$

Proof. From the results of Lemma 4, 12, 13, 14 and using the procedure adopted in [12], it follows that

$$e(x_i) \leq \begin{cases} \begin{pmatrix} CN^{-1}\ln N \\ CN^{-1}\ln N \end{pmatrix}, & \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}, \\ \begin{pmatrix} CN^{-1}(\ln N)^2 \\ CN^{-1}(\ln N)^2 \end{pmatrix}, & \sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}, \end{cases} \quad \forall\ x_i \in\ \Omega^N. \tag{25}$$

To prove the desired error at the point of discontinuity $x_i = x_d$:
Consider the case $\sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}$. Define the discrete barrier function $\phi_{j*}(x_i)$ for $j = 1, 2$ to be the solution of

$$\varepsilon\delta^2\phi_{j*}(x_i) + \mu\alpha_j^*(x_i)D^*\phi_{j*}(x_i) - \beta_j(x_i)\phi_{j*}(x_i) = 0,$$
$$\phi_{j*}(x_0) = 0,,\ \phi_{j*}(x_d) = 1\text{and } \phi_{j*}(x_N) = 0.$$

We can prove that

$$D^-\phi_{j*}(x_i) \geq\ 0,\ \text{for } 1 \leq i \leq d-1 \ \text{ and}$$
$$D^+\phi_{j*}(x_i) \leq\ 0,\ \text{for } 1 \leq i \leq d+1.$$

Note that $\bar{L}^N\phi_{j*}(x_i) \leq 0$ for all $x_i \in \Omega^N$. using the procedure adopted from [6] we can prove the following result at the point $x_d = d$,

$$\begin{aligned} \left(D^+\phi_{j*} - D^-\phi_{j*}\right)(x_d) &= \frac{\phi_{j*}(x_d + h_4) - 1}{h_4} - \frac{\phi_{j*}(x_d + h_3) - 1}{h_3} \\ &\leq \left(D^+y_2(x_d) - D^-y_1(x_d)\right) \pm \frac{C(N^{-1}\ln N)^2}{max(h_3, h_4)} \\ \left(D^+\phi_{j*} - D^-\phi_{j*}\right)(x_d) &\leq -C/\sqrt{\varepsilon}. \end{aligned}$$

Consider the barrier function for $j = 1, 2$

$$\psi_{j1}^{\pm}(x_i) = C_3 N^{-1} \ln N + C_4 \frac{h}{\sqrt{\varepsilon}} \phi_{j*}(x_i) \pm e(x_i), \quad \forall\ x_i \in \bar{\Omega}^N.$$

Now, $\psi_{j1}^{\pm}(x_0) \geq 0$, $\psi_{j1}^{\pm}(x_N) \geq 0$ and $\bar{L}^N \psi_{j1}^{\pm}(x_i) \leq 0, \quad x_i \in \Omega^N$,
$(D^+ - D^-)\psi_{j1}^{\pm}(x_d) \leq 0, \quad i = d.$

Hence applying the discrete minimum principle, we get $\psi_{j1}^{\pm}(x_i) \geq 0\ \forall\ x_i \in \bar{\Omega}^N$. For sufficiently large N we derive

$$|(\bar{U} - \bar{u})(x_i)| \leq CN^{-1}(\ln N), \quad \sqrt{\alpha}\mu \leq \sqrt{\rho\varepsilon}. \tag{26}$$

In the second case when $\sqrt{\alpha}\mu \geq \sqrt{\rho\varepsilon}$, consider the discrete barrier function $\psi_{j2}(x_i) = \psi(x_i) \pm e(x_i)$ for $j = 1, 2$ defined in the interval $(d - \tau_2, d + \tau_3)$ where

$$\psi(x_i) = CN^{-1}(\ln N)^2 + \begin{cases} \dfrac{CN^{-1}\tau_2}{\mu^2}(x_i - d - \tau_2), & x_i \in (d - \tau_2, d], \\ \dfrac{CN^{-1}\tau_3\mu^2}{\varepsilon^2}(d + \tau_3 - x_i), & x_i \in [d, d + \tau_3). \end{cases}$$

It could be seen that $\psi_2(d - \tau_3) > 0$, $\psi_{j2}(d + \tau_3) > 0$ and $\bar{L}^N \psi_{j2}(x_i) < 0$
and $D^+\psi_{j2}(x_i) - D^-\psi_{j2}(x_i) < 0$.

Applying the discrete minimum principle to $\psi_{j2}(x_i)$, we find that $\psi_{j2}(x_i) \geq 0$. Hence

$$|(U - u)(x_i)| \leq \begin{cases} \dfrac{CN^{-1}\tau_2^2}{\mu^2} \text{ for } x_i \in (d - \tau_2, d + \tau_3) \\ \dfrac{CN^{-1}\tau_3^2\mu^2}{\varepsilon^2} \text{ for } x_i \in (d - \tau_2, d + \tau_3) \end{cases} \leq CN^{-1}(\ln N)^2. \tag{27}$$

Therefore by combining (26) and (27) we obtain the desired result. □

6 Numerical Example

In order to find the applicability of the present method, we have considered the problems of singularly perturbed two parameter BVP with discontinuous convection coefficient and source term.

*Example*1

$$-\varepsilon \bar{y}_1''(x) - \mu \mathbf{A}(x)\bar{y}'(x) + \mathbf{B}(x)\bar{y}(x) = \bar{f}(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$\bar{y}(0) = (1, 1)^T \quad \bar{y}(1) = (0, 0)^T,$$

where

$$\mathbf{A}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ for } 0 < x < 1 \ \ \mathbf{B}(x) = \left\{ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \right.,$$

$$f_1(x) = \begin{cases} 0.5, & \text{for } 0 < x < 0.5 \\ -0.6, & \text{for } 0.5 < x < 1 \end{cases} \quad and\ f_2(x) = \begin{cases} 0.2, & \text{for } \ 0 < x < 0.5 \\ -2 + x, & \text{for } 0.5 < x < 1 \end{cases}$$

*Example*2

$$-\varepsilon \bar{y}_1''(x) - \mu \mathbf{A}(x)\bar{y}'(x) + \mathbf{B}(x)\bar{y}(x) = \bar{f}(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$\bar{y}(0) = (0, 0)^T \quad \bar{y}(1) = (1, 1)^T,$$

$$\mathbf{A}(x) = \begin{pmatrix} 1 + x & 0 \\ 0 & 2 + x \end{pmatrix} \text{ for } 0 < x < 1 \ \ \mathbf{B}(x) = \left\{ \begin{pmatrix} 2 + x & -(1 + x) \\ -(1 + x) & x + 2 \end{pmatrix} \right.,$$

$$f_1(x) = \begin{cases} 1 + exp(x), & \text{for } 0 < x < 0.5 \\ -(2 + x^2), & \text{for } 0.5 < x < 1 \end{cases} \quad and \ \ f_2(x) = \begin{cases} exp(x), & \text{for } 0 < x < 0.5 \\ -sin(\pi x/2), & \text{for } 0.5 < x < 1 \end{cases}$$

Table 1. Maximum point-wise errors E^N order of convergence R^N for y_1 of Example 1 when $\mu = 2^{-10}$.

ε_1	64	128	256	512	1024	2048
2^0	7.086e-004	3.522e-004	1.756e-004	8.766e-005	4.380e-005	2.189e-005
2^{-2}	1.270E-003	6.245E-004	3.097E-004	1.542E-004	7.694E-005	3.843E-005
2^{-4}	2.092e-003	1.049e-003	5.255e-004	2.629e-004	1.315e-004	6.577e-005
2^{-6}	5.170e-003	2.569e-003	1.280e-003	6.391e-004	3.193e-004	1.596e-004
2^{-8}	6.498e-003	3.212e-003	1.598e-003	7.974e-004	3.983e-004	1.990e-004
− − −	− − −	− − −	− − −	− − −	− − −	− − −
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-28}	1.545e-001	1.003e-001	6.119e-002	3.571e-002	2.021e-002	1.120e-002
2^{-30}	1.562e-001	1.017e-001	6.209e-002	3.630e-002	2.054e-002	1.139e-002
2^{-32}	1.567e-001	1.020e-001	6.232e-002	3.644e-002	2.063e-002	1.144e-002
2^{-34}	1.568e-001	1.021e-001	6.238e-002	3.648e-002	2.065e-002	1.145e-002
− − −	− − −	− − −	− − −	− − −	− − −	− − −
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-50}	1.568e-001	1.021e-001	6.240e-002	3.649e-002	2.066e-002	1.145e-002
E^N	1.568e-001	1.021e-001	6.240e-002	3.649e-002	2.066e-002	1.145e-002
R^N	0.6189	0.7104	0.7740	0.8207	0.8515	0.8718

Table 2. Maximum point-wise errors E^N and orders of convergence R^N for y_2 of Example 1 when $\mu = 2^{-10}$.

ε	64	128	256	512	1024	2048
2^0	3.291e-004	1.664e-004	8.366e-005	4.194e-005	2.100e-005	1.051e-005
2^{-2}	1.264e-003	6.372e-004	3.199e-004	1.603e-004	8.023e-005	4.013e-005
2^{-4}	4.160e-003	2.087e-003	1.045e-003	5.228e-004	2.615e-004	1.308e-004
2^{-6}	7.675e-003	3.825e-003	1.909e-003	9.539e-004	4.768e-004	2.383e-004
2^{-8}	9.087e-003	4.510e-003	2.248e-003	1.122e-003	5.608e-004	2.803e-004
− − −	− − −	− − −	− − −	− − −	− − −	
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-28}	1.399e-001	8.986e-002	5.526e-002	3.232e-002	1.831e-002	1.015e-002
2^{-30}	1.403e-001	9.016e-002	5.545e-002	3.244e-002	1.838e-002	1.019e-002
2^{-32}	1.404e-001	9.024e-002	5.550e-002	3.247e-002	1.840e-002	1.020e-002
2^{-34}	1.405e-001	9.026e-002	5.551e-002	3.248e-002	1.840e-002	1.021e-002
− − −	− − −	− − −	− − −	− − −	− − −	
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-36}	1.405e-001	9.026e-002	5.551e-002	3.248e-002	1.840e-002	1.021e-002
2^{-50}	1.405e-001	9.026e-002	5.551e-002	3.248e-002	1.840e-002	1.021e-002
E^N	1.405e-001	9.026e-002	5.551e-002	3.248e-002	1.840e-002	1.021e-002
R^N	0.6384	0.7013	0.7732	0.8198	0.8497	

Table 3. Maximum point-wise errors E^N and orders of convergence R^N for y_1, y_2 of Example 1 for various μ with ε varying from $2^0 - 2^{-50}$

μ	N=	64	128	256	512	1024	2048
2^-15	E_1^N	6.078e-002	3.973e-002	2.439e-002	1.434e-002	8.152e-003	4.531e-003
	R^N	0.6134	0.7039	0.7662	0.8148	0.8473	0.8689
	E_2^N	7.867e-002	5.161e-002	3.205e-002	1.880e-002	1.068e-002	5.937e-003
	R^N	0.6082	0.6873	0.7696	0.8158	0.8471	0.8697
2^{-7}	E_1^N	1.590e-001	1.033e-001	6.302e-002	3.683e-002	2.082e-002	1.061e-002
	R^N	0.6222	0.7130	0.7749	0.8229	0.9725	0.9867
	E_2^N	1.439e-001	9.216e-002	5.674e-002	3.318e-002	1.876e-002	9.561e-003
	R^N	0.6429	0.6998	0.7741	0.8227	0.9724	0.9860

Table 4. Maximum point-wise errors E^N and orders of convergence R^N for y_1 of Example 2 when $\mu = 2^{-8}$.

ε	64	128	256	512	1024	2048
2^0	7.572e-004	3.632e-004	1.779e-004	8.799e-005	4.376e-005	2.182e-005
2^{-2}	2.613e-003	1.375e-003	7.044e-004	3.566e-004	1.794e-004	8.998e-005
2^{-4}	1.024e-002	5.193e-003	2.614e-003	1.311e-003	6.565e-004	3.285e-004
2^{-6}	2.421e-002	1.176e-002	5.784e-003	2.867e-003	1.427e-003	7.121e-004
2^{-8}	4.625e-002	2.187e-002	1.058e-002	5.202e-003	2.578e-003	1.283e-003
− − −	− − −	− − −	− − −	− − −	− − −	− − −
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-32}	4.992e-001	4.048e-001	2.750e-001	1.860e-001	1.120e-001	5.952e-002
2^{-34}	4.993e-001	4.048e-001	2.750e-001	1.861e-001	1.120e-001	5.953e-002
2^{-36}	4.993e-001	4.048e-001	2.750e-001	1.861e-001	1.120e-001	5.953e-002
− − −	− − −	− − −	− − −	− − −	− − −	− − −
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-50}	4.993e-001	4.048e-001	2.750e-001	1.861e-001	1.120e-001	5.953e-002
E^N	4.993e-001	4.048e-001	2.750e-001	1.861e-001	1.120e-001	5.953e-002
R^N	0.3027	0.5578	0.5634	0.7326	0.9118	

Table 5. Maximum point-wise errors E^N and orders of convergence R^N for y_2 of Example 2 when $\mu = 2^{-8}$.

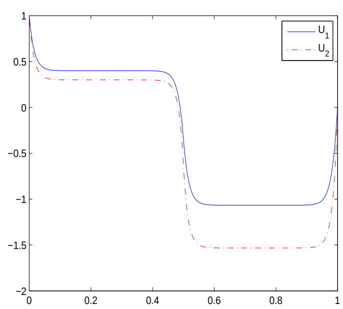
ε	64	128	256	512	1024	2048
2^0	7.241e-004	3.884e-004	2.010e-004	1.022e-004	5.156e-005	2.589e-005
2^{-2}	2.722e-003	1.442e-003	7.415e-004	3.759e-004	1.892e-004	9.495e-005
2^{-4}	9.360e-003	4.808e-003	2.435e-003	1.225e-003	6.146e-004	3.078e-004
2^{-6}	2.132e-002	1.048e-002	5.188e-003	2.581e-003	1.287e-003	6.429e-004
2^{-8}	3.330e-002	1.563e-002	7.548e-003	3.706e-003	1.836e-003	9.136e-004
− − −	− − −	− − −	− − −	− − −	− − −	− − −
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-28}	3.969e-001	3.074e-001	2.205e-001	1.473e-001	8.892e-002	4.768e-002
2^{-30}	3.970e-001	3.076e-001	2.206e-001	1.474e-001	8.897e-002	4.771e-002
2^{-32}	3.970e-001	3.076e-001	2.206e-001	1.474e-001	8.898e-002	4.772e-002
2^{-34}	3.970e-001	3.076e-001	2.206e-001	1.474e-001	8.899e-002	4.772e-002
− − −	− − −	− − −	− − −	− − −	− − −	− − −
− − −	− − −	− − −	− − −	− − −	− − −	− − −
2^{-50}	3.970e-001	3.076e-001	2.206e-001	1.474e-001	8.899e-002	4.772e-002
E^N	3.970e-001	3.076e-001	2.206e-001	1.474e-001	8.899e-002	4.772e-002
R^N	0.3681	0.4796	0.5817	0.7280	0.8990	

Table 6. Maximum point-wise errors E^N and orders of convergence R^N for y_1, y_2 of Example 2 for various μ with ε varying from $2^0 - 2^{-50}$

μ	N=	64	128	256	512	1024	2048
2^-15	E_1^N	1.514e-001	9.416e-002	5.600e-002	3.214e-002	1.802e-002	9.936e-003
	R^N	0.6852	0.7497	0.8011	0.8348	0.8589	0.8757
	E_2^N	1.174e-001	7.329e-002	4.362e-002	2.506e-002	1.406e-002	7.755e-003
	R^N	0.6797	0.7486	0.7996	0.8338	0.8584	0.8758
2^{-10}	E_1^N	2.705e-001	1.841e-001	1.146e-001	6.734e-002	3.828e-002	2.125e-002
	R^N	0.5551	0.6839	0.7671	0.8149	0.8491	0.8709
	E_2^N	2.163e-001	1.448e-001	9.042e-002	5.374e-002	3.075e-002	1.717e-002
	R^N	0.5790	0.6793	0.7506	0.8054	0.8407	0.8669

7 Conclusion

A weakly coupled system of two Singularly Perturbed Diffusion-Convection-Reaction Problems SPDCRPs) having two small parameters (ε, μ) multiplying the diffusion and convection term with discontinuity over source term is studied. Theoretical bounds on the derivatives, regular and singular components of the solution were derived. An upwind finite difference scheme was constructed on Shishkin mesh to obtain almost first-order convergence. Parameter uniform error bounds for the numerical approximation were illustrated. Numerical examples considered for study support the theoretical results derived.



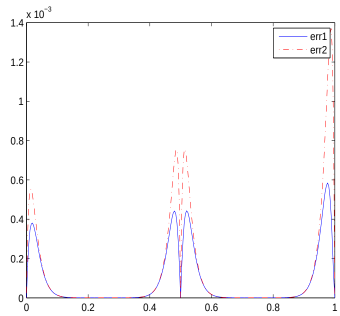
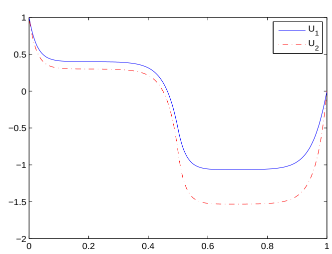


Figure 1: (a) Numerical Solution Figure 2: (b) Error

Figures 1,2 represent Plot of Numerical Solution and Error for $\varepsilon_1 = 2^{-12}$, $\varepsilon_2 = 2^{-20}$ when $N = 256$ for $Example1$



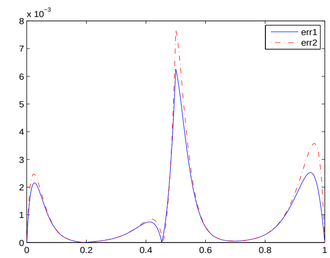
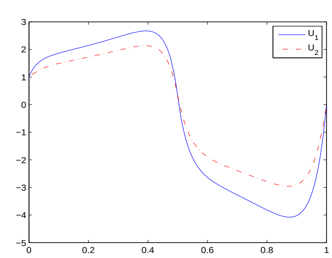


Figure 3: (a) Numerical Solution Figure 4: (b) Error

Figures 3,4 represent Plot of Numerical Solution and Error for $\varepsilon_1 = 10^{-6}$, $\varepsilon_2 = 10^{-4}$ when $N = 256$ for $Example1$



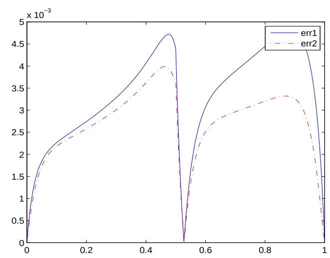


Figure 5: (a) Numerical Solution Figure 6: (b) Error

Figures 5,6 represent Plot of Numerical Solution and Error for $\varepsilon_1 = 2^{-10}$, $\varepsilon_2 = 2^{-16}$ when $N = 256$ for $Example2$

References

[1] P. M. Basha and V. Shanthi. Fitted mesh method for a weakly coupled system of singularly perturbed reaction–convection–diffusion problems with discontinuous source term. *Ain Shams Engineering Journal*, 2016.

[2] M. Chandru, T. Prabha, and V. Shanthi. A hybrid difference scheme for a second-order singularly perturbed reaction-diffusion problem with non-smooth data. *International Journal of Applied and Computational Mathematics*, 1(1):87–100, 2015.

[3] M. Chandru, T. Prabha, and V. Shanthi. A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data. *Journal of Computational and Applied Mathematics*, 309:11–27, 2017.

[4] M. Chandru and V. Shanthi. Fitted mesh method for singularly perturbed robin type boundary value problem with discontinuous source term. *International Journal of Applied and Computational Mathematics*, 1(3):491–501, 2015.

[5] E. P. Doolan, J. J. H. Miller, and W. H. Schilders. *Uniform numerical methods for problems with initial and boundary layers*, volume 1. Boole Press, 1980.

[6] P. Farrell, J. Miller, E. ORiordan, and G. Shishkin. Singularly perturbed differential equations with discontinuous source terms. In *Proceedings of Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems, Lozenetz, Bulgaria*, pages 23–32, 1998.

[7] P. A. Farrell, A. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin. *Robust computational techniques for boundary layers*. CRC Press, 2000.

[8] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin. Singularly perturbed convection–diffusion problems with boundary and weak interior layers. *Journal of Computational and Applied Mathematics*, 166(1):133–151, 2004.

[9] P. A. Farrell, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin. Singularly perturbed differential equations with discontinuous source terms. In *Proc. Lozenetz*, 2000.

[10] N. Madden, M. Stynes, and G. Thomas. On the application of robust numerical methods to a complete-flow wave-current model. *Proceedings of BAIL*, 2004, 2004.

[11] S. Matthews, E. O'Riordan, and G. Shishkin. A numerical method for a system of singularly perturbed reaction–diffusion equations. *Journal of Computational and Applied Mathematics*, 145(1):151–166, 2002.

[12] E. O'Riordan, M. L. Pickett, and G. I. Shishkin. Singularly perturbed problems modeling reaction-convection-diffusion processes. *Computational Methods in Applied Mathematics*, 3(3):424–442, 2003.

[13] T. Prabha, M. Chandru, and V. Shanthi. Hybrid difference scheme for singularly perturbed reaction-convection-diffusion problem with boundary and interior layers. *Applied Mathematics and Computation*, 314:237–256, 2017.

[14] H. G. Roos, M. Stynes, and L. Tobiska. *Numerical Methods for Singularly Perturbed Differential Equations*. Springer, 1996.

[15] V. Shanthi, N. Ramanujam, and S. Natesan. Fitted mesh method for singularly perturbed reaction-convection-diffusion problems with boundary and interior layers. *Journal of Applied Mathematics and Computing*, 22(1-2):49–65, 2006.

[16] A. Tamilselvan, N. Ramanujam, and V. Shanthi. A numerical method for singularly perturbed weakly coupled system of two second order ordinary differential equations with discontinuous source term. *Journal of computational and Applied Mathematics*, 202(2):203–216, 2007.