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Some results on G-fuzzy normed linear space

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Abstract

The aim of this paper is to introduce G-fuzzy norm and to study it's underlying topology. Decomposition theorem from a G-fuzzy norm to a family of G-norm is established.

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Key Words and Phrases: G-fuzzy norm, G-fuzzy metric, Fuzzy norm, Quasi G-norm, G-norm.

1 Introduction

Gähler ([5] [6]) initiated and investigated the notion of 2-metric spaces and 2-normed spaces in sxities. Gähler claimed that 2-metric is more general form of the standard metric. But different authors tested that there is no relation between these two functions. For instance Ha et. al ([7]) shows that there are some 2-metric which are not continuous function of its variables. In the year 1984, B. C. Dhange ([4]) generalized the concept of usual metric and invent the idea of D-metric to translate results from usual metric to D-metric. But from topological aspect it is found that D-metric spaces is incorrect ([10]). Finally in ([11]) Mustafa and Sims presented an idea of G-metric in which tetrahedral inequality is replaced by an

inequality involving repetition of indices and studied the relation between G-metric topology and metric topology. Following the concept of G-metric, in 2012 Sun and Yang ([15]) developed the idea of Q-fuzzy metric and studied some fixed point theory in such spaces. On the other hand to generalize the concept of normed linear space, in 2014 K. A. Khan ([8]) proposed the concept of generalized norm say G-norm. Following the idea of Q-fuzzy metric and G-norm, the notion of G-fuzzy normed linear space is introduced in this paper, which can be considered as a generalization of fuzzy normed linear space introduced by Bag and Samanta in 2012 ([3]). Underlying topology of G-fuzzy normed linear space is also studied.
The paper is organized as follows:
Section 2 some preliminary results are discussed. In section 3, G-fuzzy norm is introduced and its underlying topology is studied. In section 4, idea of generating space of quasi G-norm and G-norm family is introduced and some basic results are studied. In section 5, A decomposition theorem from the G-fuzzy norm into a family of quasi G-norm is formulated. A G-fuzzy norm is originated from a generating space of quasi G-norm family. In section 6, the definition of equipotent G-fuzzy norm is given and relation between a G-fuzzy norm and the G-fuzzy norm formulated from its generating space of quasi G-norm family is established.

2 Preliminaries

Definition 2.1. *[11] A G-metric or generalized metric is an ordered pair (X, G) such that X is a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ is a function satisfying:*

$(G1)$ *If $x = y = z$ then $G(x, y, z) = 0$;*

$(G2)$ *For all $x, y \in X$, with $x \neq y$, $G(x, x, y) > 0$;*

$(G3)$ *$G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;*

$(G4)$ *For all $x, y, z \in X$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots\ldots$ and*

$(G5)$ *$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.*

Definition 2.2. *[11] In a G-metric space (X, G) a sequence $\{x_n\} \subseteq X$ is G-convergent to x if it converges to x in the G-metric topology, $\tau(G)$, where $\tau(G)$ is generated by the family of $\mathcal{B} = \{B_G(x, r) = \{y \in X : G(x_0, y, y) < r\} : x \in X, r > 0\}$*

Note 2.3. *[11] $d_G(x, y) = G(x, y, y) + G(x, x, y)$ defines a metric on X, for any G-metric G on X,*

Proposition 2.4. *[11] For a sequence $\{x_n\}$ in G-metric space (X, G) and point $x \in X$ the following statements are equivalent.*

(1) $\{x_n\}$ *is G-convergent to* x.

(2) As $n \to \infty$, $d_G(x_n, x) \to 0$.

(3) As $n \to \infty$, $G(x_n, x_n, x) \to 0$.

(4) As $n \to \infty$, $G(x_n, x, x) \to 0$.

(5) As $m, n \to \infty$, $G(x_m, x_n, x) \to 0$.

Definition 2.5. *[8] An ordered pair $(X, \|.,.,.\|)$ is called a G-normed space if X is a real vector space and $\|.,.,\| : X^3 \to \mathbb{R}$ is function on X satisfies the following conditions:*

$(N1)$ $\|x, y, z\| \geq 0$ *and* $x = y = z = 0$ *if and only if* $\|x, y, z\| = 0$,

$(N2)$ $\|x, y, z\|$ *is invariant under permutation of* x, y *and* z,

$(N3)$ $\forall\ \alpha \in \mathbb{R}$ *and* $x, y, z \in X$, $\|\alpha x, \alpha y, \alpha z\| = |\alpha|\|x, y, z\|$,

$(N4)$ $\|x + x', y + y', z + z'\| \leq \|x, y, z\| + \|x', y', z'\|$ $\forall x, y, z, x', y', z' \in X$,

$(N5)$ $\|x + y, \theta, z\| \leq \|x, y, z\|\ \forall x, y, z \in X$, *where* θ *be the origin of the vector space* X.

Definition 2.6. *[9] A t-norm is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ satisfying following conditions:*
(1) $*$ *is commutative and associative;*
(2) $a * 1 = a\ \forall a \in [0, 1]$;
(3) For each $a, b, c, d \in [0, 1]$, $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$.

Definition 2.7. *[3] A triplet $(U, N, *)$ is called a fuzzy normed space if U is a linear space over the field $\mathbb{F}$ ($\mathbb{C}$ or $\mathbb{R}$) and a fuzzy subset N of $X \times \mathbb{R}$ ($\mathbb{R}$-the set of all real numbers) is called a fuzzy norm on U if*
$(N1)$ $\forall t \in \mathbb{R}$ *with* $t \leq 0,\ N(x,t) = 0$;
$(N2)$ $(\forall t \in \mathbb{R},\ t > 0,\ N(x,t) = 1)\ iff\ x = \underline{0}$;
$(N3)$ $\forall t \in \mathbb{R},\ t > 0,\ N(cx,t) = N(x,\ \frac{t}{|c|})\ if\ c \neq 0$;
$(N4)$ $\forall s,t \in \mathbb{R};\ x,u \in U$;
$N(x+u, s+t) \geq N(x,s) * N(u,t)$;
$(N5)$ $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$.

Note 2.8. *[3] Let (U, N) be a fuzzy normed linear space. Define a set $B(x,\alpha,t)$ as* $B(x,\alpha,t) = \{y \in X : N(x-y,t) > 1-\alpha\}$

Theorem 2.9. *[2] Let (U, N) be a fuzzy normed linear space. Define $\tau = \{O \subseteq U : x \in O\ iff\ \exists\ 0 < \alpha < 1\ and\ t > 0\ such\ that\ B(x,\alpha,t) \subseteq O\}$, then τ is a topology on (U, N).*

Theorem 2.10. *[12] Let $(U, N, *)$ be a fuzzy normed linear space where $'*'$ is continuous t-norm and $N(x,.)$ is left continuous $\forall x \in X$. $\tau_N = \{O \subseteq U : x \in O\ iff\ \exists\ 0 < \alpha < 1\ t > 0\ such\ that\ B(x,\alpha,t) \subseteq O\}$, then τ_N is a topology on $(U, N, *)$.*

Note 2.11. *(i) In Theorem 2.9 if $(U, N, *)$ be generalized fuzzy normed linear space, then τ is also a topology on U. (ii) In Theorem 2.10 if $'*'$ is continuous at each point of $[0,1] \times \{1\}$, then τ_N is also a topology on U.*

Definition 2.12. *[13] Let X be a linear space over $\mathbb{K}$ (the field of real or complex numbers) and θ the origin of X. Let $Q = \{\ |.|_\alpha : \alpha \in (0,1]\}$ be a family of mappings from X into $[0,\infty)$. (X, Q) is called a generating space of quasi-norm family and Q a quasi-norm family if following conditions are satisfied:*

(QN1) For all $\alpha \in (0,1)$ $|x|_\alpha = 0$ *if only if* $x = \theta$;

(QN2) For $x \in X$, $\forall \alpha \in (0,1)$ and $c \in \mathbb{K}$, $|cx|_\alpha = |c||x|_\alpha$;

(QN3) for any $\alpha \in (0,1)$ there exists $\beta \in (0,\alpha]$ such that
$|x+y|_\alpha \leq |x|_\beta + |y|_\beta$ for $x, y \in X$;

(QN4) $|x|_\alpha$ *is a non-increasing w.r.t* $\alpha \in (0,1)$*., for any* $x \in X$.

(X,Q) *is called a generating space of sub-strong quasi-norm family, strong quasi norm family and semi-norm family respectively, if* $(QN3)$ *is strengthened to* $(QN3u)$, $(QN3t)$, $(QN3e)$, *where*

$(QN3u)$ *for any* $\alpha \in (0,1)$ *there exists a* $\beta \in (0,\alpha]$ *such that* $|\sum_{i=1}^{n} x_i|_\alpha \leq \sum_{i=1}^{n} |x_i|_\beta$ *for any* $n \in Z^+$ *(set of all positive integers) and* $x_i \in X$ $(i = 1,2,...,n)$;

$(QN3t)$ *for any* $\alpha \in (0,1)$ *there exists a* $\beta \in (0,\alpha]$ *such that* $|x+y|_\alpha \leq |x|_\alpha + |y|_\beta$ *for* $x,y \in X$;

$(QN3e)$ *for any* $\alpha \in (0,1)$, *it holds that* $|x+y|_\alpha \leq |x|_\alpha + |y|_\alpha$ *for* $x,y \in X$.

Proposition 2.13. *[13] Let* $(X,N,*)$ *be a fuzzy normed linear space and* $Q = \{|.|_\alpha : \alpha \in (0,1)\}$ *be the quasi-norm family induced by* N*. If we assume* $(T5) : \forall \alpha,\beta \in (0,1),\ \alpha * \beta > 0$, *then* $|x+y|_{1-(1-\alpha)*(1-\beta)} \leq |x|_\alpha + |y|_\beta,\ \forall x,y \in X,\ \forall \alpha,\beta \in (0,1)$.

Definition 2.14. *[15] A 3-tuple* $(X,Q,*)$ *is called a Q-fuzzy metric space where* X *a nonempty set ,* $*$ *a continuous* t*-norm and* Q *is a fuzzy subset of* $X^3 \times (0,\infty)$ *satisfying the following conditions for each* $x,y,z,a \in X$ *and* $t,s > 0$:

- (i) $Q(x,x,y,t) > 0\ \forall x,y \in X$ *with* $x \neq y$, *and* $Q(x,x,y,t) \geq Q(x,y,z,t),\ \forall\ x,y,z \in X$ *with* $z \neq y$,
- (ii) $Q(x,y,z,t) = 1$ *if and only if* $x = y = z$,
- (iii) $Q(x,y,z,t) = Q(p(x,y,z),t)$, *(symmetry) where* p *is a permutation function.*
- (iv) $Q(x,y,z,t+s) \geq Q(x,a,a,t) * Q(a,y,z,s)\ \forall\ x,y,z,a \in X$,
- (v) $Q(x,y,z,.) : (0,\infty) \to [0,1]$ *is continuous.*

Note 2.15. *Through Out this paper Q-fuzzy metric space is denoted by G-fuzzy metric space.*

5

Definition 2.16. *[15] A G-fuzzy metric space $(X, Q, *)$ is said to be symmetric if $Q(x, y, y, t) = Q(x, x, y, t)$, for all $x, y \in X$.*

Definition 2.17. *[15] A sequence $\{x_n\}$ in X converges to x if and only if $Q(x_n, x_n, x, t) \to 1$ as $n \to \infty$ for each $t > 0$.*

3 G-fuzzy norm

In this section G-fuzzy norm is introduced and its underlying topology is studied.

Definition 3.1. *A 3-tuple $(X, G_N, *)$ is called a G-fuzzy normed linear space if X is a linear space, $'*'$ is a general t-norm and G_N is a fuzzy subset of $X^3 \times \mathbb{R}$, satisfying the following conditions: for each $x, y, z \in X, c \in F$*

(G_N1) $\forall t \in \mathbb{R},\ t \leq 0,\ G_N(x, y, z, t) = 0,$

(G_N2) $(\forall t \in \mathbb{R}, t > 0,\ G_N(x, y, z, t) = 1)\ iff\ x = y = z = \theta,$

(G_N3) $G_N(x, y, z, t) = G_N(p(x, y, z), t), (symmetry)$ *where p is a permutation function,*

(G_N4) $\forall t \in \mathbb{R}, t > 0,\ G_N(cx, cy, cz, t) = G_N(x, y, z, \frac{t}{|c|}), if c \neq 0,$

(G_N5) $\forall\ s, t \in \mathbb{R},\ x, y, z, x', y', z' \in X; G_N(x+x', y+y', z+z', t+s) \geq G_N(x, y, z, t) * G_N(x', y', z', s),$

(G_N6) $\lim_{t\to\infty} G_N(x, y, z, t) = 1,$

(G_N7) $G_N(x + y, \theta, z, t) \geq G_N(x, y, z, t),\ \forall x, y, z \in X,\ \forall t \in \mathbb{R}$

Remark 3.2. *$G_N(x, y, z, .)$ is non-decreasing on $\mathbb{R}$ directly follows form (G_N2) and (G_N5).*

Example 3.3. *In the linear space $X = C[0, 1]$ of real valued continuous function on $[0, 1]$ define a function $\|., ., .\| : X^3 \to \mathbb{R}$ by $\|f, g, h\| = max_{0\leq t\leq 1}\{|f(t)| + |g(t)| + |h(t)|\}$ $(f, g, h \in C[0, 1])$. Then $(X, \|., ., .\|)$ is a G-normed space. (please see [8]). Define $a * b = ab$ and*

$$G_N(f, g, h, t) = \begin{cases} [exp(\frac{\|f,g,h\|}{t})]^{-1}, & \forall f, g, h \in C[0, 1], and\ t \in (0, \infty) \\ 0, & if\ t \leq 0 \end{cases}$$

Then $(X, G_N, *)$ is a G-fuzzy normed linear space.

Proof.

(G_N1) If $t \leq 0$ then from the definition we have, $G_N(f,g,h,t) = 0, \forall f,g,h \in C[0,1]$.

(G_N2) $\forall t > 0, G_N(f,g,h,t) = 1 \Rightarrow [exp(\frac{\|f,g,h\|}{t})]^{-1} = 0 \Leftrightarrow [exp(\frac{\|f,g,h\|}{t})] = 1 \Leftrightarrow \|f,g,h\| = 0 \Leftrightarrow f = g = h = \theta$.

(G_N3) and (G_N4) are obvious.

(G_N5) We have, $\forall\ f,g,h,f_1,g_1,h_1 \in C[0,1], \forall\ s,t > 0$,
$\|f+f_1, g+g_1, h+h_1\| \leq \|f,g,h\| + \|f_1,g_1,h_1\|$
$\Rightarrow \frac{\|f+f_1,g+g_1,h+h_1\|}{s+t} \leq \frac{\|f,g,h\|+\|f_1,g_1,h_1\|}{s+t} \Rightarrow \frac{\|f+f_1,g+g_1,h+h_1\|}{s+t} \leq \frac{\|f,g,h\|}{t} + \frac{\|f_1,g_1,h_1\|}{s}$
$\Rightarrow exp(\frac{\|f+f_1,g+g_1,h+h_1\|}{s+t}) \leq exp(\frac{\|f,g,h\|}{t} + \frac{\|f_1,g_1,h_1\|}{s})$
$\Rightarrow exp(\frac{\|f+f_1,g+g_1,h+h_1\|}{s+t}) \leq exp(\frac{\|f,g,h\|}{t})exp(\frac{\|f_1,g_1,h_1\|}{s})$
$\Rightarrow [exp(\frac{\|f+f_1,g+g_1,h+h_1\|}{s+t})]^{-1} \geq [exp(\frac{\|f,g,h\|}{t})]^{-1}[exp(\frac{\|f_1,g_1,h_1\|}{s})]^{-1}$
$\Rightarrow G_N(f+f_1, g+g_1, h+h_1, t+s) \geq G_N(f,g,h,t) * G_N(f_1,g_1,h_1,s)$

(G_N6) From definition of G_N, it is clear that $\lim_{t\to\infty} G_N(f,g,h,t) = 1$.

(G_N7) Now, $\|f+g,\theta,h\| \leq \|f,g,h\|$
$\Rightarrow \frac{\|f+g,\theta,h\|}{t} \leq \frac{\|f,g,h\|}{t}$
$\Rightarrow [exp(\|f+g,\theta,h\|)]^{-1} \geq [exp(\|f,g,h\|)]^{-1}$
$\Rightarrow G_N(f+g,\theta,h,t) \geq G_N(f,g,h,t),\ \forall f,g,h \in C[0,1]$.

Hence $(X, G_N, *)$ is a G-fuzzy normed linear space. □

Proposition 3.4. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space and $*$ is a continuous t-norm and $G_N(x,y,z,.)$ is continuous $\forall t > 0$. Then the function $Q : X^3 \times (0,\infty) \to [0,1]$ defined by $Q(x,y,z,t) = G_N(x-y, y-z, z-x, t)$ is Sun and Yang type symmetric G-fuzzy metric space.*

Proof. (i) We have $Q(x,y,z,t) = G_N(x-y, y-z, z-x, t) = G_N(y-z, z-x, x-y, t)$
$\leq G_N(y-x, \theta, x-y, t) = G_N(x-y, \theta, y-x, t) = Q(x,x,y,t)$ by (G_N7).
Thus $Q(x,y,z,t) \leq Q(x,x,y,t)\ \forall x,y,z \in X$.
Again $Q(x,y,y,t) = Q(x,x,y,t)\ \forall x,y \in X$.

(ii) $\forall t \in \mathbb{R}, t > 0,\ Q(x,y,z,t) = 1 \Leftrightarrow$
$\forall t \in \mathbb{R}, t > 0, G_N(x-y, y-z, z-x, t) = 1 \Leftrightarrow x = y = z.$

(iii) Now by (G_N3) and (G_N4) it follows that $Q(x,y,z,t) = Q(p(x,y,z),t)$.

(iv) $\forall x,y,z,a \in X$ and $t,s > 0$,
$Q(x,y,z,t+s) = G_N(x-y, y-x, z-x, ,t+s)$
Now $Q(x,a,a,t) = G_N(x-a, \theta, a-x, t),\ Q(a,y,z,s) = G_N(a-y, y-z, z-a, s)$
By (G_N5), $G_N(x-a,\theta,a-x,t) * G_N(a-y,y-z,z-a,s) \leq G_N(x-y,y-z,z-x,t+s) = Q(x,y,z,t+s)$.

(v) Now by the given condition, $Q(x,y,z,.) : (0,\infty) \to [0,1]$ is continuous.

□

Theorem 3.5. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space. Define $N_{G_N} : X \times \mathbb{R} \to [0,1]$ by*
$N_{G_N}(x,t) = G_N(x,-x,\theta,t),\ \forall\ (x,t) \in X \times \mathbb{R}$. Then $(X, N_{G_N}, *)$ is a fuzzy normed linear space.

Proof.

$(N1)$ Since $G_N(x,-x,\theta,t) = 0,\ \forall\ t \leq 0$ so $N_{G_N}(x,t) = 0,\ \forall\ t \leq 0$.

$(N2)$ Since $(\ \forall t > 0,\ G_N(x,-x,\theta,t)) = 1$ iff $x = \theta$ thus
$(\ \forall t > 0,\ N_{G_N}(x,t) = 1)$ iff $x = \theta$.

$(N3)$ $\forall t \in \mathbb{R},\ t > 0,\ N_{G_N}(cx,t) = G_N(cx,-cx,\theta,t) = G_N(x,-x,\theta,\frac{t}{|c|}) = N_{G_N}(x,\frac{t}{|c|}),\ if\ c \neq 0.$

$(N4)$ $\forall\ s,t \in \mathbb{R};\ x,u \in X;$
$N_{G_N}(x+u, s+t) = G_N(x+u, -x-u, \theta, s+t) \geq G_N(x,-x,\theta,s) * G_N(u,-u,\theta,t) = N_{G_N}(x,s) * N_{G_N}(u,t).$

$(N5)$ $\lim_{t\to\infty} N_{G_N}(x,t) = \lim_{t\to\infty} G_N(x,-x,\theta,t) = 1.$

Thus N_{G_N} is a fuzzy norm in X and $(X, N_{G_N}, *)$ is a fuzzy normed linear space. We call N_{G_N} is a fuzzy norm induced by G_N. □

Definition 3.6. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space. For a given $x,\ z \in X,\ t > 0$ and $0 < \alpha < 1$, we define an open ellipse $E_{G_N}(x,z,\alpha,t)$ to be a subset of X given by*
$E_{G_N}(x,z,\alpha,t) = \{y \in X : G_N(x-y, y-z, z-x, t) > 1 - \alpha\}$

Definition 3.7. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space. For a given $x \in X,\ t > 0,\ 0 < \alpha < 1$, we define an open ball $B_{G_N}(x, \alpha, t)$ to be a subset of X given by $B_{G_N}(x, \alpha, t) = \{y \in X : G_N(x - y, y - x, \theta, t) > 1 - \alpha\}$*

Note 3.8. *For an open ball $B_{G_N}(x, \alpha, t)$, we can say that $B_{G_N}(x, \alpha, t) = E_{G_N}(x, x, \alpha, t)$.*

Remark 3.9. $B_{G_N}(x, \alpha, t) \neq \phi,\ \forall x \in X,\ \forall \alpha \in (0, 1),\ \forall t > 0.$

Theorem 3.10. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space. Define $\tau_{B_{G_N}} = \{G \subseteq X : x \in G\ iff\ \exists\ 0 < \alpha < 1,\ t > 0$ such that $B_{G_N}(x, \alpha, t) \subseteq G\}$. Then $\tau_{B_{G_N}}$ is a topology on X.*

Proof. Clearly $\phi, X \in \tau_{B_{G_N}}$.
$\tau_{B_{G_N}}$ is closed under finite intersection follows from the implication,
$G_N(x - y, y - x, \theta, \overset{n}{\underset{i=1}{\wedge}} t_i) > 1 - \overset{n}{\underset{i=1}{\wedge}} \alpha_i$
$\Rightarrow G_N(x - y, y - x, \theta, t_i) > 1 - \alpha_i, \forall\ i = \{1, 2, ..., n\}$,
which holds by virtue of the fact that $G_N(x, y, z, .)$ is a non decreasing function. It can be easily check that $\tau_{B_{G_N}}$ is closed under arbitrary union. Hence $\tau_{B_{G_N}}$ is a topology on X. □

Note 3.11. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space. We consider the following conditions:*
(G_N8) $G_N(x, y, z, .)$ is lower semi continuous
for each $(x, y, z) \neq (\theta, \theta, \theta)$.
$(T4)$ $'*'$ is continuous at each point of $[0, 1] \times \{1\}$.

Remark 3.12. *If G_N satisfies (G_N8) and underlying t-norm $*$ satisfies $(T4)$, then $\{B_{G_N}(x, \alpha, t) : x \in X, 0 < \alpha < 1, t > 0\}$ (Definition 3.7) forms a base for $\tau_{B_{G_N}}$.*

Lemma 3.13. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space where $'*'$ satisfies $(T4)$ and G_N satisfies (G_N8). Then for each $z \in E_{G_N}(x, y, \alpha, t)\ \exists\ t_z > 0, 0 < \alpha_z < 1$ such that $B_{G_N}(z, \alpha_z, t_z) \subseteq E_{G_N}(x, y, \alpha, t)$.*

Proof. Let $z \in E_{G_N}(x, y, \alpha, t)$.
$\Rightarrow G_N(x - z, z - y, y - x, t) > 1 - \alpha$
Now by lower semi continuity of $G_N(x, y, z, .)$, $\exists t_0 > 0$ with $0 <$

$t_0 < t$ such that
$G_N(x - z, z - y, y - x, t_0) > 1 - \alpha$.
Let $\beta_0 = G_N(x - z, z - y, y - x, t_0)$.
Choose $0 < \alpha_0 < 1$ such that $\beta_0 * (1 - \alpha_0) > (1 - \alpha)$.
Now consider the open ball $B_{G_N}(z, \alpha_0, t - t_0)$.
Let $u \in B_{G_N}(z, \alpha_0, t - t_0)$.
$\Rightarrow G_N(z - u, u - z, \theta, t - t_0) > 1 - \alpha_0$
Now $G_N(x-u, u-y, y-x, t) \geq G_N(x-z, z-y, y-x, t_0) * G_N(z-u, u-z, \theta, t-t_0) \geq \beta_0 * (1-\alpha_0) > (1-\alpha)$
Hence $B_{G_N}(z, \alpha_z, t_z) \subseteq E_{G_N}(x, y, \alpha, t),\ \forall z \in E_{G_N}(x, y, \alpha, t)$. □

Remark 3.14. (1) $E_{G_N}(x, y, \alpha, t) = \underset{z \in E_{G_N}(x,y,\alpha,t)}{\cup} B_{G_N}(z, \alpha_z, t_z)$.

(2) $E_{G_N}(x, y, \alpha_1, t_1) \subseteq E_{G_N}(x, y, \alpha_2, t_2)$, $\forall t_1 \leq t_2$ *and* $\alpha_1 \leq \alpha_2$, $\forall x, y \in X$.

Theorem 3.15. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space where underlying t-norm $*$ satisfies $(T4)$ and G_N satisfies (G_N8). Define $\tau_{E_{G_N}} = \{G \subseteq X : x \in G\ iff\ \exists\ z \in X,\ t > 0,\ 0 < \alpha < 1$ such that $E_{G_N}(x, z, \alpha, t) \subseteq G\}$. Then $\tau_{E_{G_N}}$ is a topology on X.*

Proof. (i) Obviously $X, \phi \in \tau_{E_{G_N}}$.

(ii) Now let $G_1, G_2 \in \tau_{E_{G_N}}$ and $x \in G_1 \cap G_2$.
Then $\exists\ z_1, t_1, \alpha_1$ and z_2, t_2, α_2 such that $E_{G_N}(x, z_1, \alpha_1, t_1) \subseteq G_1$ and $E_{G_N}(x, z_2, \alpha_2, t_2) \subseteq G_2$.
Now $x \in E_{G_N}(x, z_1, \alpha_1, t_1) \cap E_{G_N}(x, z_2, \alpha_2, t_2)$
Then by the Lemma 3.13 $\exists\ t^{'} > 0, 0 < \alpha^{'} < 1$ and $\exists\ t^{''} > 0, 0 < \alpha^{''} < 1$ such that
$B_{G_N}(x, \alpha^{'}, t^{'}) \subseteq E_{G_N}(x, z_1, \alpha_1, t_1)$ and $B_{G_N}(x, \alpha^{''}, t^{''}) \subseteq E_{G_N}(x, z_2, \alpha_2, t_2)$
Let $t = min\{t^{'}, t^{''}\}$ and $\alpha = min\{\alpha^{'}, \alpha^{''}\}$.
Now by Lemma 3.13, Remark 3.9 and Note 3.8, $E_{G_N}(x, x, \alpha, t) \subseteq E_{G_N}(x, z_1, \alpha_1, t_1) \cap E_{G_N}(x, z_2, \alpha_2, t_2)$.
$\therefore G_1 \cap G_2 \in \tau_{E_{G_N}}$.

(iii) Now $G_i \in \tau_{E_{G_N}},\ i \in \Delta \Rightarrow \underset{i \in \Delta}{\cup} G_i \in \tau_{E_{G_N}}$- follows directly.
$\therefore \tau_{E_{G_N}}$ is a topology on X. □

Theorem 3.16. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space where underlying t-norm $*$ satisfies $(T4)$ and G_N satisfies*

(G_N8). . *Then the smallest topology generated by the open balls* $\{B_{G_N}(x,\alpha,t):\ x\in X,\ t>0\in\mathbb{R},\ 0<\alpha<1\}$ *is equal to* $\tau_{E_{G_N}}$.

Proof. The proof directly follows from the Remark 3.14, Lemma 3.13 and Note 3.8. □

Theorem 3.17. *A G-fuzzy norm* G_N *and its induced fuzzy norm* N_{G_N} *give the same topology. Moreover, if* G_N *satisfies* $(G_N\ 8)$ *and it's underlying t-norm satisfies* $(T4)$ *then* $\tau_{B_{G_N}}=\tau_{N_{G_N}}=\tau_{E_{G_N}}$.

Proof. Consider an arbitrary G-fuzzy normed linear space $(X,G_N,*)$. N_{G_N} be the fuzzy norm induced by G_N.
We have $\mathcal{B}_G=\{B_{G_N}(x,\alpha,t):x\in X,\ t>0,\ 0<\alpha<1\}$ is a base of τ_{G_N}. Now for each $x\in X$, $t>0$, and $0<\alpha<1$

$$\begin{aligned}B_{G_N}(x,\alpha,t)&=\{y\in X:G_N(x-y,-x+y,\theta,t)>1-\alpha\}\\&=\{y\in X:G_N(x-y,-(x-y),\theta,t)>1-\alpha\}\\&=\{y\in X:N_{G_N}(x-y,t)>1-\alpha\}\\&=B_{N_{G_N}}(x,\alpha,t).\end{aligned}$$

Thus $\tau_{B_{G_N}}=\tau_{N_{G_N}}$.
Second part follows from Theorem 3.16. □

Thus every G-fuzzy normed linear space topologically equivalent to an induced fuzzy normed linear space with respect to some conditions.

4 Generating space of quasi G-norm family

In this section, concept of Generating space of 'quasi G-norm family' and 'G-norm family' are introduced.

Definition 4.1. *Let X be a linear space over the field $\mathcal{F}$ (real or complex) and θ be the origin of X.*
Let $Q=\{\|.,.,.\|_\alpha,\alpha\in(0,1)\}$ *be a family of mappings from* X^3 *into* $[0,\infty)$. *Q, is called a quasi G-norm family ($QGNF$) and (X,Q) is called generating space of quasi G-norm family ($GSQGNF$) if the following conditions are satisfied:*

(QN1) $\forall\alpha\in(0,1)$, $\|x,y,z\|_\alpha=0$ *iff* $x=y=z=\theta$.

(QN2) $\|cx, cy, cz\|_\alpha = |c|\|x, y, z\|_\alpha,\ \forall x, y, z \in X,$
$\forall \alpha \in (0, 1)$ *and* $\forall c \in \mathcal{F}$.

(QN3) $\|x, y, z\|_\alpha$ *is invariant under permutations of* x, y *and* $z \in X,\ \forall \alpha\ \in (0, 1)$.

(QN4) *For any* $\alpha \in (0, 1)\ \exists \beta \in (0, \alpha]$ *such that*
$\|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha \leq \|x, y, z\|_\beta + \|x^{'}, y^{'}, z^{'}\|_\beta$

(QN5) $\|x, y, z\|_\alpha \geq \|x + y, \theta, z\|_\alpha,\ \forall x, y, z \in X,\ \alpha \in (0, 1)$.

(QN6) *For any* $x, y, z \in X$, $\|x, y, z\|_\alpha$ *is a non-increasing for* $\alpha \in (0, 1)$.

Again (X, Q) is called a generating space of sub-strong quasi G-norm family, strong quasi G-norm family and semi G-norm family respectively, if $(QN4)$ is strengthened to $(QN4u)$, $(QN4t)$ and $(QN4e)$, where

$(QN4u)$ For any $\alpha \in (0, 1),\ n \in \mathbb{N}\ \exists \beta \in (0, \alpha]$ such that
$\|\sum\limits_{i=1}^{n} x_i, \sum\limits_{i=1}^{n} y_i, \sum\limits_{i=1}^{n} z_i\|_\alpha \leq \sum\limits_{i=1}^{n} \|x_i, y_i, z_i\|_\beta$ and $x_i, y_i, z_i \in X$ $(i = 1, 2, ..., n)$;

$(QN4t)$ For any $\alpha \in (0, 1)\ \exists \beta \in (0, \alpha]$ such that
$\|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha \leq \|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\beta$

$(QN4e)$ For any $\alpha \in (0, 1)$, it holds that,
$\|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha \leq \|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\alpha$

Definition 4.2. *If* Q *is a family of* G*-norms on* X*, i.e. for each* $\alpha \in (0, 1)$, $\|., ., .\|_\alpha$ *is a* G*-norm on* X*, then* (X, Q) *is called a generating space of* G*-norm family* $(GSGNF)$.

Remark 4.3. *From the definition*

(i) $(QN4e) \Rightarrow (QN4t)$

(ii) $(QN4)$ *and* $(QN6) \Rightarrow (QN4u)$

(iii) $(QN4u) \Rightarrow (QN4)$

Example 4.4. *Let* $X = \mathbb{R}$ *be a linear space. For* $x, y, z \in X$*, define* $\|x, y, z\|_\alpha = \frac{1}{\alpha}(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|})^2,\ \forall \alpha \in (0, 1)$.
Then $Q = \{\|., ., .\|_\alpha; \alpha \in (0, 1)\}$ *is a* $(QGNF)$ *and* (X, Q) *is a* $(GSQGNF)$.

Solution:-

$(QN1)$ Let $x = y = z = \theta$, then $\|x, y, z\|_\alpha = 0, \forall\alpha \in (0, 1)$.
Conversely let for $x, y, z \in X$, $\|x, y, z\|_\alpha = 0, \forall\alpha \in (0, 1)$
$\Rightarrow \frac{1}{\alpha}(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|})^2 = 0 \Rightarrow x = y = z = \theta$.

$(QN2)$ Let $x, y, z \in X$ *and* $c \in \mathcal{F}$,
$\|cx, cy, cz\|_\alpha = \frac{1}{\alpha}(\sqrt{|cx|} + \sqrt{|cy|} + \sqrt{|cz|})^2, \ \forall\alpha \in (0, 1)$
$= |c|\frac{1}{\alpha}(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|})^2, \ \forall\alpha \in (0, 1)$
$= |c|\|x, y, z\|_\alpha$

$(QN3)$ From the definition it is clear that $\|x, y, z\|_\alpha$ is invariant under permutations of x, y *and* $z \in X, \ \forall\alpha \in (0, 1)$.

$(QN4)$ Let $x, y, z, x^{'}, y^{'}, z^{'} \in X$ and $\alpha \in (0, 1)$
$3(\|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\alpha) - \|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha$
$= \frac{3}{\alpha}\{[|x| + |y| + |z| + 2\sqrt{|x||y|} + 2\sqrt{|y||z|} + 2\sqrt{|z||x|}] + [|x^{'}| + |y^{'}| + |z^{'}| + 2\sqrt{|x^{'}||y^{'}|} + 2\sqrt{|y^{'}||z^{'}|} + 2\sqrt{|z^{'}||x|^{'}}]\} - \frac{1}{\alpha}\{|x + x^{'}| + |y + y^{'}| + |z + z^{'}| + 2\sqrt{|x + x^{'}||y + y^{'}|} + 2\sqrt{|y + y^{'}||z + z^{'}|} + 2\sqrt{|z + z^{'}||x + x^{'}|}\}$
$\geq \frac{3}{\alpha}[|x| + |x^{'}| + |y| + |y^{'}| + |z| + |z^{'}| + 2\{\sqrt{|x||y|} + \sqrt{|y||z|} + \sqrt{|z||x|} + \sqrt{|x^{'}||y^{'}|} + \sqrt{|y^{'}||z^{'}|} + \sqrt{|z^{'}||x^{'}|}\}] - \frac{1}{\alpha}[|x| + |x^{'}| + |y| + |y^{'}| + |z| + |z^{'}| + 2\sqrt{\{|x| + |x^{'}|\}\{|y| + |y^{'}|\}} + 2\sqrt{\{|y| + |y^{'}|\}\{|z| + |z^{'}|\}} + 2\sqrt{\{|x| + |x^{'}|\}\{|z| + |z^{'}|\}}]$
$= \frac{1}{\alpha}[\{\sqrt{|x| + |x^{'}|} - \sqrt{|y| + |y^{'}|}\}^2 + \{\sqrt{|y| + |y^{'}|} - \sqrt{|z| + |z^{'}|}\}^2 + \{\sqrt{|z| + |z^{'}|} - \sqrt{|x| + |x^{'}|}\}^2 + 6\sqrt{|x||y|} + 6\sqrt{|y||z|} + 6\sqrt{|z||x|} + 6\sqrt{|x^{'}||y^{'}|} + 6\sqrt{|y^{'}||z^{'}|} + 6\sqrt{|z^{'}||x^{'}|}] \geq 0$
Thus $\|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha \leq 3(\|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\alpha)$.
Now if we put $\beta = \frac{\alpha}{3} \in (0, \alpha)$ then,
$\|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha \leq \|x, y, z\|_\beta + \|x^{'}, y^{'}, z^{'}\|_\beta$

$(QN5)$ Let $x, y, z \in X$ *and* $\alpha \in (0, 1)$,
$\|x, y, z\|_\alpha = \frac{1}{\alpha}(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|})^2$
And $\|x + y, \theta, z\|_\alpha = \frac{1}{\alpha}(\sqrt{|x + y|} + \sqrt{|z|})^2 \leq \frac{1}{\alpha}(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|})^2 (As \sqrt{|x + y|} \leq \sqrt{|x| + |y|} \leq \sqrt{|x|} + \sqrt{|y|})$
Therefore $\|x, y, z\|_\alpha \geq \|x + y, \theta, z\|_\alpha, \ \forall\alpha \in (0, 1)$.

$(QN6)$ It is clear that $\|x, y, z\|_\alpha$ is non-increasing $\forall\alpha \in (0, 1)$.

13

Hence (X, Q) is a $(GSQNF)$.

Remark 4.5. *In the above example* $Q = \{\|.,.,.\|_\alpha; \alpha \in (0,1)\}$ *is* $(GSQGNF)$ *but not a* $(GSGNF)$ *i.e.* $\|.,.,.\|_\alpha$ *is not a G-norm* $\forall\alpha \in (0,1)$.
We take, $x = y = 1, z = 0$ *and* $x^{'} = y^{'} = 0, z^{'} = 1 \in \mathcal{R}$.
Then $\|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha = \|1,1,1\|_\alpha = \frac{9}{\alpha}$
Again $\|x, y, z\|_\alpha = \|1,1,0\| = \frac{4}{\alpha}$, $\|x^{'}, y^{'}, z^{'}\| = \|0,0,1\| = \frac{1}{\alpha}$
So, $\|x+x^{'}, y+y^{'}, z+z^{'}\|_\alpha \neq \|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\alpha \; \forall x, y, z, x^{'}, y^{'}, z^{'} \in X$ *for any* $\alpha \in (0,1)$.

5 Decomposition theorem

Theorem 5.1. *(First Decomposition theorem) Let* $(X, G_N, *)$ *be a G-fuzzy normed linear space with* $*$ *is lower semi continuous. For* $\alpha \in (0,1)$ *we define,* $\|x, y, z\|_\alpha = \wedge\{t > 0; G_N(x, y, z, t) \geq 1-\alpha\}$ *and* $Q = \{\|.,.,.\|_\alpha; \alpha \in (0,1)\}$ *Then* (X, Q) *is a* $(GSQGNF)$.

Proof. $(QN1)$ Let $x = y = z = \theta$ and $\alpha_0 \in (0,1)$. Then by (G_N2),
$G_N(x, y, z, t) = 1, \forall t > 0 \Rightarrow G_N(x, y, z, t) \geq 1 - \alpha_0, \forall t > 0 \Rightarrow \|x, y, z\|_{\alpha_0} = 0$
Since $\alpha_0 \in (0,1)$ is arbitrary so, $\|x, y, z\|_\alpha = 0, \; \forall\alpha \in (0,1)$.
Conversely let, $\|x, y, z\|_\alpha = 0 \; \forall\alpha \in (0,1)$.
$\Rightarrow \wedge\{t > 0; G_N(x, y, z, t) \geq 1 - \alpha\} = 0, \; \forall\alpha \in (0,1)$.
Let $\epsilon > 0$ be given.
$\Rightarrow \wedge\{t > 0; G_N(x, y, z, t) \geq 1 - \alpha\} < \epsilon, \; \forall\alpha \in (0,1)$.
$\Rightarrow G_N(x, y, z, \epsilon) \geq 1 - \alpha, \; \forall\alpha \in (0,1)$
$\Rightarrow G_N(x, y, z, \epsilon) = 1$
Since $\epsilon > 0$ is arbitrary then, $G_N(x, y, z, t) = 1 \forall t > 0$
$\Rightarrow x = y = z = \theta$
Hence $\|x, y, z\|_\alpha = 0 \; \forall\alpha \in (0,1)$ iff $x = y = z = \theta$.
So, $(QN1)$ holds.

$(QN2)$ Let $x \in X, c \in \mathcal{F}$ and $\alpha \in (0,1)$. Then

$$\begin{aligned}
\|cx, cy, cz\|_\alpha &= \wedge\{t > 0; G_N(cx, cy, cz, t) \geq 1 - \alpha\} \\
&= \wedge\{t > 0; G_N(x, y, z, \tfrac{t}{|c|}) \geq 1 - \alpha\} \\
&= \wedge\{|c|.\tfrac{t}{|c|} > 0; G_N(x, y, z, \tfrac{t}{|c|}) \geq 1 - \alpha\} \\
&= |c|. \wedge \{\tfrac{t}{|c|} > 0; G_N(x, y, z, \tfrac{t}{|c|}) \geq 1 - \alpha\} \\
&= |c|\|x, y, z\|_\alpha
\end{aligned}$$

14

$(QN3)$ By (G_N3), $G_N(x,y,z,t) = G_N(p(x,y,z)t)$.
Thus $\|x,y,z\|_\alpha$ is invariant under permutation of x,y,z, $\forall\alpha \in (0,1)$.

$(QN4)$ Since $*$ is lower semi continuous, for any $\alpha \in (0,1), \exists\beta \in (0,\alpha]$ such that $(1-\beta)*(1-\beta) \geq (1-\alpha)$
Now, $\|x,y,z\|_\beta + \|x^{'},y^{'},z^{'}\|_\beta =$
$\wedge\{t > 0; G_N(x,y,z,t) \geq 1-\beta\} + \wedge\{s > 0; G_N(x^{'},y^{'},z^{'},s) \geq 1-\beta\}$
$\geq \wedge\{t+s > 0; G_N(x,y,z,t) \geq 1-\beta, G_N(x^{'},y^{'},z^{'},s) \geq 1-\beta\}$
$\geq \wedge\{t+s > 0; G_N(x+x^{'},y+y^{'},z+z^{'},t+s) \geq (1-\beta)*(1-\beta) \geq 1-\alpha\}$
$\geq \|x+x^{'},y+y^{'},z+z^{'}\|_\alpha$
Hence $(QN4)$ is satisfied.

$(QN5)$ By (G_N7) $G_N(x+y,\theta,z,t) \geq G_N(x,y,z,t),\ \forall x,y,z \in X$.
Then $\{t > 0; G_N(x,y,z,t) \geq 1-\alpha\} \subseteq \{t > 0; G_N(x+y,\theta,z,t) \geq 1-\alpha\}$
$\Rightarrow \wedge\{t > 0; G_N(x,y,z,t) \geq 1-\alpha\} \geq \wedge\{t > 0; G_N(x+y,\theta,z,t) \geq 1-\alpha\}$
$\Rightarrow \|x,y,z\|_\alpha \geq \|x+y,\theta,z\|_\alpha,\ \forall\alpha \in (0,1)$.
So, $(QN5)$ is satisfied.

$(QN6)$ Let $\alpha_1,\alpha_2 \in (0,1),\ x,y,z \in X$ with $\alpha_1 > \alpha_2$ then $1-\alpha_2 > 1-\alpha_1$
$\Rightarrow \{t > 0; G_N(x,y,z,t) \geq 1-\alpha_2\} \subseteq \{t > 0; G_N(x,y,z,t) \geq 1-\alpha_1\}$
$\Rightarrow \wedge\{t > 0; G_N(x,y,z,t) \geq 1-\alpha_2\} \geq \wedge\{t > 0; G_N(x,y,z,t) \geq 1-\alpha_1\}$
$\Rightarrow \|x,y,z\|_{\alpha_2} \geq \|x,y,z\|_{\alpha_1}$
So, $(QN6)$ is satisfied.

Thus $Q = \{\|.,.,.\|_\alpha; \alpha \in (0,1)\}$ is a $(QGNF)$ and (X,Q) is a $(GSQGNF)$. □

Proposition 5.2. *Let $(X,G_N,*)$ be a G-fuzzy normed linear space with $*$ satisfying $(T5)$ and $Q = \{\|.,.,.\|_\alpha : \alpha \in (0,1)\}$ be the $(QGNF)$ induced by G_N. Then $\|x+x^{'},y+y^{'},z+z^{'}\|_{1-(1-\alpha)*(1-\beta)} \leq \|x,y,z\|_\alpha + \|x^{'},y^{'},z^{'}\|_\beta,\ \forall x,y,z,x^{'},y^{'},z^{'} \in X,\ \forall\alpha,\beta \in (0,1)$.*

Proof. From definition, we have
$\|x,y,z\|_\alpha + \|x^{'},y^{'},z^{'}\|_\beta = \wedge\{t > 0; G_N(x,y,z,t) \geq 1-\alpha\} + \wedge\{s >$

$0; G_N(x^{'}, y^{'}, z^{'}, s) \geq 1 - \beta\}$
$\geq \wedge\{t + s > 0; G_N(x, y, z, t) \geq 1 - \alpha, G_N(x^{'}, y^{'}, z^{'}, s) \geq 1 - \beta\}$
Now $G_N(x, y, z, t) \geq (1 - \alpha)$ and $G_N(x^{'}, y^{'}, z^{'}, s) \geq (1 - \beta)$
$\Rightarrow G_N(x+x^{'}, y+y^{'}, z+z^{'}, t+s) \geq G_N(x, y, z, t) * G_N(x^{'}, y^{'}, z^{'}, s) \geq (1 - \alpha) * (1 - \beta)$
Thus $\|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\beta \geq \wedge\{t + s > 0; G_N(x + x^{'}, y + y^{'}, z + z^{'}, t + s) \geq (1 - \alpha) * (1 - \beta)\} = \|x + x^{'}, y + y^{'}, z + z^{'}\|_{1-(1-\alpha)*(1-\beta)}$
Hence $\|x+x^{'}, y+y^{'}, z+z^{'}\|_{1-(1-\alpha)*(1-\beta)} \leq \|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\beta$, $\forall x, y, z, x^{'}, y^{'}, z^{'} \in X$, $\forall \alpha, \beta \in (0, 1)$. □

Remark 5.3. *In the Theorem 5.1, if t-norm* $*$ *is defined by* $a * b = min\{a, b\}$ $\forall a, b \in [0, 1]$ *then* (X, Q) *is a generating space of semi G-norm family.*

Proof. Conditions $(QN1), (QN2), (QN3), (QN5), (QN6)$ are similar as Theorem 5.1.
For $(QN4e)$, choose $\alpha \in (0, 1)$.
Now $\|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\alpha$
$= \wedge\{t > 0; G_N(x, y, z, t) \geq 1 - \alpha\} + \wedge\{s > 0; G_N(x^{'}, y^{'}, z^{'}, s) \geq 1 - \alpha\}$
$\geq \wedge\{t + s > 0; G_N(x, y, z, t) \geq 1 - \alpha, G_N(x^{'}, y^{'}, z^{'}, t) \geq 1 - \alpha\}$
Now $G_N(x, y, z, t) \geq 1 - \alpha$ and $G_N(x^{'}, y^{'}, z^{'}, s) \geq 1 - \alpha$
$\Rightarrow G_N(x + x^{'}, y + y^{'}, z + z^{'}, s + t) \geq (1 - \alpha) * (1 - \alpha) = 1 - \alpha$
Thus $\|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\alpha \geq \|x + x^{'}, y + y^{'}, z + z^{'}\|_\alpha$,
$\forall \alpha \in (0, 1)$, $\forall x, y, z, x^{'}, y^{'}, z^{'} \in X$.
Therefore (X, Q) is a generating space of semi G-norm family. □

Moreover if we assume that
$(QN7)$ $\forall x, y, z \in X$, $(x, y, z) \neq (\theta, \theta, \theta)$ implies $\|x, y, z\|_\alpha > 0$, $\forall \alpha \in (0, 1)$. Then $(X, Q)$is a $(GSGNF)$.

Theorem 5.4. *Let* (X, G_N, min) *be a G-fuzzy normed linear space. For* $\alpha \in (0, 1)$ *define,* $\|x, y, z\|_\alpha = \wedge\{t > 0; G_N(x, y, z, t) \geq 1 - \alpha\}$ *and* $Q = \{\|., ., .\|_\alpha; \alpha \in (0, 1)\}$. *Then* (X, Q) *is a* $(GSGNF)$ *iff* G_N *satisfied the following conditions:* $G_N(x, y, z, t)$ *is continuous at* $t = 0, \forall\ x, y, z \in X, (x, y, z) \neq (\theta, \theta, \theta)$.

Proof. Suppose $G_N(x, y, z, t)$ is continuous at $t = 0$, $\forall\ x, y, z \in X$, $(x, y, z) \neq (\theta, \theta, \theta)$. By the Remark 5.3, (X, Q) is a generating space of semi G-norm family. To complete the proof, we have to show that Q satisfies the condition $(QN7)$. i.e. If $x, y, z \in X$ with

$(x, y, z) \neq (\theta, \theta, \theta)$ then $\|x, y, z\|_\alpha > 0$.
If possible let $\exists\ (x, y, z) \neq (\theta, \theta, \theta) \in X$ and $\alpha_0 \in (0, 1)$ such that,
$\|x, y, z\|_{\alpha_0} = 0$
$\Rightarrow G_N(x, y, z, t) \geq 1 - \alpha_0\ \forall t > 0$
This contradicts the fact that $G_N(x, y, z, t)$ is continuous at
$t = 0, \forall (x, y, z) \neq (\theta, \theta, \theta)$.
Thus $\forall\ x, y, z \in X$ with $(x, y, z) \neq (\theta, \theta, \theta)$, $\|x, y, z\|_\alpha > 0$.
Hence (X, Q) is a $(GSGNF)$ and Q, a G-norm family.
Conversely suppose G_N does not satisfy the given condition.
Then $\exists\ (x, y, z) \neq (\theta, \theta, \theta)$ such that $G_N(x, y, z, .)$ is not continuous at $t = 0$. i.e. $\exists\ \alpha_0 \in (0, 1)$ such that, $G_N(x, y, z, t) \geq 1 - \alpha_0\ \forall\ t > 0$.
$\Rightarrow \|x, y, z\|_{\alpha_0} = 0$.
Hence Q does not satisfy the condition $(QN7)$.
So, (X, Q) is not a $(GSGNF)$. □

Theorem 5.5. *(Second decomposition theorem) Let (X, Q) be a $(GSQGNF)$, where $Q = \{\|., ., .\|_\alpha; \alpha \in (0, 1)\}$ and t-norm $'*'$ is lower semi continuous and satisfying $(T5)$.*
Assume that $\|x + x', y + y', z + z'\|_{1-(1-\alpha)*(1-\beta)} \leq \|x, y, z\|_\alpha + \|x', y', z'\|_\beta$, $\forall x, y, z, x', y', z' \in X$, $\alpha, \beta \in (0, 1)$. We define a function $G'_N : X^3 \times \mathbb{R} \to [0, 1]$ as

$$G'_N(x, y, z, t) = \begin{cases} \vee\{\alpha \in (0, 1) : \quad \|x, y, z\|_{1-\alpha} \leq t\} & for\ t > 0 \\ 0, \quad for\ t \leq 0. \end{cases}$$

Then $(X, G'_N, *)$ is a G-fuzzy normed linear space.

Proof.

(G_N1) follows immediately.

(G_N2) Let $\forall t > 0\ G'_N(x, y, z, t) = 1$
Then $\vee\{\alpha \in (0, 1); \|x, y, z\|_{1-\alpha} \leq t\} = 1,\ \forall t > 0$
Thus for any $t > 0$ and $\epsilon \in (0, 1)\ \exists\ \alpha_0(t, \epsilon) > \epsilon$ such that $\|x, y, z\|_{1-\alpha_0} \leq t$
Since $t > 0$ is arbitrary, $\|x, y, z\|_\alpha$ is non-increasing for $\alpha \in (0, 1)$ then, $\|x, y, z\|_{1-\epsilon} = 0,\ \forall \epsilon \in (0, 1) \Rightarrow x = y = z = \theta$ $(by\ QN1)$.
Conversely if $x = y = z = \theta$ then for any
$t > 0,\ \|x, y, z\|_{1-\alpha} < t,\ \forall \alpha \in (0, 1)$
$\Rightarrow G'_N(x, y, z, t) = 1, \forall\ t > 0$
Thus $(\forall t \in \mathbb{R}, t > 0,\ G'_N(x, y, z, t) = 1)\ iff\ x = y = z = \theta$.

(G_N3) Since $\forall x, y, z \in X, \ \forall \alpha \in (0,1)$
$\|x, y, z\|_\alpha$ is invariant under permutation of x, y, z.
Thus $G_N^{'}(x, y, z, t) = G_N^{'}(p(x, y, z), t)$, where p is a permutation function.

(G_N4) $\forall\ t(>0) \in \mathbb{R}, \ for\ c(\neq 0) \in \mathcal{F}$
We have, $G_N^{'}(cx, cy, cz, t) = \vee\{\alpha \in (0,1); \|cx, cy, cz\|_\alpha \leq t\}$
$= \vee\{\alpha \in (0,1); \|x, y, z\|_\alpha \leq \frac{t}{|c|}\}$
$= G_N^{'}(x, y, z, \frac{t}{|c|})$

(G_N5) We have to show that $\forall s, t \in \mathbb{R}$,
$G_N^{'}(x+x^{'}, y+y^{'}, z+z^{'}, t+s) \geq G_N^{'}(x, y, z, t) * G_N^{'}(x^{'}, y^{'}, z^{'}, s)$,
If possible suppose that, $\exists\ x, y, z, x^{'}, y^{'}, z^{'}, t\ and\ s$ such that
$G_N^{'}(x+x^{'}, y+y^{'}, z+z^{'}, t+s) < G_N^{'}(x, y, z, t) * G_N^{'}(x^{'}, y^{'}, z^{'}, s)$,
Then $\exists\ 1-\alpha_0 \in (0,1)$ such that
$G_N^{'}(x+x^{'}, y+y^{'}, z+z^{'}, t+s) < 1-\alpha_0 < G_N^{'}(x, y, z, t) * G_N^{'}(x^{'}, y^{'}, z^{'}, s)$,
Since $*$ is lower semi continuous,
then $\exists\ (1-\alpha_1) < G_N^{'}(x, y, z, s),\ (1-\alpha_2) < G_N^{'}(x^{'}, y^{'}, z^{'}, t)$ such that
$(1-\alpha_1) * (1-\alpha_2) \geq (1-\alpha_0)$
$\Rightarrow \|x, y, z\|_{\alpha_1} \leq t,\ \|x, y, z\|_{\alpha_2} \leq s$
Now by the given condition $\|x+x^{'}, y+y^{'}, z+z^{'}\|_{1-(1-\alpha_1)*(1-\alpha_2)} \leq \|x, y, z\|_{\alpha_1} + \|x^{'}, y^{'}, z^{'}\|_{\alpha_2}$
Since $\alpha_0 \geq 1-(1-\alpha_1)*(1-\alpha_2)$ and $\|., ., .\|_\alpha$ is non-increasing w.r.t α then,
$\|x+x^{'}, y+y^{'}, z+z^{'}\|_{\alpha_0} \leq \|x, y, z\|_{\alpha_1} + \|x^{'}, y^{'}, z^{'}\|_{\alpha_2} \leq t+s$
Thus $G_N^{'}(x+x^{'}, y+y^{'}, z+z^{'}, t+s) \geq 1-\alpha_0$ - a contradiction.
Thus, $G_N^{'}(x+x^{'}, y+y^{'}, z+z^{'}, t+s) \geq G_N^{'}(x, y, z, t) * G_N^{'}(x^{'}, y^{'}, z^{'}, s)$,

(G_N6) Now $t > \|x, y, z\|_{1-\alpha} \Rightarrow G_N^{'}(x, y, z, t) = \vee\{\beta \in (0,1)\} : \ \|x, y, z\|_{1-\beta} \leq t\} \geq \alpha$, So, $\lim\limits_{t\to\infty} G_N^{'}(x, y, z, t) = 1$

(G_N7) We have, $\forall\ \alpha \in (0,1), \ \forall t > 0\ \|x, y, z\|_{1-\alpha} \geq \|x+y, \theta, z\|_{1-\alpha}$
$\Rightarrow \{\alpha \in (0,1) : \ \|x, y, z\|_{1-\alpha} \leq t\} \subseteq \{\alpha \in (0,1) : \|x+y, \theta, z\|_{1-\alpha} \leq t\}$
$\Rightarrow \vee\{\alpha \in (0,1) : \ \|x, y, z\|_{1-\alpha} \leq t\} \leq \vee\{\alpha \in (0,1) : \|x+y, \theta, z\|_{1-\alpha} \leq t\}$
$\Rightarrow G_N^{'}(x, y, z, t) \leq G_N^{'}(x+y, \theta, z, t)$

18

If $t \leq 0,\ G_N^{'}(x, y, z, t) = 0 = G_N^{'}(x + y, \theta, z, t)$.
Thus $(X, G_N^{'}, *)$ is a G-fuzzy normed linear space.

□

6 Relation between G_N and $G_N^{'}$

Definition 6.1. *Let X be a linear space and G_N be a G-fuzzy norm on X. Define $G_N(x, y, z, t+) = G_{N+}(x, y, z, t) = \lim_{s \downarrow t} G_N(x, y, z, s)$ and $G_N(x, y, z, t-) = G_{N-}(x, y, z, t) = \lim_{s \uparrow t} G_N(x, y, z, s)$.*

Theorem 6.2. *Let X be a linear space and G_{N1}, G_{N2} be two G-fuzzy norms on X. Then $\forall\ x \in X,\ \forall t \in \mathbb{R},\ G_{N1}(x, y, z, t+) = G_{N2}(x, y, z, t+)$ and $G_{N1}(x, y, z, t-) = G_{N2}(x, y, z, t-)$ iff $\|x, y, z\|_\alpha^1 = \|x, y, z\|_\alpha^2, \forall \alpha \in (0, 1)$, where $\|., ., .\|_\alpha^1, \|., ., .\|_\alpha^2$ denote the corresponding quasi G-norm families of G_{N_1} and G_{N_2}.*

Proof. First we suppose that, $\|x, y, z\|_\alpha^1 = \|x, y, z\|_\alpha^2,\ \forall\ \alpha \in (0, 1)$.
If possible suppose that for some $t = t_0 \in \mathbb{R},\ G_{N1}(x, y, z, t_0+) \neq G_{N2}(x, y, z, t_0+)$. Without loss of generality we may assume that, $G_{N1}(x, y, z, t_0+) < G_{N2}(x, y, z, t_0+)$.
Choose $\beta \in (0, 1)$ such that, $G_{N1}(x, y, z, t_0+) < 1-\beta < G_{N2}(x, y, z, t_0+)$.
Then $\exists\ \epsilon > 0$ *such that for* $t_0 < t < t_0 + \epsilon$,
$G_{N1}(x, y, z, t) < G_{N2}(x, y, z, t)$
We have,
$\|x, y, z\|_\alpha^1 = \wedge\{t > 0 : G_{N1}(x, y, z, t) \geq 1 - \alpha\}$.........................(i)
$\|x, y, z\|_\alpha^2 = \wedge\{t > 0 : G_{N2}(x, y, z, t) \geq 1 - \alpha\}$.......................$(ii)$
Thus we have, $\|x, y, z\|_\beta^1 \geq t_0 + \epsilon$ and $\|x, y, z\|_\beta^2 \leq t_0$ - which is a contradiction of the hypothesis.
Therefore $G_{N1}(x, y, z, t+) = G_{N2}(x, y, z, t+),\ \forall t \in \mathbb{R}$
Similarly, $G_{N1}(x, y, z, t-) = G_{N2}(x, y, z, t-),\ \forall t \in \mathbb{R}$.
Conversely suppose that,
$G_{N1}(x, y, z, t+) = G_{N2}(x, y, z, t+),\ G_{N1}(x, y, z, t-) = G_{N2}(x, y, z, t-)$ hold $\forall t \in \mathbb{R}$. We have to show that $\|x, y, z\|_\alpha^1 = \|x, y, z\|_\alpha^2$ $\forall \alpha \in (0, 1)$.
If possible let $\exists\ \alpha_0 \in (0, 1)$ such that $\|x, y, z\|_{\alpha_0}^1 \neq \|x, y, z\|_{\alpha_0}^2$.
With out loss of generality suppose that, $\|x, y, z\|_{\alpha_0}^1 > \|x, y, z\|_{\alpha_0}^2$
Choose k_1, k_2, k_3 such that $\|x, y, z\|_{\alpha_0}^1 > k_1 > k_2 > k_3 > \|x, y, z\|_{\alpha_0}^2$..................$(iii)$
Then by using (iii) we get,

$G_{N1}(x,y,z,k_1) < 1-\alpha_0$ and $G_{N2}(x,y,z,k_3) \geq 1-\alpha_0$.........................(iv)
Now by using (iii) $(1-\alpha_0) > G_{N1}(x,y,z,k_1) > G_{N1}(x,y,z,k_2+)$.
Again, $G_{N2}(x,y,z,k_2-) > G_{N2}(x,y,z,k_3) \geq (1-\alpha_0)$
Thus, combining the above two results,
$G_{N2}(x,y,z,k_2+) \geq G_{N2}(x,y,z,k_2-) > 1-\alpha_0 > G_{N1}(x,y,z,k_2+)$
- a contradiction to the assumption.
Therefore $\|x,y,z\|^1_\alpha = \|x,y,z\|^2_\alpha,\ \forall \alpha \in (0,1)$ □

Definition 6.3. *Let X be a linear space and G_{N1}, G_{N2} be two G-fuzzy norms on X. G_{N1} and G_{N2} are said to be equipotent if, $G_{N1}(x,y,z,t-) = G_{N2}(x,y,z,t-)$ and $G_{N1}(x,y,z,t+) = G_{N2}(x,y,z,t+),\ \forall\ x,y,z \in X$.*

Theorem 6.4. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space, where $*$ is lower semi continuous and satisfying $(T5)$ and $\|x,y,z\|_\alpha = \wedge\{t > 0 : G_N(x,y,z,t) \geq 1-\alpha\},\ \alpha \in (0,1)$. A function $G'_N : X^3 \times \mathbb{R} \to [0,1]$ be defined as*

$$G'_N(x,y,z,t) = \begin{cases} \vee\{\alpha \in (0,1) :\ \|x,y,z\|_{1-\alpha} \leq t\} & for\ t > 0 \\ 0, & for\ t \leq 0. \end{cases}$$

Then G'_N is a G-fuzzy norm on X and G_N and G'_N are equipotent.

Proof. By Theorem 5.5, it follows that G'_N is a G-fuzzy norm on X.
We have $\|x,y,z\|_\alpha = \wedge\{t > 0 : G_N(x,y,z,t) \geq (1-\alpha)\},\ \alpha \in (0,1)$.
We have to show that
$G_N(x,y,z,t-) = G'_N(x,y,z,t-)$ and $G_N(x,y,z,t+) = G'_N(x,y,z,t+),\ \forall x,y,z \in X$ and $\forall t \in \mathbb{R}$.
If possible suppose that for some $t = t_0 \in \mathbb{R}$ and some $x,y,z \in X$,
$G_N(x,y,z,t_0-) \neq G'_N(x,y,z,t_0-)$.
Without loss of generality we may suppose that, $G_N(x,y,z,t_0-) < G'_N(x,y,z,t_0-)$.
Choose $\beta \in (0,1)$ such that $G_N(x,y,z,t_0-) < 1-\beta < G'_N(x,y,z,t_0-)$.
Then $\exists \epsilon > 0$ such that $t_0 - \epsilon < t < t_0$, $G_N(x,y,z,t) < 1-\beta < G'_N(x,y,z,t)$.
Now for $t_0 - \epsilon < t < t_0$, $G_N(x,y,z,t) < 1-\beta \Rightarrow \|x,y,z\|_\beta \geq t_0$
Now for $t_0 - \epsilon < t < t_0$, $G'_N(x,y,z,t) > 1-\beta \Rightarrow \|x,y,z\|_\beta \leq t < t_0$.
Thus we arrive at a contradiction.
Therefore $G_N(x,y,z,t_0-) = G'_N(x,y,z,t_0-)$.
Similarly, $G_N(x,y,z,t_0+) = G'_N(x,y,z,t_0+)$.
Hence G_N and G'_N are equipotent. □

However under the following condition
($\mathbf{G_N 9}$) for $(x, y, z) \neq (\theta, \theta, \theta)$, $G_N(x, y, z, .)$ is a continuous function of $\mathbb{R}$ (set of all real numbers),
the relation between G_N, $G_N^{'}$ becomes the relation of identity.
In fact, we have the following theorem:

Theorem 6.5. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space satisfying $(G_N 9)$ for any lower semi continuous t-norm $*$ satisfying $(T5)$. Let $\|x, y, z\|_\alpha = \wedge\{t > 0 : G_N(x, y, z, t) \geq 1 - \alpha\}$, $\alpha \in (0, 1)$, $G_N^{'} : X^3 \times \mathbb{R} \to [0, 1]$ be a function defined by*

$$G_N^{'}(x, y, z, t) = \begin{cases} \vee\{\alpha \in (0,1) : \ \|x, y, z\|_{1-\alpha} \leq t\} & for\ t > 0 \\ 0, & for\ t \leq 0. \end{cases}$$

Then

(*i*) *$Q = \{\|., ., .\|_\alpha : \alpha \in (0, 1)\}$ is a Quasi G-norm family on X satisfying*
$\|x + x^{'}, y + y^{'}, z + z^{'}\|_{1-(1-\alpha)*(1-\beta)} \leq \|x, y, z\|_\alpha + \|x^{'}, y^{'}, z^{'}\|_\beta$,
$\forall x, y, z, x^{'}, y^{'}, z^{'} \in X$, $\forall \alpha, \beta \in (0, 1)$.

(*ii*) *$G_N^{'}$ is a G-fuzzy norm on X.*

(*iii*) $G_N^{'} = G_N$

To prove the theorem first we prove the following lemma.

Lemma 6.6. *Let $(X, G_N, *)$ be a G-fuzzy normed linear space, $(x_0, y_0, z_0) \neq (\theta, \theta, \theta)$ and $Q = \{\|., ., .\|\alpha : \alpha \in (0, 1)\}$ be the corresponding quasi G-norm family on X corresponding to the G-fuzzy norm G_N. Then*

(1) If $G_N(x_0, y_0, z_0, .)$ is upper semi continuous and if for $t_0 > 0$, $G_N(x_0, y_0, , z_0, t_0) = 1 - \alpha_0 \in (0, 1)$ then $G_N(x_0, y_0, z_0, \|x_0, y_0, z_0\|_{\alpha_0}) = (1 - \alpha_0)$.

(2) If $G_N(x_0, y_0, z_0, .)$ is continuous, then for any $\alpha \in (0, 1)$, $G_N(x_0, y_0, z_0, \|x_0, y_0, z_0\|_\alpha) = 1 - \alpha$.

(3) If $G_N(x_0, y_0, z_0, .)$ is continuous and strictly increasing for $t > 0$, then $G_N(x_0, y_0, z_0, t) = 1 - \alpha \Leftrightarrow \|x_0, y_0, z_0\|_\alpha = t$.

Proof. (1) From the definition we get, $\|x_0, y_0, z_0\|_{\alpha_0} = \wedge\{t > 0 : G_N(x_0, y_0, z_0, t) \geq 1 - \alpha_0\}$.

Since $G_N(x_0, y_0, z_0, t_0) = 1 - \alpha_0 \Rightarrow \|x_0, y_0, z_0\|_{\alpha_0} \leq t_0$
Since $G_N(x_0, y_0, z_0, .)$ is non-decreasing then,
$G_N(x_0, y_0, z_0, \|x_0, y_0, z_0\|_{\alpha_0}) \leq (1 - \alpha_0)$.
If possible suppose that,
$G_N(x_0, y_0, z_0, \|x_0, y_0, z_0\|_{\alpha_0}) < (1 - \alpha_0)$
Then by the upper semi continuity of $G_N(x_0, y_0, z_0, .)$,
$\exists\ t' > \|x_0, y_0, z_0\|_{\alpha_0}$ such that, $G_N(x_0, y_0, z_0, t') < (1 - \alpha_0)$.
Then $\|x_0, y_0, z_0\|_{\alpha_0} > t' > \|x_0, y_0, z_0\|_{\alpha_0}$ - a contradiction.
So, $G_N(x_0, y_0, z_0, \|x_0, y_0, z_0\|_{\alpha_0}) = (1 - \alpha_0)$.

(2) Since $G_N(x_0, y_0, z_0, .)$ is continuous by (G_N1) and (G_N6) for each $\alpha \in (0, 1)$ $\exists\ t > 0$ such that $G_N(x_0, y_0, z_0, t) = 1 - \alpha_0$. Then by (1), $G_N(x_0, y_0, z_0, \|x_0, y_0, z_0\|_{\alpha}) = (1 - \alpha)$, $\forall \alpha \in (0, 1)$.

(3) By (1) and (2) and using strict increasing property (3) holds.
□

Proof. We consider the following cases:
Let $(x_0, y_0, z_0, t_0) \in X^3 \times \mathbb{R}$.
Case-I
$t_0 \leq 0$, then $G_N(x_0, y_0, z_0, t_0) = 0 = G'_N(x_0, y_0, z_0, t_0)$
Case-II
$(x_0, y_0, z_0) = (\theta, \theta, \theta),\ t_0 > 0$, then $G_N(x_0, y_0, z_0, t_0) = 1 = G'_N(x_0, y_0, z_0, t_0)$
Case-III
$(x_0, y_0, z_0) \neq (\theta, \theta, \theta),\ t_0(> 0) \in \mathbb{R}$ such that $G_N(x_0, y_0, z_0, t_0) = 0$
For $\alpha \in (0, 1)$; $\|x_0, y_0, z_0\|_{\alpha} = \wedge\{t > 0 : G_N(x_0, y_0, z_0, t) \geq 1 - \alpha\}$
Now by Lemma (2) of 6.6 $G_N(x_0, y_0, z_0, \|x_0, y_0, z_0\|_{\alpha}) = (1 - \alpha)$.
Since $G_N(x_0, y_0, z_0, t_0) = 0 < 1 - \alpha$, $\forall \alpha \in (0, 1) \Rightarrow t_0 < \|x_0, y_0, z_0\|_{\alpha}$, $\forall\ \alpha \in (0, 1)$.
So, $G'_N(x_0, y_0, z_0, t_0) = \vee\{\alpha_0 \in (0, 1) :\ \|x, y, z\|_{1-\alpha_0} \leq t_0\}$
$= \vee\ \phi = 0$
Therefore $G_N(x_0, y_0, z_0, t_0) = 0 = G'_N(x_0, y_0, z_0, t_0)$
Case-IV
Let $(x_0, y_0, z_0) \neq (\theta, \theta, \theta),\ t_0(> 0) \in \mathbb{R}$ such that
$0 < G_N(x_0, y_0, z_0, t_0) < 1$.
Again suppose that $G_N(x_0, y_0, z_0, t_0) = 1 - \alpha_0 \in (0, 1)$.
Now $G'_N(x_0, y_0, z_0, t) = \vee\{\alpha \in (0, 1) :\ \|x_0, y_0, z_0\|_{1-\alpha} \leq t\}$
and $\|x_0, y_0, z_0\|_{\alpha} = \wedge\{t > 0 : G_N(x_0, y_0, z_0, t) \geq 1 - \alpha\}$
Since $G_N(x_0, y_0, z_0, t_0) = 1 - \alpha_0 \Rightarrow \|x_0, y_0, z_0\|_{\alpha_0} \leq t_0$

22

$\Rightarrow G'_N(x_0, y_0, z_0, t_0) \geq 1 - \alpha_0 = G_N(x_0, y_0, z_0, t_0)$................(1)
For $\alpha \in (0, \alpha_0)$, let $\|x_0, y_0, z_0\|_\alpha = t'$.
By the lemma (2) of 6.6,
$G_N(x_0, y_0, z_0, t') = 1 - \alpha > 1 - \alpha_0 = G_N(x_0, y_0, z_0, t_0)$
Since $G_N(x, y, z, .)$ is monotonically increasing w.r.t t, thus $t' > t_0$.
So, for each $\alpha \in (0, \alpha_0)$, $\|x_0, y_0, z_0\|_\alpha = t' \nleq t_0$
So, $G'_N(x_0, y_0, z_0, t_0) \leq 1 - \alpha_0 = G_N(x_0, y_0, z_0, t_0)$................(2)
From (1) and (2), $G_N(x_0, y_0, z_0, t_0) = G'_N(x_0, y_0, z_0, t_0)$
Case V
When $(x_0, y_0, z_0) \neq (\theta, \theta, \theta)$ and $t_0(> 0) \in \mathbb{R}$
such that $G_N(x_0, y_0, z_0, t_0) = 1$.
Note that, $G'_N(x_0, y_0, z_0, t) = \begin{cases} \vee\{\alpha \in (0,1) : \ \|x_0, y_0, z_0\|_{1-\alpha} \leq t\} & for\ t > 0 \\ 0, & for\ t \leq 0. \end{cases}$
and $\|x_0, y_0, z_0\|_\alpha = \wedge\{t > 0 : G_N(x_0, y_0, z_0, t) \geq 1 - \alpha\}$,
$\forall \alpha \in (0, 1),\ x_0, y_0, z_0 \in X$.
We have $\|x_0, y_0, z_0\|_{1-\alpha} \leq t_0,\ \forall \alpha \in (0, 1) \Rightarrow G'_N(x_0, y_0, z_0, t_0) = 1$.
Thus $G_N(x_0, y_0, z_0, t_0) = G'_N(x_0, y_0, z_0, t_0)$.
Hence $G_N(x, y, z, t) = G'_N(x, y, z, t)\ \forall (x, y, z, t) \in X^3 \times \mathbb{R}$. □

7 Conclusion

Previously several authors introduced the idea of generalized fuzzy normed linear space viz. D-fuzzy metric, D^*-fuzzy metric etc. Following the concept G-normed space, in this paper G-fuzzy normed linear space is introduced. Decomposition theorem from a G-fuzzy normed linear space into a family of G-normed space have been established. Underlying topology of G-fuzzy normed linear space is also defined. There is a wide scope of research to develop fuzzy functional analysis by using the concept of G-fuzzy normed linear space.

References

[1] T. Bag, S. K. Samanta, Finite dimensional fuzzy normed linear spaces, *The Journal of Fuzzy Mathematics*, **11(3)** (2003), 687-705.

[2] T. Bag and S. K. Samanta, Fuzzy bounded linear operators, *Fuzzy sets and Systems*, **151(3)** (2005), 513-547.

[3] T. Bag and S. K. Samanta,Finite dimensional fuzzy normed linear spaces, *Ann. Fuzzy Math. Inform*, **6(2)** (2013), 271-283.

[4] Bapurao C Dhage, Generalised metric space and mappings with fixed point,(1992).

[5] Siegfried Gähler, 2-metrische räume and ihre topologische strukture, *Mathematische Nachrichten*, **26(1-2)** (1963), 115-184.

[6] Siegfried Gähler, Lineare 2-normierte räume, *Mathematische Nachrichten*, **28(1-2)** (1964), 1-43.

[7] Ki Sik Ha, Yeol Je Cho, and Albert White, Strictly convex and strictly 2-convex 2-normed spaces, *Mathematica Japonica*, **33(3)**(1988) 375-384.

[8] K. A. Khan, Generalized normed spaces and fixed point theorems, *Journal of Mathematics and Computer Science*, **13** (2014), 157-167.

[9] George J Klir, Bo Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, New Delhi (1997).

[10] Zead Mustafa and Brailey Sims, Some remarks concerning D-metric spaces, *Proceedings of the International Conferences on Fixed Point Theory and Applications*, Valencia(Spain) (July 2003), 189-198.

[11] Zead Mustafa and Brailey Sims, A new approach to generalized metric spaces, *Journal of Nonlinear and convex Analysis*, **7(2)** (2006), 289-297.

[12] Sorin Nădăban and Ioan Dzitac, Atomic decompositions of fuzzy normed linear spaces for wavelet applications, *Informatica*, **25(4)** (2014), 643-662.

[13] G. Rano, T. Bag, and S. K. Samanta, Bounded linear operators in generating spaces of quasi-norm family, *Journal of Fuzzy Mathematics*, **21(1)** (2013), 51-58.

[14] G. Rano, T. Bag, and S.K. Samanta, Relation between fuzzy normed linear space and generating space of quasi-norm family, *The Journal of Fuzzy Mathematics*, **21(3)** (2013), 667-688.

[15] G. Sun and K. Yang, Generalized fuzzy metric spaces with properties, *Research Journal of Applied Sciences, Engineering and Technology*, **2(7)** (2010), 673-678.