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RESTRICTING THE GAP BETWEEN THE ERROR TERMS OF Ω-RESULTS FOR $(N_3 - \tau x)$ AND THE ERROR TERMS OF O-RESULTS FOR $(N - \tau x)$ ON R.H

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Abstract

This article showed a study of restricting the gap between the error terms of **Ω**-results for $(N_{\upsilon} - \tau x)$ and the error terms of O-results for $(N - \tau x)$ on Riemann Hypothesis. Note that the **Ω**-results is a generalized **Ω**-results for N_p as counting function of Beurling. Her a generalized prime system $\mathcal{P}$ is a sequence of positive reals $p1,\ p2,\ p3,\ \ldots$ satisfying $1 < p1 \leq p2 \leq p3 \leq \ldots$ and for which $p_k \to \infty$ as $k \to \infty$. The $\{p_k\}$ are called generalized primes (or Beurlingprimes) with the products $p_1^{\beta_1}.p_2^{\beta_2} \ldots p_k^{\beta_k}$ (where $k \in \mathbb{N}$ and $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{N} \cup \{0\}$) forming the generalized integers (or Beurling integers).

Keywords**:** *analytic number theory; Beurling prime system; Riemann zeta function*

1. Introduction

Remark

Note that her we write

1. $\log_2 x$ is mean $\log\log x$, $\log_3 x$ means $\log\log\log x$ and $\log_4 x$ means $\log\log\log\log x$ for short.
2. R.H. means Riemann hypothesis.

In 1937, Beurling [5] defined his systems (or generalized system) as any sequencesof reals satisfies the following:

$\mathcal{P} = \{p_k\}_{k=1}^{\infty}$ such that p_k greater than 1, strictly increasing function of reals tending to ∞ as k tending to ∞. The generalized (or Beurlings) integers defined to be in this manner as $\{n_k\}_{k=1}^{\infty}$ where $n_k = \prod_{i=1}^{k} p^{a_i}$, where p_i are Beurling primes and a_i are natural numbers or could be zero.

In this sense, Beurling defined the counting function of Beurling prime and Beurling integers as follows:

$$N_P\,(x) = \sum_{\substack{n\leq x\\ n\in\{n_k\}}} 1 \quad and \quad \pi_P\,(x) = \sum_{\substack{p\leq x\\ p\in\{p_k\}}} 1\ ,$$

Beurling's problem proved that the generalized prime number theorem as follows, there is a system of generalized prime satisfied

$N_P(x) - \rho x = O\left(\frac{x}{(\log x)^{\alpha}}\right)$ and $\pi_P(x) = \frac{x}{\log x} + O\left(\frac{x}{\log x}\right)$.

The condition of α must be greater than $\frac{3}{2}$ is sharp. This has been proved by Diamond [4] where he gave an example with α equal $\frac{3}{2}$ but the g-prime number theorem does not work.

Many other authors have studied these systems (see for example [4],[7], [8], [10], [11] to name just a few) and found such connections.

In 2006, Hilberdink (see [9]) extended Diamond's result in [1970] (to $\alpha = 1$ case) as follows: if the relation

$\psi(x) = x + O(x(\log x)^{-\eta})$ for $\eta < 1$. Then

$N_P(x) - \rho x = O\left(x \exp\{-c\sqrt{\log x \log_2 x}\}\right)$ for some $\rho, c > 0$.

In this respect, the second author of this article has showed recently study in [3] there is a such system in which the generalized Chebyshev's function $\psi_{\mathfrak{z}}(x)$ would be the integer part of x minus 1. Where the g-primes very closed to being discrete, they investigate how regular the corresponding g- integer counting function $\mathrm{N}(x)$ is.

Letting $\zeta_{\mathfrak{z}}(s) = \widehat{\mathrm{N}}(s)$ denote the Mellin transform of N, they find that

$$-\frac{\zeta_{\mathfrak{z}}^{`}}{\zeta_{\mathfrak{z}}}(s) = \widehat{\psi_{\mathfrak{z}}}(s) = \zeta(s) - 1,$$

Where $\zeta(s)$ is the Riemann zeta function.They investigate the behavior of $N_{\mathfrak{z}}(x)$ as $x \to \infty$. It is immediate from Diamond's work [6] that

$N_{\mathfrak{z}}(x) \sim \tau x$, for some $\tau > 0$

And find O-results and Ω-results for $(N_{\mathfrak{z}} - \tau x)$, first they find O-results for $(N_{\mathfrak{z}} - \tau x)$ using result in [9] with

$$\psi_{\mathfrak{z}}(x) = x + O(1),$$

They get

$N_{\mathfrak{z}}(x) - \tau x = O\left(x \exp\left\{-c\sqrt{\log x \log_2 x}\right\}\right)$, fore some $c > 0$.

And they have

$N_{\mathfrak{z}}(x) - \tau x = O\left(x \exp\left\{-b(\log x)^{\frac{3}{5}}(\log_2 x^{\frac{2}{5}}\right\}\right)$, for some $b > 0$.

Furthermore, on the Riemann Hypothesis this can be improved to

$N_{\mathfrak{z}}(x) - \tau x = O\left(x \exp\left\{-\frac{(1-\epsilon)\log x \log_3 x}{4\log_2 x}\right\}\right)$, fore every $\epsilon > 0$.

Moreover, they turn their attention to find lower bound for $(N_{\mathfrak{z}}(x) - \tau x)$, they have

$$N_{\mathfrak{z}}(x) - \tau x = \Omega(x^{1-\delta}), \text{ for every} \delta > 0$$

And

$$N_{\mathfrak{z}}(x) - \tau x = \Omega(xe^{-ck(x)}) \text{for every } c > 1,$$

Where

$$k(x) = \frac{\log x \log_4 x}{\log_3 x}.$$

Set $\Delta(x) = N_{\mathfrak{z}}(x) - \tau x$: A comparison of the O-results and Ω-results (based on Theorem 6.1 and Theorem 6.3, see[1]), they have shown that

$$\Delta(x) = \Omega\left(x\exp\{-\frac{c \log x \log_4 x}{\log_3 x}\}\right) \text{ for every } c > 1,$$

While on the Riemann Hypothesis,

$$\Delta(x) \ll x\exp\{-\frac{(1-\epsilon) \log x \log_3 x}{4\log_2 x}\} \text{ for every } \epsilon > 1.$$

This shows that there is a small gap between these results which reflects the great difficulty in determining the behaviors of $\zeta_{\mathfrak{z}}(s)$ in the strip $\frac{1}{2} < \sigma < 1$. For more details see [2].

In our work, we restricting the gap between the error terms of Ω-results for $(N_{\mathfrak{z}} - \tau x)$ and the error terms of O-results for $(N - \tau x)$ on R.H.

2.Main result

Now moving our attention into reducing the gap between O-results of $(N - \tau x)$ as R.H holds and the Ω-results of $(N_{\mathfrak{z}} - \tau x)$ in general.Using the method adapting by Al-Maamori [1]. Where Al-Maamori shows in his sharing paper [1] that for σ running between $\frac{3}{4}$ and $(1 - \frac{\log_3 T}{2\log_2 T})$, we have the maximum of modals of Riemann zeta function for t greater than 1 less than T must be greater than

$$exp\left\{(1 + o(1)\frac{(\log T)^{1-\sigma}}{16(1-\sigma)\log_2 T}\right\},$$

For T greater than or equale T_0 independent of σ.

We will need the following preposition, which will be used in the proof of the main theorem.

2.1 Preposition *(special case)*

For σ running between $\frac{3}{4}$ and $(1-\sqrt{\frac{\log_3 T}{\log_2 T}})$ and for T greater than or equal T_0 independent of σ, we have the maximal of the modals of Riemann zeta function for t greater than 1 less than T must be greater than

$$exp\left\{(1+o(1)\frac{(\log T)^{1-\sigma}}{16(1-\sigma)\log_2 T}\right\}.$$

Proof: clear by preposition (2.4) in [1].

2.2 Theorem

We have

$$N_{\mathfrak{z}}(x)-\tau x=\Omega(xe^{-cf_x}) \text{ for every } c>1,$$

Where $f_x=\log x\sqrt{\frac{\log_4 x}{\log_3 x}}$.

Proof:

If $N_{\mathfrak{z}}(x)-\tau x=\Omega(xe^{-cf_x})$ is not true, then $N_{\mathfrak{z}}(x)-\tau x=O(xe^{-cf_x})$ for some (c>1).Now applying theorem (see [3, theorem 2.2]) with $h(t)=cf_x$ and using the same style of prove to obtain:

$$\zeta_{\mathfrak{z}}(\sigma+it)=O(t^{\delta}) \text{ for some } \delta>0$$

and for

$1-((1-\epsilon)c\sqrt{\frac{\log_3 T}{\log_2 T}})\ \leq\sigma\leq 1-\frac{\log t}{t}$, any $\epsilon>0$, since $\frac{h\left(\frac{e^t}{t}\right)}{t}\sim c\sqrt{\frac{\log_3 T}{\log_2 T}}$), as in [1].

Furthermore,

$$Re\{\log\zeta_{\mathfrak{z}}(\sigma+it)\}=\log|\zeta_{\mathfrak{z}}(\sigma+it)|\leq\int_{\sigma}^{2}|\frac{\zeta_{\mathfrak{z}}^{`}}{\zeta_{\mathfrak{z}}}(v+it)|dv+O(1)\ll\log t.$$

Since $\zeta(s)=O(\log t)$ for $1-\frac{a}{\log t}\leq\sigma\leq 2$ for any $(a>0)$, see theorem (3.5) in [12].

Let

$$\beta(t)=(1-\epsilon)c\sqrt{\frac{\log_3 T}{\log_2 T}}.$$

Therefore, for $1-\beta(t)\leq\sigma\leq 2$

$$\log|\zeta_{\mathfrak{z}}(\sigma+it)|\leq B\log t \text{ for some } B>0.$$

Consider the circles with centre $\theta+it$ for $(\theta>1)$ and radii $R_1=\theta-1+\beta(t)-\gamma(t)$ and $R_2=\theta-1+\beta(t)-2\gamma(t)$, (with $\gamma(t)=\frac{1}{\log_2 t}$).

Apply the Borel- Caratheodory theorem [12] which states:

Let f(z) be a holomorphic function on $|z| \le R$ and let $F(r) = \sup_{|z|=r} |f(z)|$ and $G(r) = \sup_{|z|=r} Re\, f(z)$ then for $0 < r < R$:

$$F(r) \le \frac{2r}{R-r} G(r) + \frac{R+r}{R-r} |f(0)|$$

Where $F(z) = \log \zeta_{\mathfrak{z}}(z)$ for $\sigma \ge 1 - \beta(t) + \gamma(t)$, we get

$$\log |\zeta_{\mathfrak{z}}(\sigma + it)| \ll \log t \log_2 t$$

Now let C be the circle with centre $1 - r\beta(t) + \gamma(t) + it$, for some $r \in (\frac{1}{2}, 1)$ and radius $R = r_1\beta(t)$ for some positive constant $r_1 < 1 - r$. By Cauchy's integral formula

$$\frac{\zeta_{\mathfrak{z}}'}{\zeta_{\mathfrak{z}}}(s) = \frac{1}{2\pi i}\int_C \frac{\log \zeta_{\mathfrak{z}}(z)}{(z-s)^2} dz \text{ for } s \in C.$$

Therefore, for $s \in C$, we have

(1) $|\zeta(s) - 1| = \left|\frac{\zeta_{\mathfrak{z}}'}{\zeta_{\mathfrak{z}}}(s)\right| \le \frac{1}{R} \max_{z \in C} |\log \zeta_{\mathfrak{z}}(z)|$

$$= o(\log t)^{\frac{5}{2}}$$

So, for $s \in C$, $\zeta(s) = o(\log t)^{\frac{5}{2}}$ for $\sigma \ge 1 - \beta(t)$.

However, by preposition 2.1 for

$$1 - \sigma = (1 - \epsilon)c\sqrt{\frac{\log_3 T}{\log_2 T}}.$$

We have

$$\max_{1<t<T} |\zeta(\sigma + it)| \ge exp\left\{(1 + o(1)\frac{(\log T)^{1-\sigma}}{16(1-\sigma)\log_2 T}\right\}$$

$$> exp\left\{e^{k\sqrt{\log_2 T \log_3 T}}\right\}$$

$$> e^{\frac{5}{2}\log_2 T} = (\log T)^{\frac{5}{2}}$$

Because of equation (1), This is a contradiction.

3. Conclusion

The above work explained the restricting of the gap between the error terms of **Ω**-results for $(N_{\mathfrak{z}} - \tau x)$ and the error terms of O-results for $(N - \tau x)$ on Riemann Hypothesis. Here, **Ω**-results mean the generalized **Ω**-results for the counting function of Beurling. One can easy consider that the gap is so strict and difficult to manipulate the estimation of the error terms of Beurling counting functions as x goes to infinity. The main question here is: Could we restricting the gap between the error terms of **Ω**-results for $(N_{\mathfrak{z}} - \tau x)$and the error terms of O-results for $(N - \tau x)$ on R.H. more than above? This question leads to a big future work for the authors.

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4. Appendix

The answer of the common question is states that:
For which α the above preposition still true and satisfies the theorem, is The following lemma:

4.1 Lemma

For σ running between $\frac{3}{4}$ and $(1-\sqrt{\frac{\log_3 T}{\log_2 T}})$, $\sigma < 1$ and for T greater than or equal T_0 independent of σ, we have the maximal of the modals of Riemann zeta function for t greater than 1 less than T must be greater than

$$exp\left\{(1+o(1)\frac{(\log T)^{1-\sigma}}{16(1-\sigma)\log_2 T}\right\}.$$

The following theorem shows the best possible α to restricting the gap between the error terms of Ω-results for $(N_{\mathfrak{z}} - \tau x)$ and the error terms of O-results for $(N - \tau x)$ on R.H.

4.2 Theorem

We have

$$N_{\mathfrak{z}}(x) - \tau x = \Omega(xe^{-cf_x}) \text{ for every } c > 1,$$

Where $f_x = \log x \left(\frac{\log_4 T}{\log_3 T}\right)^{\alpha}$, for all $\alpha < 1$.

Proof:

If the result is false, then $N_{\mathfrak{z}}(x) - \tau x = O(xe^{-cf_x})$ for some (c>1). Now apply theorem (see [3 , theorem 2.2] with $h(x) = cf_x$, we have $\frac{h(\frac{e^t}{t})}{t} \sim c\left(\frac{\log_3 t}{\log_2 t}\right)^{\alpha}$. The conditions of the theorem are satisfied, and so

$$\zeta_{\mathfrak{z}}(\sigma + it) = O(t^{\delta}) \text{ for some } \delta > 0$$

and for

$1 - ((1-\epsilon)c\left(\frac{\log_3 t}{\log_2 t}\right)^{\alpha}) \leq \sigma \leq 1 - \frac{\log t}{t}$, (any $\epsilon > 0$, $t > t_0(\epsilon)$).

We show this is incompatible with known Ω-result for $\zeta(\sigma + it)$. For this we use lemma (3.1) for

$$1 - \sigma = (1-\epsilon)c\left(\frac{\log_3 t}{\log_2 t}\right)^{\alpha}$$

and using the same style of prove to find:

(2) $\max_{1<t<T} |\zeta(\sigma + it)| > e^{(1+\mu)\log_2 T} = (\log T)^{1+\mu}, \forall \mu > 0.$

Now apply Borel- Caratheodory theorem for $\log \zeta_{\mathfrak{z}}(z)$ and the circles with centre $\theta + it$ for $(\theta > 1)$ and radii $R_1 = \theta - 1 + \beta(t) - \gamma(t)$ and $R_2 = \theta - 1 + \beta(t) - 2\gamma(t)$, (with $\gamma(t) = \frac{1}{\log_2 t}$). Where $\sigma \geq 1 - \beta(t) + \gamma(t)$ and $t > t_0$, we get:

$$\log |\zeta_{\mathfrak{z}}(\sigma + it)| \ll \log_3 t.$$

Now, let C be the circle with centre $1 - r\beta(t) + \gamma(t) + it$, for some $r \in (\frac{1}{2}, 1)$ and radius $R = r_1\beta(t)$ for some positive constant $r_1 < 1 - r$ and by Cauchy's integral formula We have

$$|\zeta(s) - 1| = \left|\frac{\zeta_{\mathfrak{z}}'}{\zeta_{\mathfrak{z}}}(s)\right| \leq \frac{1}{R} \max_{z \in C} |\log \zeta_{\mathfrak{z}}(z)|$$
$$= o(\log t)^{1+\mu},$$

$\forall\, \mu > 0$

Because of (2), this is a contradiction