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A study of Pitch Estimation of Musical Notesand Three Derivations of 22 Shrutis in Indian Classical Music

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Abstract: Music has evolved in different ways through ages across different countries and cultures of the world.Music comprises of several tones having a definite pitch. However, the set of musical notesused by musiciansand relationship between those notes regarding frequency ratios,in different systems of music around the globe, do vary but follow similar characteristics.Hence, the study of music from a signal processing perspective requires extraction and processing of the pitch information of these tones.In this paper,we have introduced interesting features of music notes and ratios associated with themare explored.

The main objective of this study is to address pitch extraction in musical notes using various features based methods of a signal. In addition, this paper considers Indian Classical Music systems, which advocate usage of 22-musical notesmodelor 22-Shruti modelthat is widely mentioned in music literature.The studyof musical notes and relationship among these 22 Shrutis or musical intervals by musicologists roots back to one of India’s classical texts, NatyaShastra by Bharatha Muni.Meanwhile, many scholars have presented different ideas about the Shrutis and proposed different ratios. In this paper,the shruti ratio such as, [1: 256/243: 16/15: 10/9: 9/8 : 32/27: 6/5: 5/4 : 81/64: 4/3: 27/20: 45/32 : 729/512 : 3/2 : 128/81: 8/5: 5/3: 27/16: 16/9: 9/5: 15/8 : 243/128: 2] with associated notes (S: R1’: R1 : R2’: R2 :

G2’: G2: G3: G3’: M1: M1’: M2: M2’: P : D1’: D1: D2’: D2: N2’: N2: N3 : N3’: S(high)) are considered.Three new methods along with two inspired methods are given to derive the 22 Shrutis logically and a comparison is providedto the musical frequency ratios used in Western music.

Keywords:frequency,harmonics, consonance, musical notes, musical intervals, Pythagorean scale, Natural scale

1. Introduction

Generally, Music is an essential facet of and practice in human cultures, significantly associated to itscapacity to encourage a range of intense as well as complex emotions. Studying the psychological as well as neurophysiological responses to music permits us to scrutinize and uncover the neural methods underlying the emotional impact of music. Music has two main elements that are pitch and rhythm. Rhythm is reffered is as the element of Time in the music. The Pitch is reffered as a perceptual property of sounds, which permits their ordering on a frequency-related scale. Additionally, the pitch is the quality, which create it feasible to judge sounds as "higher" as well as "lower" in the sense linked with musical melodies [14] [15] .

In this paper, we focus only on the pitch aspect of music. In Fig 1, 2 and 3, when a musical signal or a random sound signalis plottedas a function of time, at that point that musical signal is more structured [3], and [7]. A musical signal consists of a discrete set of frequencies with a definite relation among the frequencies present in it, which is based on the music knowledge available from different cultures in the world [2], [7], and [10].Generally, a musical noteis identified with a fundamental frequency and its harmonics [2], [7], and [10]. The fundamental frequency is defined as the inverse of the fundamental period and the time after in which the signal repeats and harmonics are signals whose frequency is a multiple of the fundamental frequency. A musical note can be closely approximated with fundamental frequency f_0 and its N harmonics and can be represented using the Eq. (1),

$$x(t) = \sum_{k=1}^{N} a_k \cos(2\pi k f_0 t) + \sum_{k=1}^{N} b_k \sin(2\pi k f_0 t) \quad (1)$$

Fig. 2 and 3 illustrate the musical noteswith the same fundamental frequency of 440Hzbutdifferent harmonic structure. The structure of harmonics of several instruments is known as timbre. It assists in differentiating between instruments playing the same frequency (fundamental frequency). Nevertheless, a musical note consists of harmonics that is mostly identified with its fundamental frequency, which is well-known as the pitch of the signal.The harmonics associated with notesare mentioned in this paper.Music is produced by playing musical notes with different fundamental frequencies one after another or even together. The primary interest that prompted to write this paper was to explore why only a certain set of related musical notes are exploited across various music cultures [2], [3], and [4] and why cannot any group of musical signals, which follow the eq.(1)Evoke a sense of music. Be it human voice or an instrument, when successive frequencies producedto share some interesting relationship between them produce a pleasing musical sound to human ears [1], [2], [3], [4], [5], [7], [12] and [19].Hence, a mathematical and logical interpretation of relation among the musical signals will help to understand and define music better. In the literature [1], [2], and [4], many aspects of music havebeen studiedextensively.

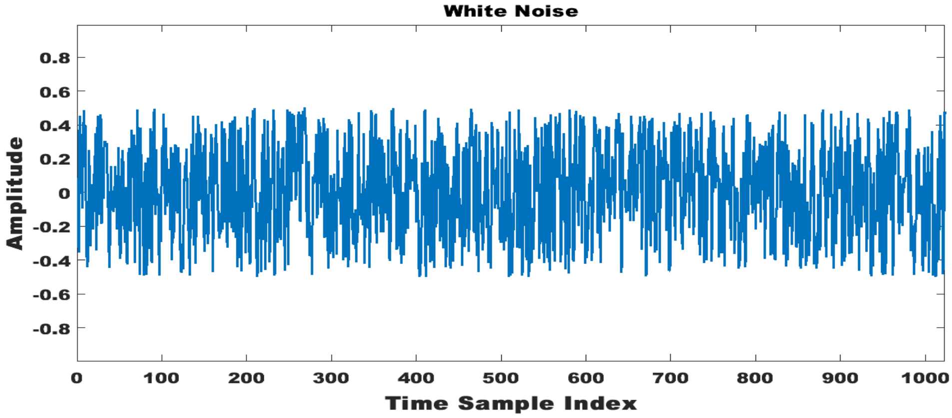

Fig.1: Graphical representation of white Noise Signal

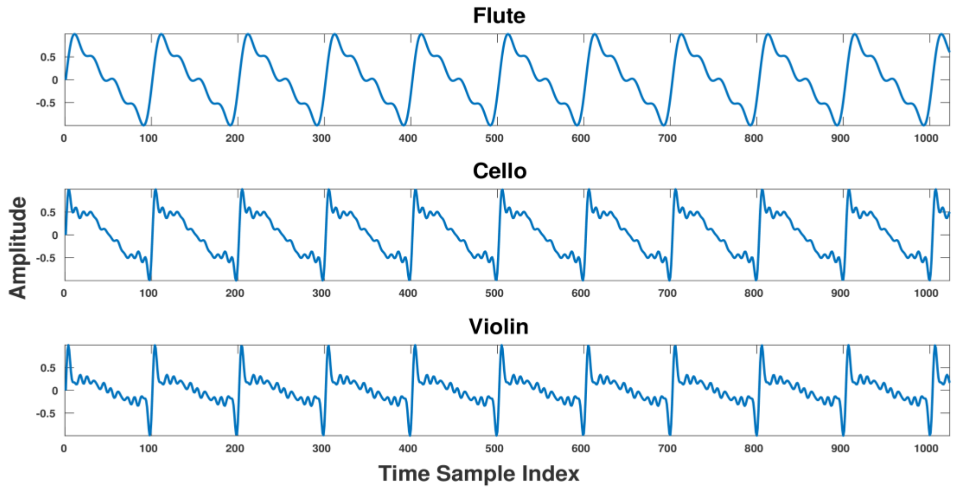

Fig.2: Graphical representation of musical signal with a fundamental frequency of 440Hz from various instruments

The application of signal processing to music signals is new approach. It could be argued to be the basis of the theremin, which is a 1920's instrument that is an oscillator and controlled by the capacitance of the player's hands near its antennae. The advancements of modern signal processing in the 1940s and 1950s led directly the first wave of electronic music in that composers namely Karlheinz Stockhausen created music exploiting signal generators, ring modulators, etc., taken straight from electronics labs. Subsequent the arrival of general-purpose digital computers in the 1960s and 1970s, it was not long before they were exploited to synthesize music by pioneers such as Max Matthews and John Pierce. Experimental music has remained a steady source of novel applications of signal processing, as well as has spawned animportant body of sophisticated methods for synthesizing as well as modifying sounds.

In this paper, we restrict ourselves to one of the most interesting and challenging aspects. In addition, the mention of 22 shrutis or musical intervals in Bharatha Muni's NatyaShastra[13],and other music cultures in the world [2], [4], [5], [8], [9], [10], [11].Shrutis in simple words can be defined as the set of musical notes, which can evoke a sense of music [1], [2], and [13].The foundation of Indian Classical Music (ICM) is the Raga system, whichis expected to select a set of notes,and it is a subset of the superset "Shruti". Raga is a collection of notes, which is similar to Western scale along with a set of rules for its rendition [1], [3], [4], [6], [8], [9], [11],[12] [16] [17]and [18]. The rules can include the pattern of ascent and descent of notes, characteristic phrases specific to a raga, the significance of various notes, the method of transition between notes etc. Together these notes of a raga with its rules help musicians to emote a certain mood [4], [5], [9], [11], and [13]. Hundreds and thousands of subsets to define various ragas is proposed, and studied over centuries in Indian music. Hence, Shruti in other words can also be defined as the notes exploited across all these ragas together.

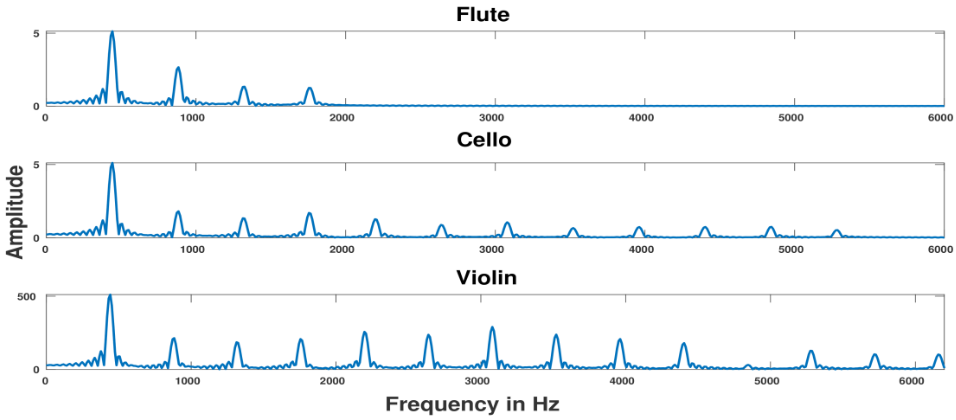

Fig.3: Graphical representation of Fourier Transform displaying the strengths of harmonics of various instruments

What are the elements which can be part of this Shruti superset has been studied for over centuries and dates back to 300 B.C. to 5th century A.D [13].Bharatha Muni's NatyaShastra is believed to be first of its kind literature on Indian Music [1], [2], and [5]. Later various scholars have also studied the relation between shrutis and have given different opinions [1], [2], [4], [7], [8], [9], and [10]. Though some scholars agreed on the number 22, they advocated different relation between these 22 shrutis.Over the course of development of music, the understanding of music notes has seen a lot of changes and the interpretations available to us may be only partially right [5]. A logical interpretation of the frequency ratios has not been found in various versions and interpretations of these scholarly works [2], and [5].

The main contribution of this paper is to present three different novel approaches and two inspired approaches in order to logically derive the 22 shrutis. In addition, it tries to bring out some interesting relationship between 22 notes.

The paper is organized as follows. In Section 2 the literature survey is discussed, whereas in section 3 we provide some background and discuss some of the previous works by musicologists on musical notes and existence of 22 shrutis. In Section 4 we have given different ways to derive the 22 musical intervals of Indian music. In Section 5, we look at interesting properties of these frequency ratios and conclusions are given in Section 6.

2. Literature Survey

Prafulla Kalapatapu *et al.* [20] presented the effect of four feature selection methodsi.e. Genetic Algorithm (GA), informationgain,forward feature selection, as well as correlation. It was performedon the basis of the four different classifiers such as Decision tree, Neural Network, K-Nearest neighbors, and Support Vector Machine. Here, the feature sets exploited were extracted features from the preprocessed songs using MIR Toolbox.It includes rhythm based, pitch based, timbre based, tonality based as well as dynamic features.

Radish Kumar Balasubramanium *et al.* [21]investigated the cepstral characteristic of voice in singers as well as non-singers.The performance analysis showedthat there was animportant difference among the means of singers and non-singers, it illustratesthat cepstralparameters were higher among the singers in comparison with non-singers.The attainedoutcomes were attributed tothe harmonic organization in the voices of singers.

Seema Ghisingh *et al.* [22] studied classical Indian Music along-withcomparative discussion of the separation quality for WesternMusic. In this paper comprise two major studies such as singing-voice analysis and music-sourceseparation. Moreover, Indian ragas were separated into lyrical composition and alaap regions that were regions of improvisation, which show the versatility of the singer.They have analyzed acousticallyin both male andfemale singing voice;also it was carried out by examining the productioncharacteristics.

Rachel M. Bittnerl et al. [23] presented a novel time-domain pitch contour tracking method on the basis of the Harmonic Locked Loops. It differs from the conventional approach concerning its method, resolution, timbreinformation as well as speed.

Zhe-Cheng Fan et al. [24]proposed a novel withan effective two-stage method to singing pitch extraction. Itengrosses singing voice separation andpitch tracking for monaural polyphonic audio music. In the first phase, it extracts singing voice from the songs exploiting deep neuralnetworks in a supervised setting. Subsequently, in the second phase it estimates the pitch by the extracted singing voice in a robust manner.

3. Understanding ofHarmonics andConsonance In Musical Notes

Basic study of oscillations and harmonics plays a major role in understanding musical signals. We take the example of a string instrument called Veena. Itis one of the first instruments used in experiments concerned with Indian music [1], [2],[4], [8], [9], [11], and [13] and look at various harmonics produced on the strings. Let us consider a string of length L with two fixed ends. The Eq. (2), gives the frequency of oscillation of a string of length L,

$$f = \frac{1}{2L}\sqrt{\frac{T}{d}} \tag{2}$$

where, T denotes the tension and d is the density of the string [2, 8].

By varying the length, tension or density of the string different frequencies can be produced. If tension and densities are constant, then length decides the frequency of oscillation. Hence, in our explanation we will be varying only length L to generate signals of different frequencies. The string in its natural position when plucked oscillates and produces a wave of a certain frequency that would look similar to the wave, which is shown in Fig. 4.Only first four harmonics produced by the stringare shown but many such harmonics are generated. The lowest frequency produced is known as the fundamental frequency of oscillation and in this case the wavelength is equal to twice the length of the string. The term $\sqrt{(T/d)}$ from eq. (2)has dimensions of velocity and is known as wave speed. If v is the wave speed, the frequency of oscillation is expressed as $f_1 = v/\lambda_1 = v/2L$ [7]. The fundamental wave has two nodes and one anti-node. This is also popularly known as the first harmonic of the signal or the instrument.

Similarly, the second harmonic has a wavelength equal to the length of the string and has 3 nodes and 2 anti-nodes. The frequency of second harmonic similarly can be expressed as $f_2 = v/\lambda_2 = v/L = 2f_1$ [7], with the wave speed v remaining constant. Over time 3rd, 4thand n higher harmonics are generatedon the string and their frequencies will be of the form $nf_1$where n stands for the harmonic number [1], [2], [5], and [6].

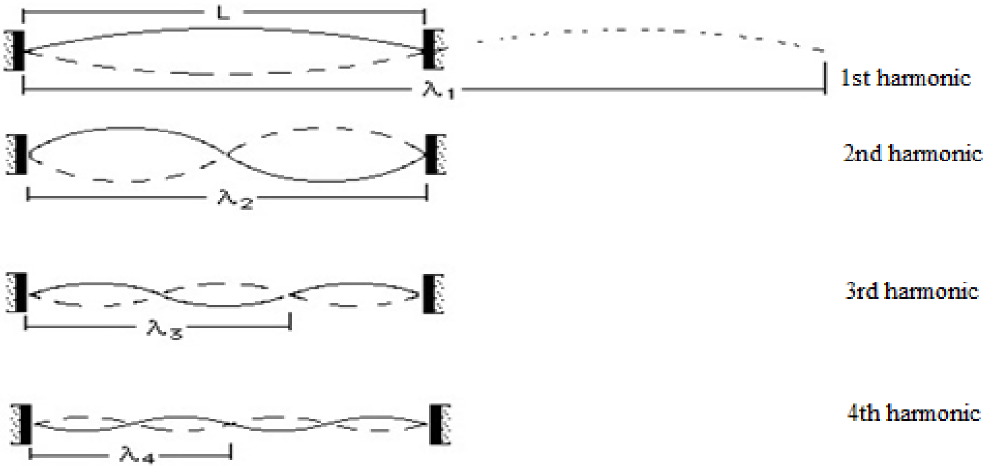

Fig.4: Diagrammatic representation of harmonics on a string

In addition, we have to observe that by pressing the string with a finger, the length of the string can be altered and plucking itwill result in a different frequencythan the fundamental to be produced. By moving the finger across and playing, many such frequencies can be produced. The frequencies produced one after another by altering the length of the string, create a sense of music only when they are in certain ratios [2], [5], and[10]. Also only frequencies played for a minimum duration of 20ms – 80ms can be perceived by human ears and transitory frequencies produced while moving from one note to another on an instrument may not be registered. Experiments were carried out with string instruments by sliding the finger on the string with one hand and simultaneously plucking with another hand and observing the musicality of frequencies produced [2], [5], [7], and [13]. Many instruments like

Veena, Sitar, Guitar, Violin etc are played using a similar principle till date. By initially choosing a frequency, different than that corresponding to complete length L of the string and moving across to generate various frequencies and maintaining the necessary musical ratios, music can still be generated. In music, usually a reference or base frequency, also known as the root is exploited. Additionally, remaining frequencies produced during a musical performance bear interesting relation with this frequency [1], [2], [3], [4], [5], [6], [7], [12], and [13]. The challenge is to understand the set of frequencies, which will sound musical by agreeingwith respect to the root frequency. The relation between frequencies for an ensemble of instruments or single instrumentsappliessimilarly.The concept of consonance explained below remains the same for multiple instruments playing musical signals together or those signals produced in succession on a single instrument.

Signal consonance plays a vital role in musicality. The frequency of a signal that can match the base frequency (root note) is the frequency itself. What do we mean by this? When an instrument say A is playing a certain frequency f_1, let us call it root frequency, and another instrument say B, we can assume two stringed instruments like Veena or any two instruments like in Fig. 2 and 3, also produces frequency f_1, irrespective of their harmonic structures, they are said to be in unison. As shown in Fig. 2 and 3, the shape of the signal may be different, due to varying strengths of harmonics, but they overlap virtually with similar time periods of repetition. We can observepositionoverlap of their harmonics clearly in frequency domain plots.

The next frequency which matches the best with the root is the one that is an octave apart. Octave frequencies are related by $f_2 = 2f_1$ and they do repeat after the same time intervals. When signal f_1 completes 1 cycle, f_2 would have completed 2 cycles though. Consider the frequencies, 100Hz and 200Hz which are octave apart as examples. The harmonics of 100Hz would be 100, 200, 300, 400, 500, 600, 700, 800 Hz and so on whereas the harmonics of

200Hz would be 200, 400, 800 Hz and so on. What we observe here is that all the harmonics of the signal with 200Hz as the fundamental frequency, overlaps with the harmonics of the signal with 100Hz as the fundamental, in other words they have harmonics in common. In general, all the harmonics of f_2 overlap with the harmonics of f_1. Because of this reason the octave signals sound pleasing to ears when played together.

Frequency $f_3$is the third harmonic of f_1 and to bring $f_3$within the octave range of f_1,we divide it by 2. For the particular example with f_1 = 100Hz we can write $f_3/2$ as 300/2 = 150Hz. The harmonics of 150Hz would be 150, 300, 450, 600 Hz and so on and when compared with harmonics of f_1 (root) we observe that every even harmonic of $f_3/2$ overlaps. Now the frequencies, 150Hz and 100Hz, in general $f_3/2$ and f_1 are in the ratio 3/2 and the note which is at a distance 3/2 from the root is known as the 5th note in music. Hence, after the unison and the octave frequencies the 5th note is observed to be the most harmonious note. Similarly, harmonics of $f_5/4$ (125 Hz for the particular example considered) will have every 4th harmonic coinciding with the harmonics of $f_{1.}$ Frequency f_4 will be just one more octave above f_2. If a frequency is harmonious with root its higher and lower octaves will also be [2], [5], [7], [10], and [12].

Harmonic signals with fundamental as nf_1 will have some of its harmonics in common with that of f_1.The overlap of harmonics and number of overlaps between signals plays a major role in producing consonance and generating musically pleasing sounds [1], [2], [3], [11], and [13].Ifa number of harmonics of two signals line up, more consonant the music signals will be and this as well explains why certain frequencies do not include musical. If none of the harmonics of two signals match then they will not sound musical.

This paper is an effort towards developing a mathematical model to explain the relation between musical frequencies.But the relations between the frequencies can still vary owing to many factors like region, culture, perception, training etc [1], [5], and [10]. Many different approaches and interpretation for musical notes may be possible [1], [2],[4], [5], [6], [7], [8], [9], [10],[11], and [13].In this paper, we start from the root note and derive the remaining notes and their ratios around the root note and analyze if 22 shrutis makes a sensible choice [1], [2], [9], and [13].The ratios can be played on an instrument like Veena as explained in section 3.8, and tested by musicians for its correctness and musicality. Thorough experimentation will always help to strengthen the interpretation and to fine-tune the model. Musicians may propose their interpretation of Shrutis and the Shrutis used in practice may not confine completely to the mathematically derived set [1], and[2].For an elaborate study of the development of music notes through ages and factors influencing them, readers can refer to scholarly works of great musicologists [4], [5], [6], [7], [8], [9], [10], [11], [12], and [13].

3.1 musical Notesand Associated Ratios in Indian and Western music

We first look at the musical notes' nomenclature followed across various systems of music and then discuss some of the ratios used in definingthesemusical notes.In music, a single frequency/note or two frequencies, which are an octaveapartcannot be played repetitively and to bring in variety and musicality, the octave spacing is divided into a number of musical notes. This division and nomenclature of notes vary across different systems of music [1], [2], [5], [6], [7], and [8], and here we will be considering only Western and Indian Classical Music.In Western music,7 natural notes are defined within an octave viz., A, B, C, D, E, F and G and 5 overlapping sharps and flats make it 12 in all. The symbol '♯' is used for sharp and '♭' for flat. With sharps and flats added the notes would be arranged as A, A♯/B♭, B, C, C♯/D♭, D, D♯/E♭, E, F, F♯/G♭, G, G♯/A♭ [2], [7], and [12]. The same set of notes repeat at a higher or lower octaveand the 13th note, which is reached after traversing a distance of 12 notes, isan octave apart. A musical note which is an octave apart will share the same name and each octave w.r.t. Particular note will be at a distance of 2/1 from the previous one. Different

Western scales pick a different subset of notes from this superset of 12 notes and arrange it in a particular order [6], [10], and [12]. The octave distance which has 12 note positions is divided equally in the ratio of $2^{1/12}$ in the equi-tempered scale division method of Western music [1], [2],[7], [10], and [12]. Many musicians in the past had advocated Just-Intonation scale division in Western music [1], [2], [7], [10], and [12]. The comparison of Just-ratios and equi-tempered ratios of musical notes in Western music areshown in Table 1[1], [2], [7], [10], and [12].The ratios in Just scale and equi-temperament do vary and in-depth comparisons and further details can be studied in the works of scholars, Ptolemy and others [2], [7], [10], and [12].

In today's Western music, the frequencies within an octave are equally divided exponentially into 12 parts to maintain uniformity in distribution of keys and their frequencies and so their ratios. Each adjacent note (piano key) is $2^{(1/12)}$ apart and the twelfth note which is an octave higher (or lower) is at a ratio of $2^{(12/12)} = 2$ (or ½). To find the frequency of the n^{th} musical note from f, all we have to do is multiply f with $2^{(n/12)}$ which gives us $f_n = 2^{(n/12)} \times f$, here the reference note gets n = 0.

Western music in its journey in the last few hundred years has been greatly influenced by Harp and Piano [1], [2], [7], [10], and [12]. In order to facilitate musicians from all over the world to play in an ensemble, standard tuning of instruments was required. The notes onapiano over various octaves have a fixed set of frequencies associated with them by following the standard 440Hz tuning for the 'A' note following middle C [2],[7], [10], and [12]. For e.g. note E in 3rd octave has a specific frequency it is tuned to which happens to be 164.814 Hz. All the instruments in anensembletune to the piano as reference [2], [10], and [12]. This allows the orchestra to sound in consonance. If an individual musician is performing he may or may not choose the standard 440Hz tuning but in an ensemble, standard tuning may be essential.

There have been various equi-tempered scale divisions defined in the history of music development [2], [3], [5], [8], [9], and [11]. Joel Mandelbaum, a music composer and musicologist, in his PhD thesis emphasized on equi-tempered scale consisting of 19 divisions [9]. He also worked on 31 temperament scale. An equal division of 22 frequency ratios spaced $\sqrt[22]{2}$ apart has also been advocated by many musicologists [5], [6], [9], and [11]. However, equi-tempered scale with 12 divisions [1], [2], [7], [10], and [12] is the most commonly used and accepted division in Western music now. We will restrict our discussion about Western music note arrangement only to 12 equi-tempered divisions.

The value of $2^{(1/12)}$ is 1.0595, which results in a lot of approximation and each note being equally separated may be the easiest but may not be the best way to lay out the musical frequencies [1],[4], [5], [7], and [8]. As many scientists and musicologists have explained the existence of music in nature and its influences on humans[1],[4], [5],[6] and [7]. The reason to have a better division of frequencies as compared to equi-tempered will be explained in detail during derivation and analysis of frequency ratios in Indian music in the coming paragraphs.

Table 1: Just-Intonation and Equi-tempered ratios

Note Number	1	2	3	4	5	6	7	8	9	10	11	12
Note	Root	Minor Second	Major Second	Minor Third	Major Third	Perfect Fourth	Diminished Fifth	Fifth	Minor Sixth	Major Sixth	Minor Seventh	Major Seventh
Just-Intonation Ratio w.r.t Root	1	16/15 = 1.06666	9/8 = 1.125	6/5 = 1.2	5/4 = 1.25	4/3 = 1.33333	45/32 = 1.40625	3/2 = 1.5	8/5 = 1.6	5/3 = 1.66666	9/5 = 1.8	15/8 = 1.875
Equi-Tempered Ratio w.r.t Root	1	$2^{1/12}$ = 1.05946	$2^{2/12}$ = 1.12246	$2^{3/12}$ = 1.18921	$2^{4/12}$ = 1.25992	$2^{5/12}$ = 1.33483	$2^{6/12}$ = 1.41421	$2^{7/12}$ = 1.49831	$2^{8/12}$ = 1.58740	$2^{9/12}$ = 1.68179	$2^{10/12}$ = 1.78180	$2^{11/12}$ =1.88775

In Indian music,an octavehas 7 notes,which are named as S, R, G, M, P, D, and N. This set of notes again repeats with higher octave S. Each of these7 notes is believed to be influenced by sounds of various animals [8], [9], [11], and [13].The elaborate names of these notes and the history of their influences can be found in the works of musicologists and historians of Carnatic music such as Sambamoorthy [11], RangaramanjuaAyyangar [8] and others [1], [2], [4], [5], [6], [8], [9], [11], and [13].S and P (root ant the fifth note) are believed to be present only in one form or position and are fixed at ratios 1 and 3/2. Whereas, the remaining notes' frequency ratios oscillate over a range [1], [3], [4], [5], [6], [7], [8], [9], [12], and [13] and we derive and discuss the ratios of these notes in the coming sections.As the other 5 notes R, G, M, D, and N occupy varied positions, the number of musical notes in an octave is more than 7. Unlike Western music the notes in Indian Music do not follow equi-tempered division but within an octave are present at different ratios regarding root note.Simple ratios like 2/1, 3/2, 4/3, 5/4, 6/5 are widely used as well as complex ratios like 16/15, 27/16, 256/243 are also found in the literature [1],[2], [4], [5], [7], [8], [10], and [13], not only in Indian music system but various music systems across the world.

The fundamental frequency of notes in Indian music is not fixed unlike in Western music [2], [5], [7], [10], [11], and [12]. Similar to root or tonic note, present in a scale of Western music, the root note position in Indian Music is occupied by S which is therefore named as "AadharaShadjam" (the foundation note) [2], [4], [5], [8], [9], [11], and [13]. In the Western music, any of the 12 notes can be named root depending on the scale but in Indian music root note is always named S irrespective of the raga [2], [3], [4], [7], [10], [11]. In Indian music, the frequency of the root node S can be any real number and all the other notes have their frequencies based on the frequency of S and are expected to be in certain musical ratios only regarding S's frequency. Each musician is free to choose S's frequency which suits his/her requirements.

Indian Classical music is broadly divided into two depending on the region. They are known as the North Indian Music or Hindustani music and the South Indian Music or Carnatic music [2], [4], [6], and [8]. But the ratios of frequencies used in both the systems of music are similar and they differ only in the style of rendition[2], [4], [6], and [8]. Both these systems are centered on the note S but use a slightly different nomenclature of notes[2], [4], and [6]. In this paper we have considered the Carnatic system of nomenclature of notes[2], [4], [6], [8] [9], and [11].

In Carnatic music, a music scholar named Venkatamakhi [2],[8], [9], and [11] in 17th century identified 12 different positions of notes (ratios) and making use of position overlap he gave 16 names to those 12 positions, as shown in Table 2 and developed a system of arrangement of notes, which formed the basis of scales of ragas known as Melakarta [2], [8], [9], and [11].From Table 2,we see that the notes, S and P are fixed at ratio 1 :3/2, whereas the remaining notes, R, G, D and N occupy 3 places each and withM at two positions makes it 16. Four of the notes overlap and share the same set of ratios. The notes used in Table 2 have specific names [2], [8], [9], and [11], for example, the traditional name for R1 is ShuddhaRishaba, for G2 is SadharanaGandhara and so on [2], [8], [9], and [11]but we will be using letter names followed by numbers to indicate a musical note [4], [6], [8], [9], and[11], the nomenclature, which is prevalent in Carnatic Music today.

To make use of all the 12 positions and to be consistent in the nomenclature of notes and avoid confusion, Venkatamakhi set a rule for the arrangement of Melakarta scales. A Melakarta raga is expected to have all the seven musical notes and repetition of particular note is not allowed [2], [7], [8], and [10]. That means a Melakarta raga cannot have two notes named as R or G or M etc. (can be any subset of R, G, M etc). As an example, if a raga makes use of say both ratios 9/8 and 6/5 from Table 2 in its scale, they will be named as R2 and G2 and not as (R2, R3) or (G1, G2). Similarly, suppose a raga uses the frequency ratio 6/5 along with the next ratio in the table which is 5/4 they will be named as R3 and G3 and not as G2 and G3. The same rule applies to all the notes and Venkatamakhi arranged the musical notes with all possible combinations of these 12 positions to develop Melakarta raga table [8], [9], and [11]. So a raga will have only one flavor of each of R, G, M, D and N notes. This was the reason to give 16 names to 12 positions which is one the ways to arrange the frequency ratios but the same ratios also found in the Hindustani system of music has a different nomenclature and overlap of positions is not presented in the literature of Hindustani music[6]. These 12 note positions can be roughly mapped against the 12Western notes. If each of the notes in Table 2 can still sound musical when slightly displaced from their respective positionsin terms ofratios, we can get more musical positions. If we can identify 1 such position for each of the 10 musical positions, excluding S and P's respective positions, from Table 2, we will have 22 musical intervals in all within an Octave [2], [8], [9], [11], and [13].

Table 2. Ratios of Notes and Frequency

Note Number	1	2	3	4	5	6	7	8	9	10	11	12
Note Name	S	R1	R2/G1	G2/R3	G3	M1	M2	P	D1	D2/N1	N2/D3	N3
Ratiow.r.t Rootnote S	1	16/15	9/8	6/5	5/4	4/3	45/32	3/2	8/5	27/16	9/5	15/8

In literature [2], [4], [5], [7], [8], [10], and [13] prior to the era of Venkatamakhi there has been mention of 22 shrutis. While we refer to the tables of musical note ratios in these scholarly works we observe that the 12 positions used by Venkatamakhi are a subset of the 22 positions mentioned. Table 2 summarizes the notes and frequency ratios exploited by Venkatamakhi for his Melakarta Raga system.If all the 22 positions are used, the Table 2will become unwieldy hence Venkatamakhi restricted himself to 12. Musicians in Carnatic music make use of a particular variant of a note depending on the need of a specific raga. For example if a raga has the note R1 in its scale, depending on the feel the raga is expected to generate, the ratio of R1 can be at 16/15 or the other ratio which is allowed for the note to occupy. In literature [1], [2], [3], [4], [8],[9], and [11] R1 was found to be at 256/243 (close to 16/15 but at a ratio of 80/81 w.r.t. 16/15), and can be named as R1'or even R1 can be used to name the note because both frequencies together will never occur in a raga.What we understand here is R1 has the privilege to oscillate only between 256/243 and 16/15 (the frequencies which have to verify in the coming sections) and any other ratio for R1 does not sound musical rather will not sound like the intended note R1. The same logic can be extended to all the other notes.

4. Derivation Of 22 Shrutis

In Indian music, various efforts have been done in the past to understand and explain the existence of notes and the relation between them. Some of the prominent work has been due to Prof. Sambamoorthy [11], Swami Prajnanda [9], RangaramanujaAyyangar [8] and others [1], [2], [5], [6], and [13]. These books agree on most of the ratios of these notes but differ in some [1], [2]. 22 Shrutis is the number which is vastly accepted by a number of musicians and often found in the history of Indian music [1], [4], [5], [7], [8], and [13]. These 22 musical intervals are also foundin Western music [1], [2], [7], [10], and [12].

4.1. Work of Pythagoras in developing the relation between musical notes

Musicologists and mathematicians like Pythagoras, Ptolemy and others [1], [3], [5], [6], [7], [8], [9], [11], and [12] have worked towards identifying the relationship between the musical notes. Pythagoras in one of his theories developed the ratios of musical notes using the concept of 5^{th} note [1], [2], [4], [5], [7], [10], [11], and [12]. He started at the root note which hence is believed to have a ratio of 1 and moved a distance of 3/2 from thereon. The ratios were obtained by repeatedly multiplying'1'with 3/2 and to make sure the ratios are within the range 1 and 2 (to find notes within an octave), they were divided or multiplied by a power of 2 to bring it into the range of interest.

The root note is at a distance of 2/3 from the 5^{th} note (3/2 x 2/3 = 1). Pythagoras also discovered that the higher octave note which is at a ratio of 2 w.r.t root, is the5^{th} note of note present at ratio 4/3 (4/3 x 3/2 = 2 or 2/ (3/2) = 4/3). In other words, a 5^{th}notebelow(obtained by dividing with 3/2) the octave note is at 4/3. Hence,he explored that moving a distance of multiples of 2/3 should also give interesting musical ratios. Again multiplying (or dividing) the resulting ratios with a power of 2 ensures that they stay within the range of 1 and 2. This is the concept of 4^{th} note traversal. Starting from ratio 1 he traveledbackward a ratio of 2/3 repeatedly. Multiplying with 2/3 or 4/3 will have a similar effect as these ratios are just octave apart.

Some of the ratios of Pythagorean workare shown in Table 3. The procedure can be continued but some of the ratios become invalid later as they are too close to the root note or the 5^{th} note. Alsoby 5^{th} note traversal concept, we do not reach the starting ratio of 1 exactly after traversing a distance of 3/2 repeatedly 12 times but reach a value close to 1.0136(or 8.1088, which is 3 octaves higher). Neither by repeated 4^{th} note traversal are we able to converge to the original ratio of 1.

The work of Pythagoras formed the basis for Western notes and their ratios for quite some time. The concept of the circle of fifths and fourths is one of the first ways to derive musical notes and till date is studied by musicians to understand the relation between notes. In the circle of fifths/fourths after twelve traversals the beginning position will be reached [2], [7], [10], and [12], but the ratios used are not based on Pythagorean work but an equi-tempered division of 12 notes is followed.

The Pythagorean concept also lays one of the foundations to develop the ratios of Indian musical notes [1], [4], [5], [7], [8], and [10]. His method also gives some of the ratios used in

Indian music [2], [5], [7], [10], [11], [12], and [13] but does not cover the entire set as we do not converge to root position 1 with continued traversal [1], [2], and [9]. However, Pythagoras' work opened gates for research of music ratios but a modification of his approach and appropriate reasoning would be required to achieve all the relevant ratios [1], [2], [5], [10], and [12].

Table 3: Pythagoras theory to derive musical note ratios

Note number	1	2	3	4	5	6	7
Ratio	1	1 x 3/2 = 3/2	3/2 x 3/2 = 9/4 = 9/8	9/8 x 3/2 = 27/16	27/16 x 3/2 = 81/32 = 81/64	81/64 x 3/2 = 243/128	243/128 x 3/2 = 729/512
Nearest decimal	1	1.5	1.125	1.6875	1.2656	1.898	1.423
After rearranging in ascending order	1	9/8	81/64	729/512	3/2	27/16	243/128
Equivalent Indian musical notes [4, 6, 8, 12]	S	R2	G3'	M2'	P	D2	N3'
Similarly multiplying with 2/3 or 4/3 gives us few more ratios							
Ratio	1	4/3	16/9	32/27	128/81	256/243	1024/729
After rearranging in ascending order	1	256/243	32/27	4/3	1024/729	128/81	16/9
Equivalent Indian musical notes [4, 6, 8, 12]	S	R1'	G2'	M1	Not defined in Indian music but close to M2	D1'	N2'

4.2 Methodologies Used to Derive 22 Shrutis

We wish to find all frequency ratios within an Octave which are in musical consonance. The arrangement of notes beyond the octave remains the same, except that they are one octave higher or lower.Since the note S in literature is named "aadharaShadja" [1], [4], [5], [6], [8], and[9] and it is believed that all the other frequencies emanate from this frequency, we make use of this concept to derive the relative frequency ratios starting with the root note.This helps us to arrive at a number which will indicate the number of acceptable musical ratios and their values.

VidyadharOke [2] has derived 22 shrutis by finding the notes S, G and P in natural and mathematical scales first, then identifying different ratios between adjacent notes. The same pattern is repeated for the entire octave to find all the possible shrutis.

Dinesh Thakur [1] has taken the North Indian music ragas Bilawal and Kaafi's scales and compared their ratios and explained the existence of 22-shruthis by making use of Syntonic Comma [1], [2], [10], and [11].

We have given 3new approaches to arrive at 22 shrutis based on the concept of consonance, symmetry and transposition. The 4^{th} approach is also based on transposition influenced by the method mentioned in Bharatha Muni's NatyaShastra [13]. The 5^{th} approach is givenas a comparison which is the approach proposed by VidyadharOke[2].

In all the approaches, the frequency ratios are multiplied or divided byappropriate power of 2, i.e. 2^n where n is an integer, to keep it within the octave range.

4.3Approach 1: Use of P and Mrelation alternately to derive the shrutis

This approach is influenced by Pythagorean tuning [2], [9], [10], and [12]. We make use of S and P relation, i.e. the 5^{th} note relation and S and M relation i.e. the 4^{th} note relation alternately obtain the further ratios [1], [2], [4], [5], and [11]. For S and P relation we multiply the ratios with 3/2 and for S and M relation we multiply them with 4/3.

Let us consider the first five harmonics (stronger harmonics and higher harmonics die off soon) of root $f_1$which are spaced as 1:2:3:4:5.By dividing with relevant powers of 2 to bring the ratios within 1 and 2we get 1:2:3/2:2:5/4 and then by rearrangingin ascending order excluding the redundantwe obtain ratios 1:5/4:3/2:2. These ratios are foundinthe literature [1], [2], [4], [5], [6], and [9] and are named as (S: G3: P: S_{high}).

Here instead of considering root note alone at ratio 1 and moving a distance of 3/2repeatedly like the Pythagorean tuning [2, 9], we first find the vector (S: G3: P: S_{high}) corresponding to ratios [1:5/4:3/2:2].Nowto find the P note i.e. the 5th note with each of the ratios in the vector we take the product of the vector (1:5/4:3/2:2) with 3/2 which gives us (3/2: **15/8**: **9/4**: 3). Dividing 9/4 by 2 and rearranging all the obtained ratios so far we get (1: 9/8 : 5/4 : 3/2 : 15/8 : 2) which correspond to notes (S: R2 : G3: P : N3 : $\mathbf{S_{high}}$) [4], [5], [6], and [8].This step has provided us with two new ratios.

We now multiply the resultant vector with4/3. The vector size keeps growing at each stage and we continue to multiply the updated vector of ratios alternately with 3/2 and 4/3 until no new ratio is obtained. Multiplying with 4/3 is equivalent to finding M note for a particular frequency ratio [1], [4], [5], [7], and [8]. By multiplying the vector of frequency ratios obtained so far with 4/3 we get (1: 9/8 :5/4 :3/2 :15/8 : 2) x 4/3 which results in [**4/3** :3/2 :**5/3** :2 : 5/2 (5/4) : 8/3(or 4/3)]. Ignoring the frequency ratios which are repeated and by arranging in ascending order we get (1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2). These notes are named as (S: R2: G3: M1: P : D2': N3 :$\mathbf{S_{high}}$) [4], [5], and [7].This way we obtain all the 7 natural notes in just two steps of P and M calculation.

This procedure is continued and is shown in clearly in Table 4. New ratios obtained after each operation is shown in bold. We multiply with the factors 3/2 and 4/3 alternately and also multiply or divide with an appropriate of the power of 2 to bring the ratios into the desired octave range.There will be rare omissions of certain frequencies for certain reasons.They are omitted as they result in frequencies which correspond to different colors or positions of S and P (ratios which are close to 1 or 3/2) but S and Pare assumed and observed to be stable[1], [4], [5], [6] [7], [8]and [13].Also if a ratio is generated close to an existing ratio the less complex value will be retained.

Table 4: Derivation of frequency ratios

Step No.	Beginning Ratios	Multiplication factor	New Ratio Vector	Complete ratio vector after rearranging and their names
1	[1: 5/4: 3/2 : 2]	3/2 finding Panchams i.e. P	[3/2: **15/8**: **9/4(9/8)**: 3]	[1: 9/8 : 5/4 : 3/2 : 15/8 : 2] [S: R2 : G3: P : N3 : $\mathbf{S_{high}}$]
2	[1: 9/8 : 5/4 : 3/2 : 15/8 : 2]	4/3 finding Madhyam i.e. M	[**4/3** : 3/2 : **5/3** : 2 : 5/2 (5/4) : 8/3(4/3)]	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2] [S: R2 : G3: M1: P : D2': N3 :$\mathbf{S_{high}}$]
3	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2]	3/2	[3/2:**27/16**: 15/8: 2: 9/4(9/8): 5/2 (5/4): **45/16** (**45/32**): 3]	[1: 9/8 : 5/4 : 4/3: 45/32: 3/2 : 5/3: 27/16: 15/8 : 2] [S: R2 : G3: M1: M2: P : D2': D2: N3 :$\mathbf{S_{high}}$]
4	[1: 9/8 : 5/4 : 4/3: 45/32: 3/2 : 5/3: 27/16: 15/8 : 2]	4/3	[4/3: 3/2: 5/3: **16/9**: 15/8: 2: **20/9**(**10/9**):9/4(9/8): 5/2(5/4): 8/3(4/3)]	[1: 10/9: 9/8 : 5/4 : 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 2] [S: R2': R2 : G3: M1: M2: P : D2': D2: N2': N3 :$\mathbf{S_{high}}$]
5	[1: 10/9: 9/8 : 5/4 : 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 2]	3/2	[(3/2: 5/3: 27/16: 15/8: 2: ~~135/64~~: 9/4(9/8): 5/2(5/4): **81/32 (81/64)**: 8/3(4/3): 45/16(45/32): 3]	[1: 10/9: 9/8 : 5/4 : 81/64: 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 2] [S: R2': R2 : G3: G3': M1: M2: P : D2': D2: N2': N3 :$\mathbf{S_{high}}$] The ratio 135/64 has been discarded as it gives unstable S
6	[1: 10/9: 9/8 : 5/4 : 81/64: 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 2]	4/3	[4/3: ~~40/27~~: 3/2 : 5/3 : 27/16: 16/9: 15/8: 2 : 20/9: 9/4: **64/27 (32/27)**: 5/2 : 8/3]	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 2] [S: R2': R2 : G2': G3: G3': M1: M2: P : D2': D2: N2': N3 :$\mathbf{S_{high}}$] 40/27 results in unstable P at 1.48 and hence is discarded

7	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 2]	3/2	[3/2: 5/3: 27/16: 16/9: 15/8: **243/128**: 2: ~~135/64~~: 9/4(9/8): 5/2(5/4): 81/32 (81/64): 8/3(4/3): 45/16(45/32): 3]	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 243/128: 2]
8	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32: 3/2 : 5/3: 27/16: 16/9: 15/8 : 243/128: 2]	4/3	[4/3: ~~40/27~~: 3/2 : **128/81**: 5/3 : 27/16: 16/9: 15/8: 2 : 20/9: 9/4: 64/27 (32/27): 5/2: 81/32(81/64): 8/3(4/3)]	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32: 3/2 : 128/81: 5/3: 27/16: 16/9: 15/8 : 243/128: 2] [S: R2': R2 : G2': G3: G3': M1: M2: P : D1': D2': D2: N2': N3 : N3': $\mathbf{S_{high}}$]
9	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32: 3/2 : 128/81: 5/3: 27/16: 16/9: 15/8 : 243/128: 2]	3/2	[3/2: 5/3: 27/16: 16/9: 15/8: 243/128: 2:~~135/64~~: 9/4(9/8): 64/27(32/27): 5/2(5/4): 81/32 (81/64): 8/3(4/3): 45/16(45/32)**):** **729/256(729/512)**: 3]	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32 : 729/512 : 3/2 : 128/81: 5/3: 27/16: 16/9: 15/8 : 243/128: 2] [S: R2': R2 : G2': G3: G3': M1: M2: M2': P : D1': D2': D2: N2': N3 : N3': $\mathbf{S_{high}}$].
10	[1: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32 : 729/512 : 3/2 : 128/81: 5/3: 27/16: 16/9: 15/8 : 243/128: 2]	4/3	[4/3: ~~40/27~~: 3/2 : 128/81: 5/3 : 27/16: 16/9: 15/8: 243/128: 2 :**512/243(256/243)**: 20/9: 9/4: 64/27 (32/27): 5/2: 81/32(81/64): 8/3(4/3)]	[1: 256/243: 10/9: 9/8 : 32/27: 5/4 : 81/64: 4/3: 45/32 : 729/512 : 3/2 : 128/81: 5/3: 27/16: 16/9: 15/8 : 243/128: 2] [S: R1': R2': R2 : G2': G3: G3': M1: M2: M2': P : D1': D2': D2: N2': N3 : N3': $\mathbf{S_{high}}$].

Simple ratios and non-recurring decimals are mostly named without " ' "and more complex ratios with " ' " like R2 and R2', G3 and G3' etc [8], [9], and [11]. Table 4 shows the derivation of frequency ratios using the S and P and S and M relation.At the end of 10th iteration from Table 4 we have found 15 note positions in addition to S and P. Continuing further in the same way would result only in repeated ratios and does not yield any new ratios. This strengthens one fact that the number of musical frequency ratios is finite and is a definite number [4], [5], [7], and [8]. What we observed at this point was that the ratio 10/9 and 9/8 are close and they constitute the possible positions for R2. Similarly 5/4 and 81/64 are extremely close and they are the allowable ratios for G3 and G3'. This prompted us to think that each of the notes should have a close neighbor at a similar distance. Using this logic, we derived the remaining ratios and validated them using the 5th note and 4th note concept, with rare exceptions [1], [8], and [13].

We observe that the notes R2, G3, M2, D2 and N3 occupy 2 positions each. This prompts us to think that R1, G2, M1, D1, N2 should also be present at two variable positions. This further makes us think that the relative distances between two positions of R1, G2, M1, D1, N2should be same as relative distances between two positions of R2,G3, M2, D2 and N3 [1], [4], [5], [7], [8], and [13]. Now, we find that R2' and R2 are present at 10/9 and 9/8 positions respectively. The distance between R1' and R1 should be same as the distance between R2' and R2, i.e. R1': R1:: R2' : R2. Using this concept we found the position of R1 at **16/15** solving for x using 256/243: x::10/9 : 9/8.This should be the ratio of R1 and we will verify this in a while. Now similarly the position of G2 can be found using G1': G1 :: G2' : G2, 32/27: y :: 5/4: 81/64 which gives us y = **6/5**. Similarly we find M1' at (729/512 x 4/3) / (45/32) which is **27/20**andD1 at (27/16 x 128/81)/(5/3) resulting in **8/5**. Also N2 is found to be at **9/5**. Now let us see if these values satisfy the P condition. Now 16/15's P is at 16/15 x 3/2 which results in 8/5 which is the position of D1. Also 6/5 x 3/2 gives us 10/9. Similarly 8/5 and 9/5 have their P placed at 6/5 and 27/20 respectively. 27/20 does not have a P as it results in unstable S [1], [8], and [13]. Again finding the M for the newly found ratios (16/15: 6/5: 27/20: 8/5: 9/5) x 4/3 we get (64/45: 8/5: 9/5: 16/15: 6/5) which correspond to (invalid: D1: N2: R1: G2). The ratio 64/45 is discarded as it is close to M2'. This validates the authenticity of the obtained ratios.

Arranging all the musical frequencies obtained so farwe get (S: R1’: R1 : R2’: R2 : G2’: G2: G3: G3’: M1: M1’: M2: M2’: P : D1’: D1: D2’: D2: N2’: N2: N3 : N3’: S(high)) at frequency ratios [1: 256/243: 16/15: 10/9: 9/8 : 32/27: 6/5: 5/4 : 81/64: 4/3: 27/20: 45/32 : 729/512 : 3/2 : 128/81: 8/5: 5/3: 27/16: 16/9: 9/5: 15/8 : 243/128: 2]. Further finding Ps and Ms (5^{th} and 4^{th} note) for this vector will give us the same set of frequencies (ratios). So we do not proceed further and this gives us 22 shruthis at this point and these values match with the one mentioned in Bharatha Muni’s NatyaShastra [13] and other works [4], [5], [6], [8], and [9].

After the 10^{th} iteration in Table 4, we can also obtain the missing ratios by multiplying with a factor of 81/80 because the two shrutis of particular note, e.g. (R2’, R2) and (G3, G3’), was observed to be separated by a distance of 81/80. The important thing we have to understand here is whether the missing ratio is at a distance of 81/80 ahead or behind i.e. ata ratio of 80/81. If a ratio was obtained using P method we move ahead by a distance of 81/80 and if a ratio was obtained using M method we move behind a distance of 81/80 to obtain the missing ratio. In Western music the ratio 81/80 is named as syntonic comma [1], [13].

With S and P holding their respective positions, each noteis present in two different colors each (non-overlapping positions) that makes it 5 x 2 = 10 note positions. Each named note or flavor of the note is present at two different positions making it 20 in all. This gives us 22 note positions in Indian music.

This is one of the approaches to derive the 22 musical intervals known asshrutis. Many efforts have been made in understanding these 22 shrutis and here we have laid down a clear mathematical foundation to explain the existence of these frequencies for centuries. We have explained how all the frequencies can be derived based on just the root note. From the 23^{rd}shruti the same set of frequencies repeats starting from S.

4.4 Approach 2:Making use of the presence of inverse ratios

In the natural note vector (S: R2: G3: M1: P : D2': N3 :**S_{high}**) corresponding to ratios (1: 9/8 : 5/4: 4/3: 3/2: 5/3: 15/8: 2) we observed that the ratios 3/2 and 4/3 were both presents. By inverting 3/2 we get a ratio of 2/3 or 4/3 which is just an octave higher w.r.t 2/3, is also a valid musical ratio. We wanted to check this property of inverse ratio being a part of the musical vector for all the frequency ratios found and validate it using the 5^{th} and the 4^{th} note concept.

Also as mentioned string instruments were widely usedforexperiments with music [1], [2], [9], [10], [11], [12], and [13]. The frequency of the signal is inversely proportional to the length of the string. Suppose the fundamental frequency of the root note corresponds to length L of the string,the ratio 3/2 corresponds to length 2/3L of the string and 4/3 corresponds tothe 3/4L length of the string. This makes us to think if the length of the string l_1 = xL produces a valid frequency, where x is a number less than 1, then l_2 = yL where y = 1/2x also gives a valid frequency.

Let us first begin with the scale obtained using Pythagoras approachfrom Table 3 which will be of the form S: R: G: M: P: D: N in the ratio of 1: 9/8: 81/64: 729/512: 3/2: 27/16: 243/128. We will also use the 7 natural notes, using Approach 1, as S: R: G: M: P: D: N at ratios 1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2. Now these two when combined we get (with proper numbering), S: R2: G3: G3': M1: M2': P: D2': D2: N3: N3' at 1: 9/8: 5/4: 81/64: 4/3: 729/512: 3/2: 5/3: 27/16: 15/8: 243/128 [1], [4], [7], [8], and [12].

Table 5: Ratios of notes ona natural scale with each other

Ratios obtained by combining Pythagorean scale and Natural scale	Ratio vector obtained by inverting all the ratios and multiplying with 2
1: 9/8: 5/4: 81/64: 4/3: 729/512: 3/2: 5/3: 27/16: 15/8: 243/128	[1: 16/9: 8/5: 128/81: 3/2: 1024/729: 4/3: 6/5: 32/27: 16/15: 256/243]

We have 11 ratios combining the two scales from Pythagoras tuning and Natural scale as shown in Table 5. Now let us invert each of the ratios and find the vector with suitable multiplication or division with powers of 2. The inverted vector will have ratios [1: 16/9: 8/5: 128/81: 3/2: 1024/729: 4/3: 6/5: 32/27: 16/15: 256/243] which correspond to [S: N2': D1: D1': P: Invalid (close to M2'): M1: G2: G2': R1: R1'].This gives us 7 new ratios.Rearranging we get 18 ratios in all (S: R1': R1 : R2 : G2': G2: G3: G3': M1: M2': P : D1': D1: D2': D2: N2': N3 : N3': S(high)) at frequency ratios [1: 256/243: 16/15: 9/8 : 32/27: 6/5: 5/4 : 81/64: 4/3: 729/512 : 3/2 : 128/81: 8/5: 5/3: 27/16: 16/9: 15/8 : 243/128: 2].

The vector has two positions of R1, G2, G3, D1, D2 and N3 and we expect R2, M1, M2 and N2 to have similar counterparts [1], [8], and [13]. Using the syntonic comma ratio of 81/80 [1], [12], [13], we obtain the counterparts of R2, M1, M2' and N2' at(9/8 x 80/81 = **10/9**), (4/3 x 81/80 = **27/20**), (729/512 x 80/81 = **45/32**)and (16/9 x 81/80 = **9/5**) respectively. The reasoning for multiplication with 81/80 or 80/81 has been discussed in section 3.3. This completes the 22-shruti vector and again each of the shrutishasP and M ratios. Also the inverse of ratios [R2': M1': M2: N2] gives us [9/5 (N2): 40/27(discarded as it corresponds to unstable P): 64/45(discarded as it is close M2): 10/9 (R1')]. Hence all the ratios with exceptions of M1', M2 and M2' satisfy the valid inverse musical ratio property [1], [4], [5], and [8]. The inverse ratios of 45/32 and 729/512 are discarded as they generate ratios whose decimal values are close and almost overlaps with that of 729/512 and 45/32 and the human perception will not able to differentiate between these ratios.

4.5. Approach 3: **Moving the root note to different positions on the natural scale**

Dinesh Thakur [1] used Bilawal scale [1], [2], [6] and Kaafi scale [1], [6] and compared the ratios and completed the 22-shruti table using the syntonic comma [1], [2], and [9]. We begin with the natural scale obtained using Approach 1and move the position of the root note to all the ratios and derive six new scales, which help us complete the 22-shruti model. The natural scale in section 3.3was obtained as S: R: G: M: P: D: N: S_{high} at ratios [1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2]. Now from each of these notes let us find the distance to adjacent notes and try to derive new ratios. Since we have derived this vector w.r.t S let us start from R. Now G is placed at 5/4 and R at 9/8. Let us find the ratio of each of the notes w.r.t. R. First we see that5/4 w.r.t 9/8 gives a ratio (5/4)/ (9/8) which simplifies to 10/9. In simple words, we move the root note to required reference note, by dividing the entire vector with its ratio. This division places the ratio of 1 at the new reference point. From Table 6in step 1 with reference note as R, we divide the natural scale with 9/8 so that the position of R gets a ratio of 1 and that becomes the new reference root note. Similarly we can continue the procedure for all the ratios as shown in Table 6. Table 6 shows how the position 1is moved at every step and new ratios at each stage are shown in bold and octave adjustment as always is taken care of.

Table6: Ratios of notes in a natural scale with each other

Step No	Reference note	Updated Ratio Vector
1	R 9/8	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2] / (9/8) = [**8/9(16/9)**: ‘1’: **10/9**: **32/27** : 4/3: ~~40/27~~: 5/3: 16/9]
2	G 5/4	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2] / (5/4) = [**4/5(8/5)**: **9/10(9/5)**: ‘1’: **16/15**: **6/5**: 4/3: 3/2: 8/5]
3	M 4/3	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2] / (4/3) = [3/4(3/2): **27/32(27/16)**: 15/16(15/8): ‘1’: 9/8: 5/4: **45/32**: 3/2]
4	P 3/2	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2] / (3/2) = [2/3(4/3): 3/4(3/2): 5/6(5/3): 8/9(16/9): ‘1’: 10/9: 5/4: 4/3]
5	D 5/3	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2] / (5/3) = [3/5(6/5): **27/40(27/20)**: 3/4(3/2): 4/5(8/5): 9/10(9/5): ‘1’: 9/8]
6	N 15/8	[1: 9/8 : 5/4 : 4/3: 3/2 : 5/3: 15/8 : 2] / (15/8) = [8/15(16/15): 3/5(6/5): 2/3(4/3): ~~32/45(64/45)~~: 4/5(8/5): 8/9(16/9): ‘1': 16/15]

The ratiosobtainedhavingvalid ratios separated by a distance of P, i.e. 3/2, also reiterates the correctness of the procedure. Rearranging the ratios obtained till step 6 we get the vector [S: R1: R2’: R2 : G2’: G2: G3: M1: M1’: M2: P : D1: D2’: D2: N2’: N2 : N3: $\mathbf{S_{high}}$]. This method also gives us 17 ratios in all at the end of 6 iterations. The procedure of shifting the root note (S) to various natural note positions can be repeated to obtain further ratios or alternately using the ratio of 81/80 between adjacent notes as discussed in approach 1 also gives the remaining musical intervals.

The shifting of the root note can be further continued till only redundant frequency ratios are obtained. Shifting the old ratios which were already shifted will yield the same ratios hence only the new ratios can be used for next set of shifting. The 10 new ratios obtained after 6thstep in Table 6 are [16/9, 10/9, 32/27, 8/5, 9/5, 16/15, 6/5, 27/16, 45/32, 27/20]. If these are shifted using the reference R which is 9/8 from natural scale, we get two new valid ratios 128/81 (D1') and 256/243 (R1'). As discussed complex ratios giving the decimal equivalent of S, P and other existing ratios will be discarded. Shifting w.r.t to G at 5/4 does not give any new valid musical ratios.Shifting with M at 4/3 results in a new ratio at 81/64 (G3'). Further shifting w.r.t P (3/2), D (5/3) and N (15/8) does not give any new valid musical ratios and can be obtained using the syntonic comma.

4.6 Approach 4: Bharat Muni's explanation of Shrutis put into Mathematics

Bharat Muni has explained the relation between shrutis by the method of transposition of frequency ratios [13]. Many scholars have interpreted Bharatha Muni's explanation about music in his treatise "NatyaShastra" and there have been many wrong interpretations of his explanations [1], [2].Bharatha Muni explained the concept of four transitions to achieve the ratios of music [1], [2], [13]. We complete the derivation of all the notes and verify the existence of those notes mathematically.

Bharat Muni experimented with two Veenas. Two veenas were tuned such that initially on both the Veenasthe ratios of S: R: G: M: P: D: N: S was 1: 10/9: 32/27: 4/3: 3/2: 5/3: 16/9:2. One veena was kept constant while the other was tuned to vary the ratios. The transposition ratios are shown in Table 7.

Table 7: Bharat Muni's method of transposition of notes

Transposition Step No	The note shifted on adjustable veena w.r.t. untouched Veena	Multiplication factor	Updated Ratios S R G M P D N by multiplying with Ratio obtained(z)	Corresponding Ratio (1/z)
1	P is shifted such that it forms a ratio of 4/3 with R	(P x z_1) / 10/9 = 4/3 gives z_1 = 80/81	80/81: 800/729: 2560/2187: 320/243: 40/27: 400/243: 1280/729: 160/81	z_1 = 81/80
2	N is shifted such that it moves to D's original ratio which is 5/3. (Note ratios from the previous step is considered)	N x z_2 = 5/3 gives (1280/729 x z_2) /1 = 5/3 gives z_2 = 243/256	15/16: 25/24: 10/9: **5/4: 45/32**: 25/16: 5/3: **15/8**	z_2 = 256/243
3	D is shifted such that it moves to P' s ratio	(D x z_3) = 3/2 gives 25/16 x z_3 = 3/2 gives z_3 = 24/25	9/10: 1: **16/15**: **6/5**:27/25: 3/2: **8/5: 9/5**	z_3 = 25/24
4	S is shifted to N	(S x z_4) = 16/9 gives 9/5 x z_4 = 16/9 or z_4 = 80/81	8/9: 80/81: 256/81: 32/27: 16/15: 40/27: **128/81**: 16/9	z_4 = 81/80

The syntonic ratio 81/80 is found twice. We arranged ratios in the order 256/243: 81/80: 25/24: 81/80 and repeatedly multiplied them starting from the root note to complete the Octave range as shown in Table 8. Multiplication with M and N are excluded.

Table 8: Multiplication of the vector [1: 256/243: 81/80: 25/24: 81/80] with ratios of S, R, G, (M), P, D and (N)

Step No	Multiplication sequence (beginning the sequence from the last element of the vector from the previous step)	Vector obtained at the end of each stage
1	1 x 256/243 = **256/243** (R1'); 256/243 x 81/80 = **16/15** (R1); 16/15 x 25/24 = **10/9** (R2'); 10/9 x 81/80 = **9/8** (R2)	[S: R1':R1:R2':R2] at [1: 256/243: 16/15: 10/9: 9/8]
2	9/8 x 256/243 = **32/27** (G2'); 32/27 x 81/80 = **6/5** (G2); 6/5 x 25/24 = **5/4** (G3); 5/4 x 81/80 = **81/64** (G3')	[R2: G2': G2: G3: G3'] at [9/8: 32/27: 6/5: 5/4: 81/64]
3	81/64 x 256/243 = **4/3** (M1); 4/3 x 81/80 = **27/20** (M1'); 27/20 x 25/24 = **45/32** (M2); 45/32 x	[G3': M1: M1': M2: M2'] at [81/64: 4/3: 27/20: 45/32:

	81/80 = **729/512** (M2')	729/512]
4	3/2 x 256/243 = **128/81** (D1'); 128/81 x 81/80 = **8/5** (D1); 8/5 x 25/24 = **5/3** (D2'); 5/3 x 81/80 = **27/16** (D2)	[P: D1': D1: D2': D2] at [3/2: 128/81: 8/5: 5/3: 27/16]
5	27/16 x 256/243 = **16/9** (N2'); 16/9 x 81/80 = **9/5** (N2); 9/5 x 25/24 = **15/8** (N3); 15/8 x 81/80 = **243/128**(N3')	[D2: N2': N2: N3:N3'] at [27/16: 16/9: 9/5: 15/8: 243/128] .

As multiplication of these ratios with M and N gives unstable P and S ratios they are discarded.

4.7Approach 5: **Use of S and R ratios in bunches repeatedly.**

VidhyadharOke [2] has done an interesting study of 22-shrutis and has also developed a harmonium with these shrutis. He compared the natural and Pythagorean scale and identified the existence of the repeated SR pattern in the musical ratios described by Bharat Muni [13] and others [4], [5], [7], and [10]. We follow a similar approach to derive and analyze the 22-shruti model in this method. Here we make use of both 4ths and 5ths Pythagoras obtained and compared with the natural scale and proceeded further finding the whole vector S R G and M with each of the positions. S, R, G, M by Pythagoras approachof finding the consecutive fifthswere spaced at distances 1: 9/8: 81/64: 729/512 and were placed at 1: 256/243: 32/27: 4/3 when consecutive fourths were found. The natural scale we obtained using the P and M method has SRGM ratios as [1: 9/8: 5/4: 4/3].

Let us consider the S, R and G values obtained using consecutive 5ths by Pythagorean tuning and make use of the simple natural M ratio and write the vector [S: R: G: M] as [1: 9/8: 81/64: 4/3]. Comparison of natural and Pythagorean scale gives us two different values of G. Both the scales are shown in Table9.

Table9: The ratios of notes S R G and M

Approach	S	R	G	M
Natural Scale	1	9/8	5/4	4/3

Pythagoras scale	1	9/8	81/64	4/3

Now let us find the ratio between adjacent notes [1]. Considering natural scale, G at 5/4 and R 9/8 form a ratio of 10/9. M at 4/3 is spaced at a distance of 16/15 from G. When Pythagoras scale is considered, G is placed at a distance of 9/8 w.r.t R and M is placed a distance of 256/243 from G. Hence we found 4 unique distance ratios here, when rearranged looks of the form S: R1':R1:R2':R2 at [1: 256/243: 16/15: 10/9: 9/8]. The last ratio multiplies this set of the ratio in the vector and this process is repeated till no new values are found. The procedure is shown in steps in Table 10.

Table 10: Multiplication of the vector [1: 256/243: 16/15: 10/9: 9/8] with ratios of S, R, G, (M), P, D and (N)

Step No	Multiplication factor (last element in the previous vector)	New Ratio Vector
1	S	[1: 256/243: 16/15: 10/9: 9/8] [S: R1':R1:R2':R2]
2	R2 9/8	[9/8: 32/27: 6/5: 5/4: 81/64] [R2: G2': G2: G3: G3']
3	G3' 81/64	[81/64: 4/3: 27/40(27/20): 45/32: 729/512] [G3': M1: M1': M2: M2']
4	P 3/2	[3/2: 128/81: 8/5: 5/3: 27/16] [P: D1': D1: D2': D2]
5	D2 27/16	[27/16: 16/9: 9/5: 15/8: 243/128] [D2: N2': N2: N3:N3'].

The problem in finding the set of ratios w.r.t M2' at 729/512 is analyzed as follows. The ratios found would be [729/512: 3/2: 243/160: 405/256: 6561/4096]. The ratios 405/256 and 6561/4096 are rejected because they are close to and almost overlap with D1 at 128/81 and D1' at 8/5. 243/160 is close to 3/2 and is discarded as it results in unstable P.

Similarly N3' at 243/128 results in ratios 243/128 x [1: 256/243: 16/15: 10/9: 9/8] which simplifies to [243/128: 2: 81/40(81/80): 135/64(135/128): 2187/1024(2187/2048)]. The ratios 135/128 and 2187/2048 are discarded as they almost overlap with R1' and R1 at 256/243 and 16/15 respectively. Since 81/80 corresponds to unstable S it is not considered.

4.8Playing musical notes on a string.

These shrutis can be played on a string by varying the length of string appropriately. First a length has to be fixed corresponding to the root note S which forms the base to all other notes as discussed. The length of the string corresponding to 'S' can be called L_S. Now the frequency of oscillation on the string which corresponds to L_S can be obtained using the equation $f_s = \frac{1}{2L_S}\sqrt{\frac{T}{d}}$ [1], [2], [9], and [11]. Now only the length of the string has to be altered to obtain frequencies to corresponding notes. To obtain note R1' which is at a ratio of 256/243 times the frequency of S, the length of the string can be obtained using the calculation shown in equation 3.

$$\frac{f_s}{f_{R1'}} = \frac{\frac{1}{2L_S}\sqrt{\frac{T}{d}}}{\frac{1}{2f_{R1'}}\sqrt{\frac{T}{d}}} = \frac{L_{R1'}}{L_S}, \text{ but } \frac{f_S}{f_{R1'}} = \frac{1}{\frac{256}{243}} = \frac{L_{R1'}}{L_S} \text{ which gives } \frac{L_{R1'}}{L_S} = \frac{243}{256} \Rightarrow L_{R1'} = \frac{243}{256} L_S \quad (3)$$

Similarly length of the string corresponding to all other notes can be derived. For e.g. $L_{G3'} = 64/81L_S$, $L_{N3} = 8/15L_S$, $L_{M2'} = 512/729L_S$, $L_{D1'} = 81/128L_S$ and so on.

On a string between two different positions infinite frequencies are possible. Based on human perception and the concept of harmony and consonance, only a discrete set of frequencies are acceptable. Hence, musical notes are played at lengths of string which correspond to the multiples of the derived 22 shrutis. Shruti in other words can also be explained as the note which corresponds to these valid lengths of the string.We have also to note that a frequency has to be sustained for around 20-40 msec for human brain and ear to perceive. So any transitory frequency will not be registered as a note by the human brain. If a frequency which does not form a pleasing ratio with root note is sustained for duration longer than human perception, then it will be termed as inharmonious or non-musical.

5. Interesting Relation Among 22 Shrutis

Using a Pythagorean scale for S: R: G: M: P: D: N: S_{high} having ratios1: 9/8: 81/64: 729/512: 3/2: 27/16: 243/128: 2 we observe each adjacent note is at a distance of 9/8 from the previous note. For e.g. G at 81/64 is at a distance of 9/8 from R at 9/8 similarly N at 243/128 forms ratio of 9/8 with D at 27/16. The distance from M to P and N to S was observed to be 256/243 but all other adjacent notes are spaced 9/8 apart. The G obtained in Pythagorean scale is at 81/64 and G in scale using alternate P and M method is at 5/4. They form a ratio of 81/80 and we observe that all the notes are spaced apart by the same distance 81/80. E.g. R1' and R1 at 256/243 and 16/15 are spaced 81/80 apart also N3 to N3' i.e. distance from 15/8 to 243/128 is 81/80. This is observed in all the notes and also formed the basis to derive the ratios.

Another interesting arrangement of these notes would be that, they appear in 2 sets of SRGM within an octave. By this we mean the ratios of SRGM when multiplied by the ratio of P which is 3/2 gives P D N S. The ratios (S: R1': R1 : R2': R2 : G2': G2: G3: G3': M1) [1: 256/243: 16/15: 10/9: 9/8 : 32/27: 6/5: 5/4 : 81/64: 4/3] when multiplied with 3/2 gives [3/2 : 128/81: 8/5: 5/3: 27/16: 16/9: 9/5: 15/8 : 243/128: 2] corresponding to (P : D1': D1: D2': D2: N2': N2: N3 : N3': S_{high})

The pairs (R1, R2'), (G2, G3), (M1', M2), (D1, D2') and (N2, N3) are separated by a ratio of 25/24. Also a ratio of 256/243 can be found between pairs (S, R1'), (R2, G2'), (G3', M1), (M2', P), (P, D1'), (D2, N2') and (N', S).

Table 11 shows some of the interesting relations between the musical intervals of Indian music. The second frequency in the pair is higher.

Table 11: Shrutis and some interesting relation between them

Shruti No.	Note Name	Rational number to represent the ratio	Pairs which satisfy the inversion property	Shrutis forming 3/2 Ratio	Shrutis forming 4/3 Ratio	Shrutis forming 5/4 Ratio	Shrutis forming 6/5 Ratio	Shrutis forming 5/3 Ratio	Shrutis forming 10/9 Ratio	Shrutis forming 9/8 Ratio
1	S	1/1		(1,14)		(1,8)		(1,17)	(1,4)	(1,5)

2	R1’	256/243	(2, 22)	(2,15)						(2,6)
3	R1	16/15	(3, 21)	(3,16)		(3,10)		(3,19)	(3,6)	(3,7)
4	R2’ / G1’	10/9	(4, 20)	(4,17)			(4,10)			(4,8)
5	R2 / G1	9/8	(5, 19)	(5,18)		(5,12)	(5,11)	(5,21)	(5, 8)	(5,9)
6	G2’ / R3’	32/27	(6, 18)	(6,19)	(6,15)					(6,10)
7	G2 / R3	6/5	(7, 17)	(7,20)	(7,16)	(7,14)		(7,23 or 1)	(7,10)	(7,11)
8	G3	5/4	(8, 16)	(8,21)	(8,17)		(8,14)			(8,12)
9	G3’	81/64	(9, 15)	(9,22)	(9,18)				(9,12)	
10	M1	4/3	(10, 14)		(10,19)	(10,17)	(10,16)			(10,14)
11	M1’	27/20			(11,20)	(11,18)			(11,14)	
12	M2	45/32			(12,21)		(12,18)			
13	M2’	729/512			(13,22)					
14	P	3/2				(14,21)	(14,20)		(14,17)	(14,18)
15	D1’	128/81								(15,19)
16	D1	8/5							(16,19)	(16,20)
17	D2’ / N1’	5/3								(17,21)
18	D2 / N1	27/16							(18,21)	(18,22)
19	N2’ / D3’	16/9								
20	N2 / D3	9/5							(20,23)	
21	N3	15/8								
22	N3’	243/128								
23/1 repeat	S high	2								

Conclusions

The main aim of paper is to describe the pitch estimation on musical notes using signal processing method that involves the preprocessing and the extraction of pitch pattern. Here, the implementation and the basic experiments and discussions are presented. Moreover, the structure of 22 shrutis is studied, which are widely mentioned in the music literature. Towards that end the paper examines musical notes and associated frequency ratios in both Indian and Western music traditions. Three methods are presented to derive 22 shrutis which are widely mentioned in music literature.Some interesting properties about ratios of frequencies of notes among 22 shrutisis also brought out in the form of a table. Even though music is an art, music does follow some rules which may appear intuitive. Therefore, a rational understanding of notes and the frequency ratios using mathematics may be difficult. However, a good mathematical model may help in building better instruments and also to synthesize music using a computer which may be indistinguishable from music produced by humans. This is a challenge as well as continued inspiration for researchers and musicologists to provide interesting insights on music understanding. More work, is needed in developing more interesting insights about the art of music in a more rational way.

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