

Fractional Solutions to Wave Equations in Higher Dimensions

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Abstract

In this paper, we formulate fractional wave differential equations in higher space-time dimension by concentrating on the fractional order partial derivatives. For a given rational number, we obtain the respective fractional solutions of wave equations in two and three dimension. Concerning the above analysis of the fractional order solutions, we anticipate stimulating applications of the above formulation in realism of physical and natural systems.

Key Words: Fractional calculus, fractional solutions, fractional wave equation.

1. Introduction

The partial differential equations whose fractional solutions we set out to study in this paper are the wave equations in higher dimensions. Notice that waves are described by the wave equation which gives how the disturbances travel in time [1, 2]. The mathematical form of such equations varies depending on the type of waves and the media where it propagates. Further, it is worth mentioning that the behavior of a quantum particle is described by waves [3], as well. In addition, gravitational waves also propagate in a three dimensional space, which emerges as a result of movements or vibrations of the gravitational field [4].

Wave equations play a fundamental role in various scientific and engineering studies. The wave equation arises among one of the important second order linear partial differential equations as far as the description of waves are concerned. Such waves are noted for occurring in diverse domains of engineering, physics and others. Examples include the propagation of light, sound waves, shock waves and water waves [5]. Furthermore, waves arise in other fields including electromagnetism, acoustics, radars and fluid dynamics, as well. Historically, in 17th century, D'Alembert introduced one dimensional wave equation, whereby, Euler extended it within the next ten years to the case of the three dimensional wave equations [6].

Mathematically, it is well known [5-7] that wave equations are an element of the hyperbolic partial differential equations that usually involve a time variable t , one or more spatial variables $x_1, x_2, \dots, x_n$ for a given scalar function $u = u(x_1, x_2, \dots, x_n, t)$ whose numerical values represent the displacement of the wave [7]. Namely, for such a given scalar function u , the wave equation reads as

$$\frac{\partial^2 u}{\partial t^2} = a^2 \nabla^2 u, \quad (1.1)$$

Where a is a fixed constant that defines the propagation of the wave.

In this paper, our goal is to extend the above Eqn. (1.1) for fractional order wave propagation. In the sequel, we focus to obtain the concerning fractional solutions in two and three spatial dimensions. The rest of paper is organized as follows. In section 2, we give a brief outline of fractional order differentiations and integrations. In section 3, we provide basic properties of fractional derivatives, e.g., left and right derivatives. In section 4, we formulate the fractional order wave equations in higher dimensions, whereby provide their solutions in section 5. Finally, we conclude our paper in section 6 with perspective directions for future research.

2. Review of the Fractional Calculus

Fractional differentiations and fractional integrations are generalizations of ordinary integral order differentiations and integrations. This

includes n^{th} derivatives and n -fold integrals when n is an integer as the particular cases. Due to this reason, it would be natural to seek geometric and physical understandings of the fractional order differential operators [8-12]. Such operators equally offer a link with the known classical interpretations of integer valued standard operations of the differentiation and integration of a smooth function $f(t)$.

The fractional derivative of $f(t)$ of the order $\mu > 0$ (whenever it exists) can be expressed in term of the fractional integral $D^{-\nu}f(t)$ as per the formula

$$D^{\mu} f(t) = [D^{-(m-\mu)} f(t)], \quad (2.1)$$

Where m is integer greater than or equal to $[\mu]$, where $[x]$ denotes the ceiling function [7]. In this concern, it is worth recalling that the fractional derivative of a monomial $f(t) = t^{\lambda}$ is defined by

$$\begin{aligned} D^{\mu} t^{\lambda} &= D^m [D^{-(m-\mu)} t^{\lambda}] \\ &= D^m \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m-\mu+1)} t^{\lambda+m-\mu} \right] \end{aligned} \quad (2.2)$$

Operating D^m on the quantity in the parenthesis, it yields

$$D^{\mu} t^{\lambda} = \frac{\Gamma(\lambda+1)(\lambda-\mu+m)(\lambda-\mu+m-1)\dots(\lambda-\mu+1)}{\Gamma((\lambda+m-\mu+1))} t^{\lambda-\mu} \quad (2.3)$$

Using the properties of Γ function [7], it follows that we have

$$D^{\mu} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} t^{\lambda-\mu} \quad (2.4)$$

For $\lambda > -1$ and $\mu > 0$, the fractional derivative of a constant function $f(t) = C$ reduces as

$$D^{\mu} C = C \lim_{\lambda \rightarrow 0} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} t^{\lambda-\mu} = \frac{C t^{-\mu}}{\Gamma(1-\mu)} \quad (2.5)$$

To illustrate the above operation, let us consider a fractional derivative of the order $1/2$ of the function $f(t) = t^{\lambda}$. In this case, we have

$$D^{1/2} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1/2)} t^{\lambda-1/2} \quad (2.6)$$

This is sometimes known as the semi derivative [8-12] of t^{λ} . Note that a fractional derivative of a constant function need not be zero as in case of standard integer order derivative. For example, the semi-derivative of $f(t) = C$ is given as

$$D^{1/2} C = C \lim_{\lambda \rightarrow 0} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1/2)} t^{\lambda-1/2} = \frac{C}{\sqrt{\pi t}} \quad (2.7)$$

3. Elementary Properties of Fractional Derivatives

In this section, we provide basis properties of fractional derivatives [8-12] as per the followings.

- The first property of the interest is that a fractional derivative need not to obey the association

$$D_t^n [Cf(t)] = C D_t^n [f(t)], \quad (3.1)$$

Where C is a real constant.

- The second property is the distributive law. Given any two real valued functions $\{f, g\}$, we have

$$\begin{aligned} D_t^n [f(t) \pm g(t)] \\ = D_t^n [f(t)] \pm D_t^n [g(t)] \end{aligned} \quad (3.2)$$

- The third property is that the fractional differential operator D_t^n obeys the Leibniz rule while taking the derivative of the product of two function $\{f, g\}$ as

$$\begin{aligned} D_t^n [f(t)g(t)] &= \sum_{k=0}^n \binom{n}{k} D_t^{n-k} [f(t)] D_t^k [g(t)] \\ &= \sum_{k=0}^n \binom{n}{k} D_t^{n-k} [g(t)] D_t^k [f(t)], \end{aligned} \quad (3.3)$$

Where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the standard binomial coefficient expressed in terms of Γ function.

Left & Right Fractional Derivatives

The fractional derivative with a lower terminal as at the left end of the interval $[a, b]$ as denoted by $D_{at}^p f(t)$ is called the left fractional derivative of $f(t)$, see [10,11] an introduction of the subject matter of fractional calculus. The fractional derivative with an upper terminal as at the right end of the interval $[a, b]$ is called the right fractional derivative of $f(t)$.

For example, given a real number p such that $k-1 \leq p < k$, the left Riemann Liouville fractional derivative read as $D_{at}^p f(t)$

$$= \frac{1}{\Gamma(k-p)} \left(\frac{d}{dt}\right)^k \int_a^t (t-\tau)^{k-p-1} f(\tau) d\tau, \quad (3.4)$$

Where $a < t < b$. Similarly, it is known [8-12] that the corresponding right Riemann Liouville fractional derivative of $f(t)$ is given by $D_{bt}^p f(t)$

$$= \frac{1}{\Gamma(k-p)} \left(\frac{d}{dt}\right)^k \int_t^b (\tau-t)^{k-p-1} f(\tau) d\tau \quad (3.5)$$

Fractional Derivative of a Basic Power Function

Let us assume that $f(x)$ is a monomial of the form: $f(x) = x^k$ then the ordinary derivative of $f(x)$ arises [7] as

$$f'(x) = \frac{d}{d(x)} f(x) = kx^{k-1} \quad (3.6)$$

After replacing fraction $k \geq 0$ with the gamma function $\Gamma(z)$ with its properties [7] as $\Gamma(z+1) = z\Gamma(z)$, we have

$$\frac{d^\alpha x^k}{dx^\alpha} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \quad (3.7)$$

By putting $k = 1$ and $\alpha = 1/2$, from Eqn.(1) it follows the following semi derivative

$$\frac{d^{1/2}x}{dx^{1/2}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} \quad (3.8)$$

A direct simplification of the above result into the following function

$$\frac{d^{1/2}x}{dx^{1/2}} = \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} \quad (3.9)$$

By repeating the above process once again, we may verify the standard integer value rule as

$$\begin{aligned} \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left(\frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} \right) &= \frac{2}{\sqrt{\pi}} \frac{\Gamma\left(1 + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2} + 1\right)} x^{\frac{1}{2} - \frac{1}{2}} \\ &= \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} \times 0! = 1 \end{aligned} \quad (3.10)$$

Herewith, we see that we indeed have the expected result of standard differentiation

$$\frac{d^{1/2}}{dx^{1/2}} \left(\frac{d^{1/2}}{dx^{1/2}} x^0 \right) = \frac{dx}{dx} = 1 \quad (3.11)$$

For a generic function $f(x)$ and $0 < \alpha < 1$, its fractional derivative is given by

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^\alpha} dt \quad (3.12)$$

For an arbitrary α , since the gamma function is undefined for arguments whose real part is a negative integer and the imaginary part is zero. It is necessary to apply the fractional derivative after integer derivative has been performed. For example, the $3/2$ derivative of $f(x)$ is obtained as per the evaluation

$$D^{\frac{3}{2}} f(x) = D^{\frac{1}{2}} D f(x) = D^{\frac{1}{2}} \frac{d}{dx} f(x) \quad (3.13)$$

Euler's Fractional Derivatives

Given $0 < \alpha < 1$ and a real valued function $f(t) = t^\beta$ as a monomial, its Euler's fractional derivative [8-12] is given by

$$\frac{d^\alpha}{dt^\alpha} [t^\beta] = D_t^\alpha [t^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \alpha \in \mathbb{R} \quad (3.14)$$

Note that the Euler's definition of fractional derivatives is used in this paper to solve the fractional wave equations.

4. Fractional Order Wave Equations In Higher Dimensions

In this section, we formulate the fractional order wave equation by extending its respective integral order formulation. The formulation of a fractional wave equation can be given as follows. By introducing fractional order derivatives[8-

12] with respect to t , we can introduce fractional order acceleration as

$$a = \frac{\partial^{2r} v}{\partial t^{2r}}, \quad (4.1)$$

where v is fractional order velocity, that is, for a given classical trajectory x as the real valued map $x: \mathbb{R} \rightarrow \mathbb{R}$, we have

$$v = \frac{\partial^r x}{\partial t^r} \quad (4.2)$$

By extending the Newton's 2nd law as in its standard case [7], it follows that the associated fractional forces acting on a concerning body can be expressed as per the resolution

$$\sum f^r = m a^r, \quad (4.3)$$

Where r is areal number. Hereby, we have the following equilibrium condition along the horizontal direction, i.e., the x -axis as $\sum f_x^r = 0$. Further, for a given angle α , as there is no acceleration along the x -direction, we have

$$(T + \Delta^r T) \cos(\alpha + \Delta^r \alpha) = T \cos \alpha \quad (4.4)$$

With small angle approximation, viz. $\cos \alpha \cong 1 \cong \cos(\alpha + \Delta^r \alpha)$, we see that the horizontal direction of the force vanishes, namely, we have $\Delta^r T = 0$. This is because of the fact that $T \cos \alpha \cong (T + \Delta^r T) \cos(\alpha + \Delta^r \alpha) = T$. Here, it signifies that the tension remains constant. Similarly, resolving force in the vertical direction, we have

$$T \sin(\alpha + \Delta^r \alpha) - T \sin \alpha = m \frac{\partial^{2r} u}{\partial t^{2r}}(x, t) \quad (4.5)$$

Hence, along the y direction with small angle approximation, viz. the angle α and its fractional change $\Delta^r \alpha$ remain very small. In other words, with the approximation $\tan \alpha \cong \sin \alpha / \cos \alpha \cong \sin \alpha$, it follows that the vertical component reads

$$[\tan(\alpha + \Delta^r \alpha) - \tan \alpha] = m \frac{\partial^{2r} u}{\partial t^{2r}}(x, t) \quad (4.6)$$

Because of the above fact, the tension T along the x -direction simplifies as

$$T \left(\frac{\partial^r u}{\partial x^r} \Big|_{(x+dx)} - \frac{\partial^r u}{\partial x^r} \Big|_x \right) = m \frac{\partial^{2r} u}{\partial t^{2r}} \quad (4.7)$$

Since the mass can be expressed as the product of the density ρ and volume $\partial^r x$, thus with the elementary mass $m = \rho \partial^r x$, it follows that the above Eqn.(4.7) read as

$$T \left(\frac{\partial^r u}{\partial x^r} \Big|_{(x+dx)} - \frac{\partial^r u}{\partial x^r} \Big|_x \right) = \rho \partial^r x \frac{\partial^{2r} u}{\partial t^{2r}} \quad (4.8)$$

Herewith, along the horizontal direction, from Eqn. (4.8) the tension satisfies

$$\frac{T}{\rho} \left(\frac{\frac{\partial^r u}{\partial x^r} \Big|_{(x+dx)} - \frac{\partial^r u}{\partial x^r} \Big|_x}{\partial^r x} \right) = \frac{\partial^{2r} u}{\partial t^{2r}} \quad (4.9)$$

Thus, as the elementary volume $\partial^r x$ goes to zero, i.e. $\partial^r x \rightarrow 0$, we have

$$\frac{T}{\rho} \lim_{\partial^r x \rightarrow 0} \left(\frac{\left. \frac{\partial^r u}{\partial x^r} \right|_{(x+dx)} - \left. \frac{\partial^r u}{\partial x^r} \right|_x}{\partial^r x} \right) = \frac{\partial^{2r} u}{\partial t^{2r}}, \quad (4.10)$$

In other words, along the x direction, it follows from Eqn. (4.10) that the wave equation reads

$$\frac{T}{\rho} \frac{\partial^{2r} u}{\partial x^{2r}} = \frac{\partial^{2r} u}{\partial t^{2r}} \quad (4.11)$$

Define a wave parameter a as per the relation $a^2 = T/\rho$, the above Eqn.(4.11) reduces as

$$\frac{\partial^{2r} u}{\partial t^{2r}} = a^2 \frac{\partial^{2r} u}{\partial x^{2r}} \quad (4.12)$$

Similarly, in a three dimensional space, by taking account of all the three components of scalar function u , we have the following fractional order wave equation

$$\frac{\partial^{2r} u}{\partial t^{2r}} = a^{2r} \left(\frac{\partial^{2r} u}{\partial x^{2r}} + \frac{\partial^{2r} u}{\partial y^{2r}} + \frac{\partial^{2r} u}{\partial z^{2r}} \right), \quad (4.13)$$

Where r is real number between in the interval $(0, 1)$. Notice that the above Eqn. (4.13) in the special case of a two spatial dimensional space reduces as

$$\frac{\partial^{2r} u}{\partial t^{2r}} = a^{2r} \left(\frac{\partial^{2r} u}{\partial x^{2r}} + \frac{\partial^{2r} u}{\partial y^{2r}} \right) \quad (4.14)$$

5. Solutions of Fractional Wave Equations

In this section, we offer solutions to the fractional order differential equations as formulated in the previous section, namely, the fractional wave equations are solved by following the setup of the fractional calculus as above. Namely, we provide solutions to fractional order wave equations as discussed in section 3.

First of all, with the choice of the semi-derivative as $r = 1/2$, for the fractional order waves in a two dimensional space, from Eqn. (4.14) we find the following general solution

$$U(x, y, t) = \left(C_2 e^{\frac{-k^2}{a} x} \right) \cdot \left(C_3 e^{\frac{-l^2}{a} y} \right) \cdot (C_1 e^{\lambda^2 t}) \quad (5.1)$$

Here, C_1, C_2 and C_3 are integration constants. On the other hand, by taking $r = 3/4$ in Eqn. (4.14), we have the following fractional wave solution

$$U(x, y, t) = \left(-2 \frac{k^2}{a^{\frac{3}{2}}} x^{\frac{3}{2}} + c_2 \right)^2 \cdot \left(-2 \frac{l^2}{a^{\frac{3}{2}}} y^{\frac{3}{2}} + c_3 \right)^2 \cdot (2\lambda^2 t^{\frac{3}{2}} + c_1)^2 \quad (5.2)$$

Similarly, by putting $r=3/4$, from Eqn. (4.14) it follows that we have

$$U(x, y, t) = \left(-\frac{3}{4} \frac{k^2}{a^{\frac{3}{4}}} x^{\frac{4}{3}} + c_2 \right)^3 \cdot \left(-\frac{3}{4} \frac{l^2}{a^{\frac{3}{4}}} y^{\frac{4}{3}} + c_3 \right)^3 \cdot \left(\frac{3}{4} \lambda^2 t^{\frac{4}{3}} + c_1 \right)^3 \quad (5.3)$$

With the above observed solutions of the Eqn. (4.14), we have the following general formula

$$U_r(x, y, t) = \left(\frac{1}{\frac{-k^2}{a^{2r}} x^{2r} + c_2} \right)^{\frac{1}{1-2r}} \cdot \left(\frac{1}{\frac{-l^2}{a^{2r}} y^{2r} + c_2} \right)^{\frac{1}{1-2r}} \cdot \left(\frac{1}{\frac{\lambda^2 t^{2r}}{2r} + c_1} \right)^{\frac{1}{1-2r}}, \quad (5.4)$$

where C_1 , C_2 and C_3 are real constants and μ is defined as a function of λ . On the other hand, for a 3-dimensional fractional order wave equation as in Eqn. (4.13), for $r = 1/2$, we find the following fractional order solution

$$U(x, y, z, t) = \left(C_2 e^{\frac{-k^2}{a} x} \right) \cdot \left(C_3 e^{\frac{-l^2}{a} y} \right) \cdot \left(C_4 e^{\frac{-m^2}{a} z} \right) \cdot (C_1 e^{\lambda^2 t}) \quad (5.5)$$

Similarly, for $r = 1/3$ in Eqn. (4.13), the corresponding fractional order wave equation yields the solution as

$$U(x, y, z, t) = \frac{1}{\left(\frac{3-k^2}{2} x^{\frac{2}{3}} + c_2 \right)^3 \cdot \left(\frac{3-l^2}{2} y^{\frac{2}{3}} + c_3 \right)^3 \cdot \left(\frac{3-m^2}{2} z^{\frac{2}{3}} + c_4 \right)^3 \cdot \left(\frac{3\lambda^2 t^{\frac{2}{3}}}{2} + c_1 \right)^3} \quad (5.6)$$

From Eqn. (4.13), it is not difficult to see that the choice $r = 1/4$ yields the following solution

$$U(x, y, z, t) = \frac{1}{\left(2 \frac{-k^2}{a^{2/3}} x^{1/2} + c_2 \right)^2 \cdot \left(2 \frac{-l^2}{a^{2/3}} y^{1/2} + c_3 \right)^2 \cdot \left(2 \frac{-m^2}{a^{2/3}} z^{1/2} + c_4 \right)^2 \cdot (2\lambda^2 t^{1/2} + c_1)^2}, \quad (5.7)$$

Where μ is defined as a function of λ . Finally, following the above specific solutions of the Eqn. (4.13) for particular values of r , we have obtained the corresponding general fractional solution, as well. In general, for all $r \in \mathbb{R}$, we find that the three space dimensional fractional wave equations as in Eqn. (4.13) give the following solution

$$U_r(x, y, z, t) = \left(\frac{1}{\frac{-k^2}{a^{2r}} x^{2r} + c_2} \right)^{\frac{1}{1-2r}} \cdot \left(\frac{1}{\frac{-l^2}{a^{2r}} y^{2r} + c_3} \right)^{\frac{1}{1-2r}} \cdot \left(\frac{1}{\frac{-m^2}{a^{2r}} z^{2r} + c_4} \right)^{\frac{1}{1-2r}} \cdot \left(\frac{1}{\frac{\lambda^2 t^{2r}}{2r} + c_1} \right)^{\frac{1}{1-2r}} \quad (5.8)$$

Where C_1 , C_2 , C_3 and C_4 are real constants and μ is defined as a function of λ .

6. Conclusion

In this paper, we provide formulation and solutions of fractional order wave equations in higher dimensions. In particular, the fractional wave equations are solved and we offer the corresponding generalized fractional solutions. In addition, we have given solutions for specific fractional waves equations, e.g. semi-wave equation. The solutions of fractional wave equations as outlined in this paper are a prototype consideration for perspective developments of fractional mechanical equilibria, fractional vibrations, fractional oscillations and fractional media.

Further considerations include developments of the fractional order elliptic, parabolic and hyperbolic differential equations. We anticipate that that our initiation of a fractional order wave is merely a path way for opening of new branches that may fill gaps of the ordinary calculus. Finally, it is worth mentioning that the fractional calculus has the right potential of endowing intriguing and useful applications that we leave open for the future research.

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