A Note on Dominator Chromatic Number of Double Wheel Graph and Friendship Graph

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Abstract

A dominator coloring is a coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of at least one other color class. In this paper, we establish the dominator chromatic number for $M(W_{n,n})$, $C(W_{n,n})$, $T(W_{n,n})$, $M(Fr^4_n)$, $C(Fr^4_n)$ and $T(Fr^4_n)$ respectively.

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1 Introduction

For notation and graph theory terminology we generally follow [11, 6]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. The degree of
a vertex \( v \in V \) is \( |N(v)| \) and is denoted \( d_G(v) \) (or simply \( d(v) \) when the context is clear). We let \( \delta(G) = \min \{d(v) : v \in V\} \) and \( \Delta(G) = \max \{d(v) : v \in V\} \) denote the minimum and maximum degrees respectively (if the context is clear, we will simply let \( \delta = \Delta(G) \) and \( \Delta = \Delta(G) \). A vertex of degree one is called a pendant, and its neighbor is called a support vertex.

A set \( S \subseteq V \) is a dominating set of \( G \) if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). A (proper) \( k \)-coloring of a graph \( G \) is a function \( c : V(G) \to \{1, 2, \ldots, k\} \) such that \( c(x) \neq c(y) \) for any edge \( xy \). The color class \( c_i \) is the set of vertices of \( G \) that are assigned with color \( i \).

A dominator coloring of a graph \( G \) is a proper coloring of \( G \) in which every vertex dominates every vertex of at least one color class; that is, every vertex in \( V(G) \) is adjacent to all other vertices in its own color class or is adjacent to all vertices from at least one (other) color class. The concept of a dominator coloring in a graph was first aforementioned by Gera et al.\([3]\) and studied further by Gera \([4, 5]\) and Chellai and Maffray \([1]\). The upper and lower bounds \( \max \{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G) \) on the dominator coloring of graphs in terms of its domination\([11]\) number and its chromatic number was established by Gera et al. \([4, 5]\).

In the following section we obtain the exact value for \( \chi_d \) for Double wheel graph and Friendship graph.

2 Dominator Chromatic Number of Cycle Related Graphs

**Theorem 2.1.** For \( n \geq 4 \), the dominator chromatic number of double wheel graph is,

\[
\chi_d(W_{n,n}) = \begin{cases} 
3 & \text{when } n \text{ is even} \\
4 & \text{when } n \text{ is odd}
\end{cases}
\]

**Proof.** Let the vertex set of double wheel graph is defined as, \( V(W_{n,n}) = \{v_0\} \cup \)
\{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}. A procedure to obtain dominator coloring of double wheel graph as follows. Consider the following case.

**Case 1.** when \( n \) is even

We define a coloring function \( c : V(W_{n,n}) \rightarrow \{1, 2, 3\} \). The vertex of \( W_{n,n} \) colored as follows.

\[
c(V(W_{n,n})) = \begin{cases} 
  c_1 & \text{for } v_0 \\
  c_2 & \text{for } v_i, u_j : 1 \leq i \leq n \text{ and } i \text{ is odd} \\
  c_3 & \text{for } v_i, u_j : 1 \leq i \leq n \text{ and } i \text{ is even} \\
  c_4 & \text{for } u_n, v_n. 
\end{cases}
\]

**Case 2.** when \( n \) is odd

\[
c(V(W_{n,n})) = \begin{cases} 
  c_1 & \text{for } v_0 \\
  c_2 & \text{for } v_i, u_j : 1 \leq i \leq n - 2 \text{ and } i \text{ is odd} \\
  c_3 & \text{for } v_i, u_j : 1 \leq i \leq n - 1 \text{ and } i \text{ is even} \\
  c_4 & \text{for } u_n, v_n. 
\end{cases}
\]

By the definition of dominator coloring \( v_0 \) dominates the own color class and \( v_i, u_j : 1 \leq i \leq n \) dominates the color class \( c_1 \). Hence an easy check shows that the dominator chromatic number of double wheel graph is,

\[
\chi_d(W_{n,n}) = \begin{cases} 
  3 & \text{when } n \text{ is even} \\
  4 & \text{when } n \text{ is odd} 
\end{cases}
\]

this completes the proof of the theorem.

**Theorem 2.2.** For \( n \geq 8 \), the dominator chromatic number of Total graph of double wheel graph is,

\[
\chi_d(T(W_{n,n})) = 2n + 1
\]

**Proof.**

Let \( V(W_{n,n}) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \). By the definition of total graph, \( V(T(W_{n,n})) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{v_i \cup u_j : 1 \leq i \leq n\} \).
\[ u'_{ij} : 1 \leq i \leq n \cup e''_{ij} : 1 \leq i \leq n \sum \text{ where } e_i \text{ is the vertex set of } T(W_{n,n}) \text{ corresponding to the edge } v_{i}v_{i+1} \text{ of } W_{n,n}, 1 \leq i \leq n-1 \text{ and } v_{n}v_{1}. \]

Since the induced sub-graphs \( v_{0}, e'_{i}, e''_{i} : 1 \leq i \leq n \) forms a clique of order \( 2n + 1 \). Now we define a coloring function

\[ c : V(T(W_{n,n})) \rightarrow \{1, 2, 3, \ldots, n, n+1, \ldots, 2n, 2n+1\}. \]

The vertex of \( V(T(W_{n,n})) \) colored as follows

\[ c(V(T(W_{n,n}))) = \begin{cases} 
\text{c}^{i} \text{ for } e'_{i} : 1 \leq i \leq n \\
\text{c}_{n+i} \text{ for } e''_{i} : 1 \leq i \leq n \\
\text{c}_{2n+1} \text{ for } v_{0} 
\end{cases} \]

The remaining vertices are colored by the following cases

**Case 1 when n is even**

\[ c_{2} \text{ for } u_{i} : 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \]
\[ c_{4} \text{ for } u_{i} : 2 \leq i \leq n \text{ and } i \text{ is even} \]
\[ c_{6} \text{ for } u''_{i} : 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \]
\[ c_{8} \text{ for } u''_{i} : 2 \leq i \leq n \text{ and } i \text{ is even} \]
\[ c_{n+2} \text{ for } v_{i} : 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \]
\[ c_{n+4} \text{ for } v_{i} : 2 \leq i \leq n \text{ and } i \text{ is even} \]
\[ c_{n+6} \text{ for } e_{i} : 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \]
\[ c_{n+8} \text{ for } e_{i} : 2 \leq i \leq n \text{ and } i \text{ is even} \]
Case 1 when \( n \) is odd

- \( c_2 \) for \( e_n, u_i : 1 \leq i \leq n - 2 \) and \( i \) is odd
- \( c_4 \) for \( u_n, u_i : 2 \leq i \leq n - 1 \) and \( i \) is even
- \( c_5 \) for \( u' : 1 \leq i \leq n - 2 \) and \( i \) is odd
- \( c_{n+2} \) for \( u_n, u_i : 1 \leq i \leq n - 2 \) and \( i \) is odd
- \( c_{n+4} \) for \( u' : 2 \leq i \leq n - 1 \) and \( i \) is even
- \( c_{n+6} \) for \( e_i : 1 \leq i \leq n - 2 \) and \( i \) is odd
- \( c_{n+8} \) for \( e_i : 2 \leq i \leq n - 1 \) and \( i \) is even

By the definition of dominator coloring the vertices \( v_0, e_1, e_3, \ldots, e_i \) \( 1 \leq i \leq n \) dominates their own color classes and the vertices \( u_i, u_i : 1 \leq i \leq n \) dominates the color class \( c_{2n+1} \). When \( n \) is odd, vertices \( e_i e_{i+1} \) dominates the color class \( c_{i+1} \) where \( i = 2, 4, 6, 8, \ldots, n-1 \) and \( u', u'_{i+1} \) dominates the color class \( c_{n+i+1} \) where \( i = 2, 4, 6, 8, \ldots, n-2 \). When \( n \) is even, \( e_i e_{i+1} \) dominates the color class \( c_{i+1} \) where \( i = 2, 4, 6, 8, \ldots, n-2 \) and \( u', u'_{i+1} \) dominates the color class \( c_{n+i+1} \) where \( i = 2, 4, 6, 8, \ldots, n-2 \). Thus \( \chi_d(T(W_n, n)) \leq 2n + 1 \).

On the other hand we cannot assign lesser-than-2\( n+1 \) color to \( V(T(W_n, n)) \) Because the induced subgraph \( v_0, e'_j, e''_i : 1 \leq i \leq n \) forms a clique of order \( 2n + 1 \). Hence

\[
\chi_d(T(W_n, n)) = 2n + 1
\]

Theorem 2.3. For \( n \geq 8 \), the dominator chromatic number of Central graph of double wheel graph is

\[
\chi_d(C(W_n, n)) = |4n/3| + 2.
\]

Proof.

Let \( V(W_n, n) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \). By the definition of central graph,
\( V(C(W_{n,n})) = \{v_0\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \sum u_i \cup \sum e_i : 1 \leq i \leq n \) . Now we define a coloring function 
\( c : V(C(W_{n,n})) \rightarrow \{1, 2, 3, \ldots, n, n + 1, \ldots, |4n/3| + 2\} \). The vertex of \( V(C(W_{n,n})) \)
colored in the following cases

**Case 1**: \( n \equiv 0 \mod 3 

\[
\begin{align*}
&\text{c}_{2i} \text{ for } u_{3i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[4n/3]+1} \text{ for } u_0 \\
&\text{c}_{[4n/3]+2} \text{ for } e_i, e_i, e_i, u_i : 1 \leq i \leq n \\
&\text{c}_{2i-1} \text{ for } u_{3i-2}, u_{3i-1}, 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[2n/3]+2i} \text{ for } u_{2i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[2n/3]+2i-1} \text{ for } u_{2i-1}, u_{2i-2}, 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i-2} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i-1}: 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
\end{align*}
\]

**Case 2**: \( n \equiv 1 \mod 3 

\[
\begin{align*}
&\text{c}_{2i} \text{ for } u_{3i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[4n/3]+1} \text{ for } u_0 \\
&\text{c}_{[4n/3]+2} \text{ for } e_i, e_i, e_i, u_i : 1 \leq i \leq n \\
&\text{c}_{2i-1} \text{ for } u_{3i-2}, u_{3i-1}, 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[2n/3]+2i} \text{ for } u_{2i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[2n/3]+2i-1} \text{ for } u_{2i-1}, u_{2i-2}, 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i-2} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i-1} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
\end{align*}
\]

**Case 3**: \( n \equiv 2 \mod 3 

\[
\begin{align*}
&\text{c}_{2i} \text{ for } u_{3i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[4n/3]+1} \text{ for } u_0 \\
&\text{c}_{[4n/3]+2} \text{ for } e_i, e_i, e_i, u_i : 1 \leq i \leq n \\
&\text{c}_{2i-1} \text{ for } u_{3i-2}, u_{3i-1}, 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[2n/3]+2i} \text{ for } u_{2i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{[2n/3]+2i-1} \text{ for } u_{2i-1}, u_{2i-2}, 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i-2} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
&\text{c}_{2i-1} \text{ for } v_{3i-1} : 1 \leq i \leq k, 1 \leq k \leq \lfloor n/3 \rfloor \\
\end{align*}
\]
Proof.

Let $V(W_{n,n}) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$. By the definition of middle graph, $V(M(W_{n,n})) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e^i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$. Now we define a coloring function $c : V(M(W_{n,n})) \rightarrow \{1, 2, 3, \ldots, n, n+1, \ldots, 2n, 2n+1, 2n+2\}$. The vertex of $V(M(W_{n,n}))$ colored as follows:

$$
\begin{align*}
\text{c}(v_{2j}) & \text{ for } v_{2j} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\text{c}(v_{4j}) & \text{ for } v_0 \\
\text{c}(e_i, e^i, e^i_j) & \text{ for } e_i, e^i, e^i_j : 1 \leq i \leq n \\
\text{c}(u_{2j-1}) & \text{ for } v_{3j-1} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\text{c}(v_{4j}) & \text{ for } v_0 \\
\text{c}(u_{2j-1}) & \text{ for } v_{3j-2} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\text{c}(e_i, e^i, e^i_j) & \text{ for } e_i, e^i, e^i_j : 1 \leq i \leq n \\
\text{c}(u_{2j-1}) & \text{ for } v_{3j-1} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\text{c}(v_{4j}) & \text{ for } v_0 \\
\text{c}(u_{2j-1}) & \text{ for } v_{3j-2} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\text{c}(e_i, e^i, e^i_j) & \text{ for } e_i, e^i, e^i_j : 1 \leq i \leq n \\
\text{c}(u_{2j-1}) & \text{ for } v_{3j-1} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\text{c}(v_{4j}) & \text{ for } v_0 \\
\text{c}(u_{2j-1}) & \text{ for } v_{3j-2} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\text{c}(e_i, e^i, e^i_j) & \text{ for } e_i, e^i, e^i_j : 1 \leq i \leq n \\
\text{c}(u_{2j-1}) & \text{ for } v_{3j-1} : 1 \leq j \leq \lfloor n/3 \rfloor \\
\end{align*}
$$

Hence an easy check shows that $\chi_d(C(W_{n,n})) = [4n/3] + 2$.

**Theorem 2.4.** For $n \geq 6$, the dominator chromatic number of Middle graph of double wheel graph is,

$$
\chi_d(M(W_{n,n})) = 2n + 2
$$
The remaining vertices are colored by the following cases

**Case 1 when** $n$ **is even**

- $c_{2n+1}$ for $u_i : 1 \leq i \leq n$
- $c_1$ for $v_i, u_i' : 1 \leq i \leq n-1$ and $i$ is odd
- $c_{2n+1}$ for $e_{n-1}, v_i : 1 \leq i \leq n-2$
- $c_1$ for $e_i : 1 \leq i \leq n-3$ and $i$ is odd
- $c_n$ for $e_i : 2 \leq i \leq n-2$ and $i$ is even

**Case 2. when** $n$ **is even**

- $c_{2n+1}$ for $u_i : 2 \leq i \leq n-1$
- $c_1$ for $u_i' : 2 \leq i \leq n-1$ and $i$ is even
- $c_n$ for $u_n, e_i : 1 \leq i \leq n-2$ and $i$ is odd
- $c_1$ for $u_1, e_i : 2 \leq i \leq n-1$ and $i$ is even

By the definition of dominator coloring the vertices $e_n, e_i' : 2 \leq i \leq n -1$ and $e_i'' : 1 \leq i \leq n$ dominates itself. The vertex $u_i$ dominates the color class of $e_i'' : 1 \leq i \leq n$ and the remaining vertices dominates any one of the color class of $e_i', e_i'' : 1 \leq i \leq n$. Hence an easy observation shows that $\chi_d(M(W_{n,n})) = 2n+2$.

**Theorem 2.5.** For $n \geq 4$, the dominator chromatic number of friendship graph is, 

$$\chi_d(F_{\frac{r(4)}{n}}) = n + 2$$

**Proof.**

Let $V(F_{\frac{r(4)}{n}}) = \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{u_{2i} : 1 \leq i \leq 2n\}$ be the vertices of friendship graph with $3n + 1$ vertices and $4n$ edges. Let $v_i : 1 \leq i \leq n$ be the outer
vertices of $F_r(4)$ and $u_i : 1 \leq i \leq 2n$ be the inside vertices of $F_r(4)$. A procedure to obtain dominator coloring of $F_r(4)_n$ graph as follows. We define a coloring $c : V(F_r(4)_n) \rightarrow \{c_1, c_2, c_3, \ldots, c_{n+1}, c_{n+2}\}$.

$$c(V(F_r(4)_n)) = \begin{cases} 
  c_i \text{ for } v_i : 1 \leq i \leq n \\
  c_i \text{ for } u_i : 1 \leq i \leq 2n \\
  c_{n+1} \text{ for } u_f : 1 \leq i \leq 2n \\
  c_{n+2} \text{ for } v_0 
\end{cases}$$

It is easy to see that the above define coloring is a dominator coloring with $c_{n+2}$ colors. By the definition of dominator coloring, vertices $v_0, u_i : 1 \leq i \leq n$ dominates itself and $u_i : 1 \leq i \leq 2n$ dominates the color class $c_{n+2}$. Hence $\chi_d(F_r(4)_n) = n + 2$.

**Theorem 2.6.** For $n \geq 4$, the dominator chromatic number of Middle graph of friendship graph is,

$$\chi_d(M(F_r(4)_n)) = 3n + 2$$

**Proof.**

Let $V(M(F_r(4)_n)) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u : 1 \leq i \leq 2n\} \cup \{e : 1 \leq i \leq 2n\}$.

$$\sum_{i \leq 2n} e_i : 1 \leq i \leq 2n$$

be the vertices of Middle graph of friendship graph. Where $e_i : 1 \leq i \leq 2n$ is the vertex set of $M(F_r(4)_n)$ corresponding to the edge $v \ u : 1 \leq i \leq 2n$ of $F_r(4)_n$ and $e_i : 1 \leq i \leq 2n$ is the vertex set of $M(F_r(4)_n)$ corresponding to the edge $v_i : 1 \leq i \leq n$, $u_i : 1 \leq i \leq 2n$, and $e_i : 1 \leq i \leq 2n$ of $F_r(4)_n$. Hence $M(F_r(4)_n)$, the vertices induced by the subgraph of $v_0, e_i : 1 \leq i \leq 2n$ is isomorphic to complete graph of order $2n + 1$. A procedure to obtain dominator coloring of $M(F_r(4)_n)$ graph as follows. We define a coloring.
\[ c : V(M(Fr^n(4))) \rightarrow \{c_1c_2c_3 \ldots c_{n+1}, c_{n+2} \ldots, c_{2n}, c_{3n+1}, c_{3n+2}\}. \]

\[ \begin{align*}
& c_i \text{ for } e_i: 1 \leq i \leq 2n \\
& c_{2n+i} \text{ for } u_i: 1 \leq i \leq n \\
& c_{2n+i} \text{ for } v_i: 2 \leq i \leq 2n \text{ and } i \text{ is even} \\
& c_{3n+1} \text{ for } e_i: 1 \leq i \leq 2n \text{ and } i \text{ is odd} \\
& c_{3n+2} \text{ for } e_i: 2 \leq i \leq 2n \text{ and } i \text{ is even} \\
& c_{3n+2} \text{ for } u_i: 1 \leq i \leq 2n \text{ and } i \text{ is odd} \\
\end{align*} \]

Above consignment is a dominator coloring of \( M(Fr^n(4)) \) with 3n+2 colors. In \( M(Fr^n(4)) \), vertex \( u_i: 1 \leq i \leq n \), \( e_i: 1 \leq i \leq 2n \) dominiates itself and \( v_i, u_i, e_i: 1 \leq i \leq 2n \) dominiates the color class \( c_i \). Thus \( \chi(M(Fr^n(4))) \leq 3n+2 \).

On the other hand we cannot assign lesser-than \( c_{3n+2} \) colors to \( M(Fr^n(4)) \). Since the sub-graph induced by the vertices \( v_{0i}, e_i: 1 \leq i \leq 2n \) forms a clique of order \( 2n+1 \) so we have to assign \( c_{2n+1} \) colors to \( v_{0i}, e_i \). To satisfies the dominator coloring definition the vertices \( u_i: 1 \leq i \leq n \) are colored by \( c_{2n+i} \) colors. We need at-least one more new color to color the remaining vertices of \( M(Fr^n(4)) \). Hence dominator coloring with lesser-than \( c_{3n+2} \) color is not possible thus \( \chi(M(Fr^n(4))) = 3n+2 \).

**Theorem 2.7.** For \( n \geq 4 \), the dominator chromatic number of Total graph of friendship graph is,

\[ \chi_d(T(Fr^n(4))) = 3n+1 \]

**Proof.**

Let \( V(T(Fr^n(4))) = \{v\} \cup \{u: 1 \leq i \leq n\} \cup \{u: 1 \leq i \leq 2n\} \cup \{e: 1 \leq i \leq 2n\} \cup \)

\( - d_i: 1 \leq i \leq 2n \) be the vertices of Total graph of friendship graph. Since the vertices \( v_{0i}, e_i: 1 \leq i \leq 2n \) forms a clique of order \( 2n+1 \). To obtain the dominator coloring of \( T(Fr^n(4)) \) with \( 3n+1 \) colors as follows. Now we define a coloring function \( c : \)
The vertices $v_0, v_i : 1 \leq i \leq n$, dominates their own color classes and $e_i : 1 \leq i \leq 2n$ dominates the color class $c_{3n+1}$ and $u e_i : 1 \leq i \leq 2n$ dominates the color class of $u : 1 \leq i \leq n$. Hence an easy observation shows that $\chi(T(F r^{(4)}_n)) = 3n + 1$

**Theorem 2.8.** For $n \geq 4$, the dominator chromatic number of Central graph of friendship graph is,

$$\chi_d(C(F r^{(4)}_n)) = 2n + 2$$

**Proof.**

Let $V(C(F r^{(4)}_n)) = \{v_0 \} \cup \{v : 1 \leq i \leq n\} \cup \{u : 1 \leq i \leq 2n\} \cup \{e : 1 \leq i \leq 2n\}$ be the vertices of Central graph of friendship graph. Since the vertices $v_i : 1 \leq i \leq n$ forms a clique of order $n$. To obtain the dominator coloring of $T(F r^{(4)}_n)$ with $2n + 2$ colors as follows. Now we define a coloring function $c : V(C(F r^{(4)}_n)) \rightarrow \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}, c_{n+2}, \ldots, c_{2n}, c_{3n+1}\}$.

- $c_i$ for $v_i : 1 \leq i \leq n$
- $c_{i/2}$ for $u_i : 2 \leq i \leq 2n$ and $i$ is even
- $c_{i+1}$ for $v_i : 1 \leq i \leq 2n - 1$ and $i$ is odd
- $c_{2n+1}$ for $u_0$
- $c_{2n+2}$ for $e_i : 1 \leq i \leq 2n$
The vertex $v_i$ dominates anyone color class $c_i : 1 \leq i \leq n$ and the vertex $e_i : 1 \leq i \leq 2n$ dominates the color class $c_{2n+1}$ and $e'_i : 1 \leq i \leq 2n$ and $i$ is even, dominates the color class $c_{i/2}$ and $v_0$ dominates itself. Next the vertices $e'_i : 1 \leq i \leq 2n - 1$ and $i$ is odd dominates the color class of $u_i : 1 \leq i \leq 2n - 1$. Hence an easy check shows that $\chi_d(C(F_n^{(4)})) = 2n + 2$.

**References**


