



$\mathcal{H}_R$ -closed sets in generalized Topological Spaces

¹A.P. PONRAJ, ²R. SHANKAR, ³D. SIVARAJ.

¹Sri Muthukumarn Institute of Technology, Anna University, Chennai, saponraj@gmail.com.

² Research Scholar(Meenakashi Academy of Higher Education and Research),

Meenakashi College of Engineering, chennai, sudasha6@gmail.com.

³sidharswamy@gmail.com.

abstract The aim of this paper is to introduce and study the properties of $\mathcal{H}_R$ -closed set in a generalized topological space (X, κ) with a hereditary class $\mathcal{H}$.

Keywords: generalized topological space, hereditary class, $f_{\mathcal{H}}$ -set, regular- $\mathcal{H}$ -closed, κ^* -closed set, semi- $\mathcal{H}$ -open, strong- β - $\mathcal{H}$ -open.

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1 Introduction

Let X be a nonempty set. A non empty subfamily κ of $\wp(X)$ is called a generalized topology on X [?] if $\emptyset \in \kappa$ and κ is closed under arbitrary union. The pair (X, κ) is called generalized topological space. Elements of κ are called κ -open sets and the complement of a κ -open set is called a κ -closed set. The largest κ -open set contained in a subset A of X is denoted by $\text{int}_{\kappa}(A)$ [?] and is called the κ -interior of A . The smallest κ -closed set containing A is called the κ -closure of A and is denoted by $\text{cl}_{\kappa}(A)$ [?].

Throughout the paper, by a space, we always mean a generalized topological space (X, κ) . A subset A is said to be κ -dense if $\text{cl}_{\kappa}(A) = X$. Every strongly κ -codense hereditary class is κ -codense but the converse is not true [?]. If $\text{cl}_{\kappa}^*(A) = A \cap A^*$ for every subset A of X , with respect to κ and a hereditary class $\mathcal{H}$ of subsets of X then $\kappa^* = \{ A \subset X / \text{cl}_{\kappa}^*(X - A) = X - A \}$ is a generalized topology [?]. Elements of κ^* are called κ^* -open sets and the complement of a κ^* -open set is called a κ^* -closed set. $\text{int}_{\kappa}^*(A)$ is the interior of A in (X, κ^*) . Let (X, κ) be a generalized topological space and $\mathcal{H}$ be a hereditary class of subsets of X . If $\text{cl}_{\kappa}^*(A) = X$, then A is called κ^* -dense.

Definition 1.1. Let (X, κ) be a generalized topological space and $\mathcal{H}$ be a hereditary class of subsets of X . A subset A of X is called an $f_{\mathcal{H}}$ -set [?] if $A \subset (\text{int}_{\kappa}(A))^*$. The family of all $f_{\mathcal{H}}$ -sets in a space (X, κ) with hereditary class will be denoted by $f_{\mathcal{H}}(X, \kappa)$.

Definition 1.2. Let (X, κ) be a generalized topological space and $\mathcal{H}$ be a hereditary class of subsets of X . A subset A of X is called a regular $\mathcal{H}$ -closed[?], if $A = (\text{int}_\kappa(A))^*$. The family of all regular $\mathcal{H}$ -closed sets is denoted by $R_{\mathcal{H}C}(X)$. Clearly, $\mathcal{H}_B(X) \subset \mathcal{H}_C(X)$.

Definition 1.3. Let (X, κ) be a generalized topological space and $\mathcal{H}$ be a hereditary class of subsets of X . A subset A of X is said to be

- (a). semi- $\mathcal{H}$ -open[?] if $A \subset \text{cl}_\kappa^*(\text{int}_\kappa(A))$,
- (b). strongly β - $\mathcal{H}$ -open [?] if $A \subset \text{cl}_\kappa^*(\text{int}_\kappa(\text{cl}_\kappa^*(A)))$.

Lemma 1.4. [?] Let (X, κ) be a GT space, $\mathcal{H}$ be a hereditary class of subsets of X and $A \subset B \subset X$. Then the following hold.

- (a). $A \subset B \subset X$. implies that $\text{cl}_\kappa^*(A) \subset \text{cl}_\kappa^*(B)$.
- (b). $(A \cup A^*)^* \subset A^*$ for every $A \subset X$.
- (C). F is κ^* -closed if and only if $F^* \subset F$.
- (d). $\kappa \subset \kappa^*$.
- (f). If $\beta = \{M - H : M \in \kappa, H \in \mathcal{H}\}$ is a base for κ^* .

2 $\mathcal{H}_R$ -CLOSED SETS

Definition 2.1. A subset A of a generalized topological space (X, κ) with a hereditary class $\mathcal{H}$ is said to be $\mathcal{H}_R$ -closed set if $A = \text{cl}_\kappa^*(\text{int}_\kappa(A))$. We will denote the family of all $\mathcal{H}_R$ -closed set sets by $\mathcal{H}_R(X)$.

The following Theorem 2.2 gives some properties of the $\mathcal{H}_R$ -closed sets whose proof is omitted.

Theorem 2.2. Let (X, κ) be a generalized topological space and $\mathcal{H}$ be a hereditary class of subsets of X . A subset A be a $\mathcal{H}_R$ -closed set of X . Then

- (a). $(\text{int}_\kappa(A))^* \subset A$, since $A = \text{int}_\kappa(A) \cup (\text{int}_\kappa(A))^*$.
- (b). If $A \neq \emptyset$, then $\text{int}_\kappa(A) \neq \emptyset$.

The following Example 2.3 and Example 2.4 shows that $f_{\mathcal{H}}$ -set and $\mathcal{H}_R$ -closed sets are independent concepts.

Example 2.3. In a generalized topological space (X, κ) with a hereditary class $\mathcal{H}$ where $X = \{1, 2, 3, 4\}$, $\kappa = \{\emptyset, \{3\}, \{1, 4\}, \{1, 3, 4\}, X\}$ and $\mathcal{H} = \{\emptyset, \{2\}, \{4\}, \{2, 4\}\}$. If $A = \{1, 4\}$,

then $\text{int}_\kappa(A) = A$ and $\text{int}_\kappa(A)^* = \{1, 2, 4\} \supset A$ and so A is $f_\mathcal{H}$ -set. But $\text{cl}_\kappa^*(\text{int}_\kappa(A)) = \text{int}_\kappa(A) \cup (\text{int}_\kappa(A))^* = \{1, 2, 4\} \neq A$. and so A is not a $\mathcal{H}_R$ -closed set.

Example 2.4. In a generalized topological space (X, κ) with a hereditary class $\mathcal{H}$ where $X = \{1, 2, 3, 4\}$, $\kappa = \{\emptyset, \{4\}, \{1, 3\}, \{1, 3, 4\}, X\}$ and $\mathcal{H} = \{\emptyset, \{1\}, \{4\}, \{1, 4\}\}$. If $A = \{4\}$, then $\text{int}_\kappa(A) = A$ and $\text{int}_\kappa(A)^* = \emptyset \not\supset A$ and so A is not a $f_\mathcal{H}$ -set. But $\text{cl}_\kappa^*(\text{int}_\kappa(A)) = \text{int}_\kappa(A) \cup (\text{int}_\kappa(A))^* = \{4\} = A$ so A is a $\mathcal{H}_R$ -closed set.

Theorem 2.5. Let (X, κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $A \subset X$. Then A is a regular- $\mathcal{H}$ -closed set if and only if A is both a $f_\mathcal{H}$ -set and a $\mathcal{H}_R$ -closed set.

Proof. Suppose A is a regular- $\mathcal{H}$ -closed set. Then $A = (\text{int}_\kappa(A))^*$ and so A is a $f_\mathcal{H}$ -set. Also, $\text{cl}_\kappa^*(\text{int}_\kappa(A)) = \text{int}_\kappa(A) \cup (\text{int}_\kappa(A))^* = \text{int}_\kappa(A) \cup A = A$ and so A is a $\mathcal{H}_R$ -closed set.

Conversely, suppose A is both a $f_\mathcal{H}$ -set and a $\mathcal{H}_R$ -closed set. Since A is a $f_\mathcal{H}$ -set, $A \subset (\text{int}_\kappa(A))^*$ and since A is a $\mathcal{H}_R$ -closed set, by Theorem 2.2 (a), $(\text{int}_\kappa(A))^* \subset A$. Therefore $A = (\text{int}_\kappa(A))^*$ and so A is a regular- $\mathcal{H}$ -closed set. $\square$

Lemma 2.6. Let (X, κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $A \subset X$. Then A is regular- $\mathcal{H}$ -closed if and only if A is both an $f_\mathcal{H}$ -set and a κ^* -closed set.

Proof. Suppose A is a regular- $\mathcal{H}$ -closed set. Then $A = (\text{int}_\kappa(A))^*$ and so A is a $f_\mathcal{H}$ -set. Since $A = (\text{int}_\kappa(A))^*$, $A^* = ((\text{int}_\kappa(A))^*)^* \subset (\text{int}_\kappa(A))^* = A$, $A^* \subset A$. Then A is κ^* -closed set.

Conversely, suppose A is both an $f_\mathcal{H}$ -set and a κ^* -closed set. Since A is a $f_\mathcal{H}$ -set, $A \subset (\text{int}_\kappa(A))^*$ and since A is a κ^* -closed set, $A^* \subset A$, $\text{int}_\kappa(A) \subset A$, $(\text{int}_\kappa(A))^* \subset A^*$, $(\text{int}_\kappa(A))^* \subset A^* \subset A$, $(\text{int}_\kappa(A))^* \subset A$. Which implies that $(\text{int}_\kappa(A))^* = A$. Hence A is regular- $\mathcal{H}$ -closed set. $\square$

Corollary 2.7. Let (X, κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $A \subset X$. Then the following are equivalent.

- (a) A is a regular- $\mathcal{H}$ -closed set.
- (b) A is both an $f_\mathcal{H}$ -set and a κ^* -closed set.
- (c) A is both an $f_\mathcal{H}$ -set and an $\mathcal{H}_R$ -closed set.

Proof. (a) and (b) are equivalent by Lemma 2.6 (a) and (c) are equivalent by Theorem 2.5. $\square$

The following Theorems 2.8 and 2.9 give characterizations of $\mathcal{H}_R$ -closed sets.

Theorem 2.8. *Let (X, κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $A \subset X$. Then the following are equivalent.*

- (a) $A \neq \emptyset$, is $\mathcal{H}_R$ -closed set.
- (b) There exists a nonempty κ -open set G such that $G \subset A = cl_\kappa^*(G)$.
- (c) There exists a nonempty κ -open set G such that $A = G \cup (G^* - G)$.

Proof. (a) $\Rightarrow$ (b). Suppose $A \neq \emptyset$, is $\mathcal{H}_R$ -closed set. Then $A = cl_\kappa^*(int_\kappa(A))$. Let $G = int_\kappa(A)$. By Theorem 2.2(b), $G \neq \emptyset$. G is the required κ -open set such that $G \subset A = cl_\kappa^*(G)$.

(b) $\Rightarrow$ (c). Since $A = cl_\kappa^*(G) = G \cup (cl_\kappa^*(G) - G) = G \cup (G^* - G)$.

(c) $\Rightarrow$ (a). $A = G \cup (G^* - G)$. implies that $A = cl_\kappa^*(G)$. Also, since $G \subset A, G \subset int_\kappa(A)$ and so $cl_\kappa^*(G) \subset cl_\kappa^*(int_\kappa(A))$. Therefore $A \subset cl_\kappa^*(int_\kappa(A))$. Again, $int_\kappa(A) \subset A$ implies that $cl_\kappa^*(int_\kappa(A)) \subset cl_\kappa^*(A) = cl_\kappa^*(G) = A$ and so $cl_\kappa^*(int_\kappa(A)) \subset A$. Therefore, $A = cl_\kappa^*(int_\kappa(A))$ which implies that A is $\mathcal{H}_R$ -closed set. $\square$

Theorem 2.9. *Let (X, κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $A \subset X$. Then the following are equivalent.*

- (a) A is $\mathcal{H}_R$ -closed set.
- (b) A is semi- $\mathcal{H}$ -open and κ^* -closed.
- (c) A is strong- β - $\mathcal{H}$ -open and κ^* -closed.

Proof. (a) $\Rightarrow$ (b). If A is $\mathcal{H}_R$ -closed, then $A = cl_\kappa^*(int_\kappa(A))$ and so A is semi- $\mathcal{H}$ -open. By Theorem 2.7(b), A is κ^* -closed.

(b) $\Rightarrow$ (c). Follows from the fact that every semi- $\mathcal{H}$ -open set is a strong- β - $\mathcal{H}$ -open set.

(c) $\Rightarrow$ (a). Suppose A is strong- β - $\mathcal{H}$ -open and κ^* -closed. Then $A \subset cl_\kappa^*(int_\kappa(cl_\kappa^*(A)))$ and $A^* \subset A$. Now, $cl_\kappa^*(int_\kappa(A)) \subset cl_\kappa^*(A) = A$ and so $cl_\kappa^*(int_\kappa(A)) \subset A$. Also, $A \subset cl_\kappa^*(int_\kappa(cl_\kappa^*(A))) = cl_\kappa^*(int_\kappa(A))$. Therefore, $A = cl_\kappa^*(int_\kappa(A))$ which implies that A is $\mathcal{H}_R$ -closed set. $\square$

Example 2.10 and Example 2.11 shows that strong- β - $\mathcal{H}$ -open and κ^* -closed are independent concepts

Example 2.10. In a generalized topological space (X, κ) with a hereditary class $\mathcal{H}$ where $X = \{1, 2, 3, 4\}$, $\kappa = \emptyset, \{1\}, \{3\}, \{1, 3\}, X$ and $\mathcal{H} = \{\emptyset, \{1\}\}$. If $A = \{3, 4\}$, then $cl_{\kappa}^*(int_{\kappa}(A)) = cl_{\kappa}^*(\{c\}) = \{2, 3, 4\} \supset A$ and so A is semi- $\mathcal{H}$ -open and hence A is a strong- β - $\mathcal{H}$ -open set. Since $cl_{\kappa}^*(A) = cl_{\kappa}^*(\{3, 4\}) = \{2, 3, 4\}$, A is not κ^* -closed.

Example 2.11. In a generalized topological space (X, κ) with a hereditary class $\mathcal{H}$ where $X = \{1, 2, 3, 4\}$, $\kappa = \emptyset, \{1\}, \{3\}, \{1, 3\}, X$ and $\mathcal{H} = \{\emptyset, \{1\}\}$. If $B = \{2, 4\}$, then $B^* = \{2, 4\} = B$ and so B is κ^* -closed. But $cl_{\kappa}^*(int_{\kappa}(cl_{\kappa}^*(B))) = cl_{\kappa}^*(int_{\kappa}(B))cl_{\kappa}^*(\emptyset) = \emptyset \not\subset B$ and so B is not a strong- β - $\mathcal{H}$ -open set.

Theorem 2.12. Let (X, κ) be a generalized topological space, $\mathcal{H}$ be a hereditary class of subsets of X and $A \subset X$. If A is strong- β - $\mathcal{H}$ -open, then $cl_{\kappa}^*(A)$ is $\mathcal{H}_R$ -closed set.

Proof. Suppose strong- β - $\mathcal{H}$ -open. Then $A \subset cl_{\kappa}^*(int_{\kappa}(cl_{\kappa}^*(A)))$ and so $cl_{\kappa}(A) \subset cl_{\kappa}(cl_{\kappa}^*(int_{\kappa}(cl_{\kappa}^*(A)))) = cl_{\kappa}^*(int_{\kappa}(cl_{\kappa}^*(A))) \subset cl_{\kappa}^*(A)$ which implies that $cl_{\kappa}^*(int_{\kappa}(cl_{\kappa}^*(A))) = cl_{\kappa}^*(A)$. Therefore $cl_{\kappa}^*(A)$ is $\mathcal{H}_R$ -closed set.

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