Transient analysis of a single server queue with chain sequence rates subject to catastrophes, failures and repairs

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Abstract

In this paper, we analyze a single server queue subject to catastrophes, server failures and non-zero repair time whose arrival and service rates are suggested by a chain sequence. Closed form expressions for the transient probabilities of system size are obtained using continued fractions technique. The corresponding steady state probabilities and some system performance measures are deduced. It is observed that the steady state probabilities do not exist in the absence of catastrophes. Further, reliability and availability of the system are analyzed. The effect of system parameters on system size probabilities are also illustrated numerically.

Keywords: Transient analysis; Catastrophes; Server failure; Chain sequence; System reliability.

Mathematics Subject Classification 60K25.

1 Introduction

Queueing models are used to analyze many real time situations depending upon the real time environment. Hence they constitute a central tool in modeling and performance analysis of computer systems, communication networks, machine plants, air traffic, manufacturing systems and so on. When simulations are not feasible either because of time or storage complexity, it is important to have an analytical tractable model and numerical techniques to analyze the queueing systems. Gross and Harris [8] provide
a detailed introduction to queueing theory. Extensive surveys on queueing models in performance analysis of computer systems can be found in Lavenberg [15] and Takagi [20].

In the classical queueing models, the server is usually assumed to work at constant speed as long as there is any work present. However, this assumption may not always be appropriate as the state of the system may affect the server productivity. Many practical situations demand systems to serve the customers with variations in the arrival or service rates or possibly both. Hence it is of interest to study queueing systems taking into consideration the state-dependent nature of the system which have applications in computer and communication networks. Earlier Morse [16] considered queues with discouragement in which the arrival rate falls according to a negative exponential law. Lam and Lee [14] analyzed a fluid flow model with linear adaptive service rates. Krishna kumar et al. [12] considered discouraged arrival and infinite server queueing systems with catastrophes and observed that the steady state solutions are same for both the queues in the absence of catastrophes whereas the steady state solutions differ when catastrophes occur in the systems. An $M/M/1$ type tandem queueing system with state dependent arrivals, service and retrials has been studied by Dudin and Nazarov [7]. Recently Atchuta Rao Sadu et al. [2] studied a forked two-node tandem queueing model with work load dependent service rates having bulk arrivals and analyzed the system size probabilities under transient and equilibrium conditions.

Many traditional studies analyze the steady state solution of queueing systems as it is simple to derive and straightforward techniques can be employed. Very few stochastic systems are known to have closed-form transient solutions for the distribution of the process. The steady state results are well suited to study the performance measures of system on a long time scale while the transient solutions are more useful for studying the dynamical behavior of systems over a finite period. Transient analysis helps us to understand the behavior of a system when the parameters involved are perturbed and it can contribute to the costs and benefits of operating a system. Further, such transient analysis is useful in obtaining optimal solutions which lead to the control of the system. A number of methods have been adopted to find the time-dependent system size probabilities of queueing models. Abate and Whitt [1] obtained the transient behavior of $M/M/1$ queue in terms of first passage time distribution. Baccelli and Massey [3] obtained the transient distribution for the queue length using sample path behavior. The transient derivation for $M/M/1$ queue was obtained by Bailey [4] using generating function method while Champernowne [5] used complex combinatorial methods. Recently Krishna kumar et al. [10] analyzed the transient and steady state behavior of queueing systems with catastrophic failure and impatient customers using continued fraction technology.

The study of queueing systems with catastrophes has been thrust into the limelight and analyzed by many authors. Practical queueing systems like computer, communication networks, neural networks and manufacturing
systems are not reliable and disasters may occur in them. The catastrophes arrive as negative customers to the system and their characteristic is to remove some or all of the regular customers in the system. (see Gelenbe and Pujolle [9]). The catastrophes may come either from outside the system or from another service station. In computer networks, if a job infected with a virus arrives, it transmits the virus to other processors inactivating them and the system can be restored to normalcy only after taking appropriate corrective measures. Hence, computer networks with a virus infection may be modeled as queueing networks with catastrophes. Excellent surveys on queueing models with catastrophes can be found in Sophia and Vijayakumar [19] and Paz and Yechiali [18].

Queueing models with server breakdowns are more realistic in computer and communication switching systems since the failure and repair of processors have a major impact on the flow of jobs that have to be handled by those processors ( Wartenhorst [22] and Vijayashree and Janani [21] ). Queueing models subject to failures and repairs are found enthralling, either from the point of view of queueing theory or of reliability theory. Yechiali [23] considered an $M/M/c$ queue with system disasters and impatience customers and derived various quality of service measures such as mean sojourn time of a served customer, proportion of customers served and rate of lost customers due to disasters. Krishna kumar et al. [13] determined the transient solution for an $M/M/1$ queue with repairable server subject to catastrophes and discussed the reliability and availability of the system.

Queueing models with chain sequence rates are very rare and are mathematically interesting since they provide closed-form solution. (see Krishna kumar et al. [11] and the references therein). In this paper, a single server catastrophic queueing model with chain sequence rates along with server break down and repair is considered. Transient solutions of the state-dependent queueing model with the above mentioned features are obtained using continued fractions. The rest of the paper is structured as follows: The mathematical model under discussion is described in the following section. Section 3 provides the detailed analysis of the transient state probabilities and failure probability. Corresponding steady state probabilities and some important performance measures are discussed in section 4. Further more, it is deduced that the steady state solutions do not exist in the absence of catastrophes. Section 5 deals with the analysis of the reliability and availability of the system under consideration. Finally, in Section 6, numerical illustrations are added to exhibit the behavior of system performance measures with respect to various parameters. To the best of our knowledge, the present paper is the first of its kind to analyze queues with catastrophes, failures and repairs with chain sequence rates. The concluding section seven highlights the contributions of this paper.
2 Model description

Consider a state-dependent single server queueing system with infinite capacity that is subject to catastrophes at the service station. The arriving customers form a single waiting line based on the order of their arrivals and the service discipline is FCFS. Let \( \{X(t), t \in \mathbb{R}^+\} \) be the number of customers in the system at time \( t \). We assume that the arrival and service rates are \( \lambda_n \) and \( \mu_n \) respectively when the number of customers in the system at time \( t \) is \( n \); in any small interval \( (t, t + \Delta t), \Delta t \geq 0 \), during the time when the server is in up state, an arrival occurs with probability \( \lambda_n \Delta t + o(\Delta t) \); a service being completed with probability \( \mu_n \Delta t + o(\Delta t) \). It is clear that in this interval neither an arrival nor service takes place with probability \( 1 - (\lambda_n + \mu_n)\Delta t + o(\Delta t) \).

Apart from arrival and service processes when the system is not empty the catastrophes also occur at the service facility as a Poisson process with rate \( \alpha \), i.e., the catastrophe occurs in the small interval \( (t, t + \Delta t) \) with probability \( \alpha \Delta t + o(\Delta t) \). Here, the system is not empty means that the server is busy serving the customers. Whenever a catastrophe occurs at the busy server all the customers in the system are wiped out immediately, the server gets inactivated and the server is subject to catastrophic failure. The catastrophes may come either from outside the system or from another service station. The repair times of failed server are i.i.d, according to an exponential distribution with mean \( \frac{1}{\eta} \). After the completion of repair, the server immediately returns to its operational state (up state) and is ready for service when a new customer arrives. In addition, it is assumed that the newly arriving customers during the repair time of failed server will be lost forever.

Let \( P_n(t) = P(X(t) = n), \ n = 0, 1, 2, \ldots \), denote the transient probabilities that there are \( n \) customers in the system at time \( t \), when the server is in operational state. \( Q(t) \) be the probability that the server is under repair at time \( t \). Queueing systems like computers, communication networks and manufacturing systems will be more appropriate if the service speed and arrival rate depends on the workload. In many practical queueing scenarios, the speed of the server and the arrival rate of new customers are influenced by the amount of work present. Pankaj [17] analyzed a state-dependent single server queue where the arrival and service are assumed to be correlated and follow bivariate Poisson processes and the author determined the effect of system parameters on the average queue length. Even in packet-switched communication networks the transmission rates are dynamically adapted based on implicit information about the buffer content (workload). The above consideration lead us to incorporate state-dependent inter arrival times and state-dependent service speed in this paper. We consider the arrival rate \( \lambda_n \) and the service rate \( \mu_n \) are suggested by a chain sequence.

Now we assume that the state-dependent arrival and service rates \( \lambda_n \)
and \( \mu_n \) satisfy the chain sequence conditions
\[
\lambda_n + \mu_n = 1, \quad \lambda_{n-1} \mu_n = \theta, \quad \text{i.e.,} \quad (1 - \mu_{n-1}) \mu_n = \theta, \quad n = 1, 2, 3, \ldots, \quad (2.1)
\]
with \( \alpha > 0, \quad \lambda_0 = 1 \) and \( \mu_0 = 0 \), so that \( \{\mu_n\} \) is the minimal parameter sequence for the constant term chain sequence \( \{\theta, \theta, \theta, \ldots\} \) and \( 0 < \theta \leq \frac{1}{4} \), so that \( \lambda_n, \mu_n > 0 \), given by
\[
\lambda_n = \frac{\tau}{2} \frac{U_{n+1}(\frac{1}{\tau})}{U_n(\frac{1}{\tau})}, \quad n = 0, 1, 2, \ldots
\]
and
\[
\mu_n = \frac{\tau}{2} \frac{U_{n-1}(\frac{1}{\tau})}{U_n(\frac{1}{\tau})}, \quad n = 1, 2, 3, \ldots
\]
such that
\[
\lambda_0 \lambda_1 \ldots \lambda_{n-1} = \frac{\tau^n}{2^n} U_n \left( \frac{1}{\tau} \right), \quad n = 1, 2, 3, \ldots,
\]
where \( U_n(.) \) is the Chebyshev polynomial of the second kind of order \( n \) (see Chihara [6]) and \( \tau = 2\sqrt{\theta} \). For notational convenience, we use \( U_n \) instead of \( U_n(\frac{1}{\tau}) \) throughout the analysis.

It is to be noted that \( \{\lambda_n\} \) is a decreasing sequence with respect to \( n \), approaching \( \frac{1 + \sqrt{1 - 4\theta}}{2} \), implying discouraged arrivals. Similarly, \( \{\mu_n\} \) is an increasing sequence approaching \( \frac{1 - \sqrt{1 - 4\theta}}{2} \), implying the tuning up for faster service after each service completion.

The following sections deals with the transient and steady state probabilities of the queueing system subject to catastrophes and server failures whose rates are governed by minimal parameter chain sequence.

### 3 Transient Analysis

The state probabilities \( P_n(t), \quad n = 0, 1, 2, \ldots \), and the failure probability \( Q(t) \) of the queueing model under study can be described by Chapman-Kolmogorov forward differential-difference equations governing the system as follows:
\[
\frac{dQ(t)}{dt} = -\eta Q(t) + \alpha [1 - Q(t) - P_0(t)] \quad (3.1)
\]
\[
\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) + \eta Q(t) \quad (3.2)
\]

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\[
\frac{dP_n(t)}{dt} = -(\lambda_n + \mu_n + \alpha)P_n(t) + \lambda_{n-1}P_{n-1}(t)
+ \mu_{n+1}P_{n+1}(t), \quad n = 1, 2, 3, \ldots \tag{3.3}
\]

Without loss of generality, assume that initially there is no customer in the system, so that

\[ P_0(0) = 1 \quad \& \quad Q(0) = 0. \]

In the following sequel, we denote by \( g^*(s) \) as the Laplace transform of \( g(.) \).

By taking Laplace transforms and using the initial conditions, the above system of equations are reduced to a system of simultaneous equations given by

\[
(s + \lambda_0)P^*_0(s) = 1 + \mu_1P^*_1(s) + \eta Q^*(s) \tag{3.4}
\]

and

\[
(s + \lambda_n + \mu_n + \alpha)P^*_n(s) = \lambda_{n-1}P^*_{n-1}(s) + \mu_{n+1}P^*_{n+1}(s), \quad n = 1, 2, 3, \ldots \tag{3.6}
\]

From (3.4) and (3.5), after some algebra we get

\[
s + \lambda_0 = \frac{1}{P^*_0(s)} + \mu_1P^*_1(s) + \frac{\eta \alpha}{s + \eta + \alpha} \left[ \frac{1}{sP^*_0(s)} - 1 \right]
\]

which on further simplification gives

\[
\frac{1}{P^*_0(s)} = \frac{1}{(s + \lambda_0) + \frac{\eta \alpha}{s + \eta + \alpha} - \mu_1P^*_1(s)} \tag{3.7}
\]

Similarly, from (3.6), we have

\[
\frac{P^*_n(s)}{P^*_{n-1}(s)} = \frac{\lambda_{n-1}}{(s + \alpha + \lambda_n + \mu_n) - \mu_{n+1}P^*_{n+1}(s)}, \quad n = 1, 2, 3, \ldots \tag{3.8}
\]

Now using (3.8) iteratively in (3.7), we get \( P^*_0(s) \) as a continued fraction as follows:

\[
P^*_0(s) = 1 + \frac{\eta \alpha}{s + \lambda_0} - \frac{\lambda_0 \mu_1}{s + \lambda_1 + \mu_1 + \alpha} - \frac{\lambda_1 \mu_2}{s + \lambda_2 + \mu_2 + \alpha} - \ldots.
\]
Substituting the chain sequence rates given in (2.1), the above equation modifies to

\[ P^*_0(s) = \frac{1 + \frac{\eta \alpha}{s + \eta + \alpha}}{(s + 1) + \frac{\eta \alpha}{s + \eta + \alpha}} \theta = \frac{\theta}{(s + \alpha + 1) - \theta} - \frac{\theta}{(s + \alpha + 1) - \theta} - \cdots \]

which can also be expressed as

\[ P^*_0(s) = \frac{1 + \frac{\eta \alpha}{s + \eta + \alpha}}{(s + 1) + \frac{\eta \alpha}{s + \eta + \alpha} - \Phi(s)} \]  

where

\[ \Phi(s) = \frac{s + \alpha + 1 - \sqrt{(s + \alpha + 1)^2 - 4\theta}}{2} \]

That is

\[ \Phi^2(s) - (s + 1 + \alpha)\Phi(s) + \theta = 0, \]

the roots of which are

\[ \beta_1(s), \beta_2(s) = \frac{(s + 1 + \alpha) \pm \sqrt{(s + 1 + \alpha)^2 - 4\theta}}{2}. \]

It can be seen that \( \beta_2(s) \) is a unique real root within \([0,1)\) for \( \alpha > 0 \) and \( 0 \leq s < 1 \) and hence we consider only \( \beta_2(s) \) for further discussion.

Now, substituting \( \beta_2(s) \) for \( \Phi(s) \) in (3.9), \( P^*_0(s) \) becomes

\[ P^*_0(s) = \frac{1 + \frac{\eta \alpha}{s + s + \eta + \alpha}}{(s + 1) + \frac{\eta \alpha}{s + \eta + \alpha} - \left[\frac{s + \alpha + 1 - \sqrt{(s + \alpha + 1)^2 - 4\theta}}{2}\right]} \]  

which can be rephrased as

\[ P^*_0(s) = \frac{1 + \frac{\eta \alpha}{s + s + \eta + \alpha} + \theta - \left[\frac{s + \alpha + 1 - \sqrt{(s + \alpha + 1)^2 - 4\theta}}{2}\right]}{2\theta - (\alpha - \frac{s + \eta + \alpha}{s + \eta + \alpha})[s + \alpha + 1 - \sqrt{(s + \alpha + 1)^2 - 4\theta}]. \]

Expanding Binomially, we have

\[ P^*_0(s) = \left(1 + \frac{\eta \alpha}{s + s + \eta + \alpha} \right) \sum_{n=0}^{\infty} (\alpha - \frac{\eta \alpha}{s + \eta + \alpha})^n \times \left[\frac{s + \alpha + 1 - \sqrt{(s + \alpha + 1)^2 - 4\theta}}{2\theta}\right]^{n+1}. \]
That is

\[ P^*_n(s) = \left(1 + \frac{\eta\alpha}{s(s + \eta + \alpha)}\right) \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\alpha^n}{k!(n-k)!} \frac{n!}{(-1)^k} \times \left(\frac{\eta}{s + \eta + \alpha}\right)^k \left[ s + \alpha + 1 - \frac{\sqrt{(s + \alpha + 1)^2 - 4\theta}}{2\theta} \right]^{n+1} , \]

which can be rewritten as

\[ P^*_0(s) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \alpha^n (-1)^k \frac{n!}{k!(n-k)!} \left(\frac{\eta}{s + \eta + \alpha}\right)^k \times \left[ s + \alpha + 1 - \frac{\sqrt{(s + \alpha + 1)^2 - 4\theta}}{2\theta} \right]^{n+1} \]

On Inversion, (3.11) yields

\[ P_0(t) = \sum_{n=0}^{\infty} \frac{(n+1)\alpha^n}{\theta^{n+1}} \frac{I_{n+1}(2\sqrt{\theta t})}{t} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-1)^k \times \frac{(n+1)!}{\theta^{n+1}} \frac{I_{n+1}(2\sqrt{\theta u})}{u} \frac{\theta^{k+1}}{(k-1)!} e^{-\eta(t-u)} du 
+ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-1)^k \frac{(n+1)!}{\theta^{n+1}} \frac{I_{n+1}(2\sqrt{\theta u})}{u} \frac{\theta^{k+1}}{(k-1)!} e^{-\eta(t-u)} du \]

where \( I_n(.) \) is the modified Bessel function of the first kind of order \( n \).

The remaining transient probabilities can be obtained as follows. Iterating (3.8) and using (2.1) and (2.2), for \( n = 1, 2, 3, \ldots \), we get continued fraction

\[ \frac{P^*_n(s)}{P^*_{n-1}(s)} = \frac{\tau}{2} \frac{U_n}{U_{n-1}} \frac{\theta}{(s + 1 + \alpha)} - \frac{(s + 1 + \alpha)}{(s + 1 + \alpha)} - \frac{(s + 1 + \alpha)}{(s + 1 + \alpha)} - \cdots \]

Proceeding as before, the above equation reduces to

\[ \frac{P^*_n(s)}{P^*_{n-1}(s)} = \frac{\tau}{2} \frac{U_n}{U_{n-1}} \frac{(s + 1 + \alpha) - \sqrt{(s + 1 + \alpha)^2 - 4\theta}}{2\theta} , n = 1, 2, 3, \ldots , \]
after recursive procedure, we have
\[ P_n^*(s) = \left(\frac{\tau}{2}\right)^n U_n \left[\frac{(s + 1 + \alpha) - \sqrt{(s + 1 + \alpha)^2 - 4\theta}}{(2\theta)^n}\right]^n P_0^*(s), \]
\[ n = 1, 2, 3, \ldots \] (3.13)

Inversion of (3.13) gives, for \( n = 1, 2, 3, \ldots \),
\[ P_n(t) = nU_n \int_0^t P_0(u)e^{-(\alpha+1)(t-u)}I_n\left(\frac{2\sqrt{\theta}(t-u)}{t-u}\right)du. \] (3.14)

Now, the failure probability \( Q(t) \) can be obtained directly by inverting (3.4) as
\[ Q(t) = \alpha \int_0^t [1 - P_0(u)]e^{-(\eta+\theta)(t-u)}du. \] (3.15)

Thus, equations (3.12), (3.14) and (3.15) completely determine all the state probabilities \( P_n(t), n = 0, 1, 2, \ldots \), and the failure probability \( Q(t) \) for the model under consideration in terms of Chebyshev polynomial \( U_n(.) \) of second kind of order \( n \) and modified Bessel function \( I_n(.) \) of first kind of order \( n \).

4 Steady State Analysis

In this section, we investigate the behavior of the steady state probabilities \( \Pi_n, n \geq 0 \) and the failure probability \( Q \) for the state-dependent queueing model with chain sequence rates. The steady state moments are also deduced which are often good approximations to the transient counterpart expressions even when time \( t \) is of moderate size.

For \( \alpha > 0 \), we have from (3.10),
\[ sp_0^*(s) = \frac{s + \frac{\eta\alpha}{\eta+\alpha}}{\frac{\eta\alpha}{\eta+\alpha} + \left[(1+\alpha) + \sqrt{(1+\alpha)^2 - 4\theta}\right]}. \] (4.1)

Taking limit as \( s \to 0 \), on both sides and after some algebraic manipulation, we get
\[ \lim_{s \to 0} sp_0^*(s) = \frac{\frac{\eta\alpha}{\eta+\alpha} \left[2 - \left((1+\alpha) + \sqrt{(1+\alpha)^2 - 4\theta}\right)\right]}{2\left(\frac{\eta\alpha}{\eta+\alpha} + \theta - \alpha\right) - \frac{\eta\alpha}{\eta+\alpha} \left[(1+\alpha) + \sqrt{(1+\alpha)^2 - 4\theta}\right]}, \]
by applying Tauberian theorem, the above equation results in
\[ \Pi_0 = \frac{1 - \rho}{1 + \left( \frac{\alpha}{\eta} - \frac{\theta(\eta + \alpha)}{\eta \alpha} \right) \rho}, \quad (4.2) \]

where
\[ \rho = \frac{(1 + \alpha) - \sqrt{(1 + \alpha)^2 - 4\theta}}{2\theta}. \quad (4.3) \]

From (3.13), for \( \alpha > 0, n = 1, 2, 3, \ldots, \) we have
\[ \lim_{s \to 0} sP^*_n(s) = \lim_{s \to 0} sP^*_0(s) \left( \frac{\tau}{2} \right)^n U_n \left( \frac{(s + 1 + \alpha) - \sqrt{(s + 1 + \alpha)^2 - 4\theta}}{(2\theta)^n} \right). \]

Using the Tauberian theorem again, one obtains for \( n = 0, 1, 2, \ldots, \)
\[ \Pi_n = \left( \frac{\tau}{2} \right)^n U_n \frac{1 - \rho}{1 + \left( \frac{\alpha}{\eta} - \frac{\theta(\eta + \alpha)}{\eta \alpha} \right) \rho} \rho^n. \quad (4.4) \]

By similar argument as before, from (3.4), we get
\[ \lim_{s \to 0} sQ^*(s) = \lim_{s \to 0} \frac{\alpha}{s + \alpha + \eta} [1 - sP^*_0(s)] \]
which implies
\[ Q = \frac{\alpha}{\alpha + \eta} [1 - \Pi_0] \]
\[ = \frac{\alpha - \theta}{\alpha + \eta} \rho \]
\[ = \frac{1}{1 + \left( \frac{\alpha}{\eta} - \frac{\theta(\eta + \alpha)}{\eta \alpha} \right) \rho} \left( 1 - \rho \right) \sum_{n=0}^{\infty} U_n \left( \frac{\tau}{2} \rho z \right)^n. \quad (4.5) \]

Thus, in this way, closed form expressions are obtained analytically for the steady state probabilities of the system size \( \Pi_n \) and the failure distribution \( Q \) for the model under consideration.

**Remark 1:** It is interesting to note that, for \( \alpha = 0, \) \( \Pi_n = 0, \quad n = 0, 1, 2, \ldots, \) that is the steady-state solution do not exist whereas, for \( \alpha \neq 0, \) that is in the presence of catastrophes, (4.4) and (4.5) ensure the existence of steady state probabilities. Now, we obtain the steady state moments of the underlying system through the steady state probability generating function \( \Pi(z) \) which is defined as
\[ \Pi(z) = Q + \sum_{n=0}^{\infty} \Pi_n z^n \]
\[ = \frac{1}{1 + \left( \frac{\alpha}{\eta} - \frac{\theta(\eta + \alpha)}{\eta \alpha} \right) \rho} \left[ \frac{(\alpha - \theta) \rho}{\eta} + (1 - \rho) \sum_{n=0}^{\infty} U_n \left( \frac{\tau}{2} \rho z \right)^n \right]. \quad (4.6) \]
It is well known that the generating function of $U_n(\cdot)$ is given by
\[ \sum_{n=0}^{\infty} U_n(x) y^n = \frac{1}{1 - 2yx + y^2}. \] (4.7)

Making use of (4.7) in (4.6), we get
\[ \Pi(z) = \frac{1}{1 + \left( \frac{\alpha \eta - \theta \eta \alpha}{\eta \rho} \right) \rho} \left[ \frac{(\alpha - \theta)\rho}{\eta} + \frac{1 - \rho}{1 - \rho \rho z + \theta (\rho z)^2} \right]. \] (4.8)

The mean and second moment of the steady state can be deduced from (4.8) as
\[ E(X) = \left. \frac{d\Pi(z)}{dz} \right|_{z=1} = \frac{1 - \rho}{1 + \left( \frac{\alpha \eta - \theta \eta \alpha}{\eta \rho} \right) \rho} \left( 1 - 2\theta \rho \right) \frac{1}{\rho \alpha^2} \]
and
\[ E(X^2) = \left. \frac{d^2\Pi(z)}{dz^2} \right|_{z=1} + E(X) \]
\[ = \frac{1 - \rho}{1 + \left( \frac{\alpha \eta - \theta \eta \alpha}{\eta \rho} \right) \rho} \frac{\alpha (1 - 4\theta \rho) + 2(1 - 2\theta \rho)^2}{\alpha^3 \rho}. \]

Similarly the higher factorial moments can be obtained by successive differentiation of (4.8). It can be observed that the mean number of customers in the system $E(X)$ includes the down time when there are no customers in the system.

Furthermore it can be proved that
\[ P(\text{Server is busy}) = \sum_{n=1}^{\infty} \Pi_n = \sum_{n=1}^{\infty} U_n \left( \frac{T \rho}{2} \right)^n. \]

Again making use of (4.7) and after some considerable simplification we get
\[ P(\text{Server is busy}) = \frac{\rho (\alpha - \theta)}{1 + \left( \frac{\alpha \eta - \theta \eta \alpha}{\eta \rho} \right) \rho} \frac{\alpha}{\alpha} \]
and
\[ P(\text{Server is idle or under repair}) = \Pi_0 + Q = \frac{\rho (\alpha - \theta)}{1 + \left( \frac{\alpha \eta - \theta \eta \alpha}{\eta \rho} \right) \rho} \frac{\alpha}{\alpha}. \]
As before the corresponding conditional probabilities can be obtained as

\[
P(\text{Server is busy} \mid \text{Server is up}) = \frac{\sum_{n=1}^{\infty} \Pi_n}{1 - Q}
= \frac{\rho(\alpha - \theta)}{\alpha - \theta \rho}
= 1 - \alpha \rho
\]

and

\[
P(\text{Server is idle} \mid \text{Server is up}) = \frac{\Pi_0}{1 - Q}
= \frac{\alpha(1 - \rho)}{\alpha - \theta \rho}
= \alpha \rho.
\]

5 System reliability and availability analysis

System reliability is a vast area which is relevant to all areas of engineering such as computer networks, telecommunication systems, design of nuclear power plants etc., where any failure can be catastrophic. The goal of reliability engineering is to identify the most likely failures in a system and then identify appropriate actions to mitigate the effects of those failures. The reliability of a system is defined as the probability that the system will operate without failure under stated conditions for a stated period of time. In this section, the reliability indices like the system availability, system reliability, mean time to failure and the failure time density function for the model under consideration are discussed.

In systems with repair, the metric for system performance is the system availability \( A(t) \). \( A(t) \) denote the probability that the system is providing service or the server is idle at time \( t \). Then

\[
A(t) = 1 - Q(t)
= \frac{\eta}{\eta + \alpha} + \frac{\alpha}{\eta + \alpha} e^{-(\eta + \alpha)t} + \alpha \int_0^t e^{-(\eta + \alpha)(t-u)} P_0(u)du. \tag{5.1}
\]

The average availability of the system in the interval \([0,t]\) is

\[
\overline{A(t)} = \frac{1}{t} \int_0^t A(u)du
= \frac{\eta}{\eta + \alpha} + \frac{\alpha}{(\eta + \alpha)^2} \frac{1 - e^{-(\eta + \alpha)t}}{t}
+ \frac{\alpha}{(\eta + \alpha)} \int_0^t [1 - e^{-(\eta + \alpha)(t-u)}] P_0(u)du. \tag{5.2}
\]
Substituting $\eta = 0$ in (3.12) and (3.15), $P_0(t)$ and $Q(t)$ reduces to

$$P_0(t) = \sum_{n=0}^{\infty} \frac{(n+1)\alpha^n}{\theta^{n+1}} e^{-\alpha(t-\frac{2}{\sqrt{\theta}n+1})}$$

and

$$Q(t) = \alpha \int_0^t [1 - P_0(u)] e^{-\alpha(t-u)} du.$$  \hspace{1cm} (5.4)

Now, using (5.3) in (5.4), the Reliability function $R(t)$ is obtained as

$$R(t) = 1 - Q(t) = e^{-\alpha t} \left[ 1 + \alpha \int_0^t e^{-u} \sum_{n=0}^{\infty} \frac{(n+1)\alpha^n}{\theta^{n+1}} I_{n+1}(2\sqrt{\theta u}) \right].$$

The mean time to system failure, $E(T)$ is given by

$$E(T) = \int_0^\infty R(t) dt = \frac{1}{\alpha} + \frac{[1 + (1 + \alpha) - \sqrt{(1 + \alpha)^2 - 4\theta}]}{2\theta - \alpha [1 + (1 + \alpha) - \sqrt{(1 + \alpha)^2 - 4\theta}]}.$$

Finally, the failure time density function $f(t)$ can be obtained as

$$f(t) = \frac{dQ(t)}{dt} = \alpha e^{-\alpha t} - \sum_{n=0}^{\infty} \frac{(n+1)\alpha^{n+1}}{\theta^{n+2}} \times \left[ e^{-\alpha(t-\frac{2}{\sqrt{\theta}n+1})} \frac{I_{n+1}(2\sqrt{\theta u})}{u} - \alpha e^{-\alpha t} \int_0^t e^{-u} \frac{I_{n+1}(2\sqrt{\theta u})}{u} du \right].$$

and hence the higher moment of the reliability of the system can be obtained using (5.5).

\section{Numerical illustrations}

In this section, the effect of the system parameters on the steady state system size probabilities $\Pi_n$, $n = 0, 1, 2, ...$, the mean number of customers in the system $E(X)$ and the mean time to system failure $E(T)$ are discussed.
numerically. In chain sequence model, the arrival and service rates are expressed in terms of Chebyshev polynomials of second kind of order $n$ and the parameters of interest are $\theta (0 < \theta < 0 : 25)$ and $\tau$. For chosen values of $\theta$ and $\tau$, $\Pi_0, \Pi_1, \Pi_2, \Pi_3, E(X)$ and $E(T)$ are plotted against the catastrophic parameter $\alpha$, for various values of $\eta$.

Figure (1) depicts the behavior of the steady state probabilities $\Pi_1, \Pi_2$ and $\Pi_3$ for $\theta = 0.2, \eta = 4$ and $\tau = 0.89$. The graphs show a downward trend towards stability for increasing values of $\alpha$. Obviously $\Pi_2$ and $\Pi_3$ attains equilibrium faster than $\Pi_1$. In figures (2) and (3), the steady state probability $\Pi_0$ of no customers in the system is drawn as a function of $\alpha$, for varying values of $\theta$ and $\eta$. $\Pi_0$ increases with the catastrophic rate $\alpha$ and reaches stability for higher values of $\alpha$. As expected $\Pi_0$ increases with increasing values of $\theta$ and the repair rate $\eta$.

Figures (4) and (5) shows the influence of the catastrophic rate $\alpha$ on the mean number of customers, $E(X)$ in the system. It is observed that, $E(X)$ increases as the repair rate $\eta$ increases and decreases as $\theta$ increases. However, in both the cases $E(X)$ rapidly decreases and attains stability for increasing values of the catastrophic rate $\alpha$. In figure (6), the mean time to system failure $E(T)$ is drawn against the catastrophic parameter $\alpha$ for different values of $\theta$. The curves show a downward trend as $\alpha$ increases and upward trend as $\theta$ increases reflecting $E(T)$ is an increasing function of $\theta$ and an decreasing function of $\alpha$.

7 Conclusions

In this paper, the continued fraction methodology is used to determine the transient solution of a state-dependent queueing system with catastrophes, server failures and repairs. Further it is assumed that the arrival and service rates are governed by chain sequence norms. Explicit expressions for transient probabilities are obtained in terms of Chebyshev polynomials and modified Bessel functions. The behavior of the system under steady state is analyzed and steady state moments are deduced. Some interesting performance measures such as reliability, availability and the mean time to system failure are obtained. It is interesting to note that the steady state distributions do not exist in the absence of catastrophes for the system under consideration. Finally, some numerical illustrations are presented for chosen parametric values. Graphes are plotted to highlight the effect of the catastrophic rate $\alpha$ and the repair rate $\eta$ on the system behavior.

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Figure (1): $\Pi$ versus $\alpha$ for $\theta = .2$, $\tau = .89$ and $\eta = 4$.

Figure (2): $\Pi_0$ versus $\alpha$ for $\eta = 2$ and for varying $\theta$.
Figure (3): $\Pi_0$ versus $\alpha$ for $\theta = .2$ and for varying $\eta$. 

Figure (4): $E(X)$ versus $\alpha$ for $\eta = 4$ and for varying $\theta$. 

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Figure (5): $E(X)$ versus $\alpha$ for $\theta = .2$ and for varying $\eta$.

Figure (6): $E(T)$ versus $\alpha$ for varying $\theta$. 

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