



Stationary Analysis of an $M/E_k/1$ Driven Fluid Queue Subject to Exponential Timer

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Abstract

This paper discusses the fluid queue with single on-off source having the service time as the Erlangian distribution of order k and controlled by an exponential timer. The buffer content distribution is obtained using renewal process. Closed form expressions for the buffer content distribution along with averages are found and results are deduced for $k = 1$, $k = 2$ and $k = \infty$. The effect of system parameters on system size probabilities are also illustrated numerically.

Keywords: buffer content; busy period; exponential timer ; Erlangian distribution.

Mathematics Subject Classification 60K25.

1 Introduction

Nowadays, stochastic fluid flow models have turned out to be useful to capture various needs in computers and modern telecommunication networks. These models describe the behaviour of a fluid level in a storage device. The flow of data in many small packets is modelled as fluid. During the active periods the source transmits information and during the idle periods no such transmission takes place. The consecutive active and idle periods of the source constitute an alternating renewal process and the input rate is modulated by this process. Anick et al. [1982] considered a single buffer with homogeneous sources and found the stationary solution using spectral decomposition method. Later many authors analyzed systems

with heterogeneous on-off sources and obtained the buffer content distribution (see Aalto(1998)). The concept of exponential timer was introduced by Katayama and Kobayashi [2007] and Katayama [2007] to analyze multi-class $M/G/1$ priority queues with server vacations and time limited-services. Timers are used as a control mechanism to restrict the service (busy periods) in various telecommunication queueing networks. Vijayashree and Anjuka (2013) studied an $M/M/1$ driven fluid queue subject to catastrophes and subsequent repair under steady state conditions. They used continued fraction methodology to obtain analytical solutions for the buffer occupancy distribution. This paper analyzes a fluid queue driven by an $M/E_k/1$ queue whose active periods are controlled by an exponential timer. The following section provides the model description of the fluid queue under consideration. Section 3 discusses the fluid queue with single on-off source having the service time as the Erlangian distribution of order k . The buffer content distribution along with averages are found and results are deduced for $k = 1, k = 2$ and $k \rightarrow \infty$. In section 4, numerical results related to the tail distribution and the average of buffer content, displaying the implication of the exponential timer are discussed with the help of graphs.

2 Model description

We consider an infinite buffer fluid queue with non negative buffer content $\{W(t), t \geq 0\}$ whose input is generated by a single source and is regulated by an alternating renewal process $X_1, Y_1, X_2, Y_2, \dots$, where the sequences $\{X_n\}$ and $\{Y_n\}$ are independent and identically distributed random variables with probability distributions $\phi_X(\cdot)$ and $\phi_Y(\cdot)$ respectively. During any (active) period of length X , there is an inflow of fluid (information) into the buffer at rate c_0 and an outflow from the buffer at rate c_1 ($c_1 < c_0$), leading to a net inflow as $a_1 (= c_0 - c_1)$. During any (silent) period Y while there is no inflow into the buffer, there is an outflow at rate c_1 from the buffer. It is further assumed that the active periods are controlled by an exponential timer which truncates the active period, stopping the fluid inflow into the buffer, depending on the lengths of the active period and exponential timer. The fluid flow system under steady-state is stable if and only if $E(V) < E(Y)$ where the random variable $V = \min(X, T)$ with X as the active period and T as the exponential timer random variable with parameter α .

The fluid content Z_n in the beginning of the n^{th} active period satisfies the modified Lindley's recurrence relation

$$Z_{n+1} = \max(0, Z_n + \tilde{V}_n - \tilde{Y}_n) \quad (2.1)$$

where $\tilde{Y} = c_1 Y$, $\tilde{V} = \min(a_1 X, a_1 T)$ and $Z_1 = 0$. Note that $\tilde{V}$ and $\tilde{Y}$ are the net input of fluid accumulated during any active period regulated

by exponential timer and the depletion of buffer content during any silent period, respectively.

Using renewal theory arguments,

When $Y's$ are exponentially distributed with parameter λ , the Laplace-Stieltjes transform of the buffer content distribution is given by (Adan and Resing 1996)

$$F^*(s) = Z^*(s) \left[\frac{E(Y)}{E(V) + E(Y)} + \frac{E(V)}{E(V) + E(Y)} \frac{1 - \Phi_{\tilde{V}}^*(s)}{sE(\tilde{V})} \right] \quad (2.2)$$

where $Z^*(s)$ is the Laplace-Stieltjes transform of the waiting time distribution in an $M/G/1$ queue with arrival rate $\frac{\lambda}{c_1}$ and service time random variable $\tilde{V}$ having probability distribution $\phi_{\tilde{V}}(\cdot)$, given by,

$$Z^*(s) = \frac{s(1 - \frac{\lambda}{c_1}E(\tilde{V}))}{s - \frac{\lambda}{c_1} + \frac{\lambda}{c_1}\Phi_{\tilde{V}}^*(s)}.$$

On substitution, (2.2) leads to

$$F^*(s) = \frac{c_1 - \lambda E(\tilde{V})}{a_1 + \lambda E(\tilde{V})} \left[\frac{a_1 s + \lambda - \lambda \Phi_{\tilde{V}}^*(s)}{c_1 s - \lambda + \lambda \Phi_{\tilde{V}}^*(s)} \right]. \quad (2.3)$$

Consider a fluid queue fed by an $M/G/1$ queue wherein the active periods are the busy periods of $M/G/1$ queue and are controlled by an exponential timer. Assuming $H(\cdot)$ as the cdf of the busy period of an $M/G/1$ queue with arrival rate λ and $H_1(\cdot)$ as the cdf of the busy period of an $M/G/1$ queue with arrival rate λ along with exponential timer α , the Laplace-Stieltjes transform of the density function and the mean $E(B_1)$ of the busy period are (refer Vijayakumar and Sophia (2013))

$$H_1^*(s, \alpha) = \frac{\alpha + sH^*(s + \alpha)}{(s + \alpha)} \quad (2.4)$$

and

$$E(B_1) = \frac{1 - H^*(\alpha)}{\alpha}. \quad (2.5)$$

Since the active periods of the fluid queue are the busy periods of $M/G/1$ queue (Virtamo and Norros 1994), using (2.4) and (2.5) in (2.3), the Laplace-Stieltjes transform of the buffer content is given by

$$F^*(s) = \frac{c_1 \alpha - \lambda + \lambda H^*(a_1 \alpha)}{a_1 \alpha + \lambda - \lambda H^*(a_1 \alpha)} \left[\frac{a_1(s + \alpha) + \lambda - \lambda H^*(a_1(s + \alpha))}{c_1(s + \alpha) - \lambda + \lambda H^*(a_1(s + \alpha))} \right]. \quad (2.6)$$

3 Fluid Queue with Active Period as Erlangian Distribution

In this section, we consider a fluid queue fed by a single on-off source with active and silent periods corresponding to the service time and idle periods of an $M/E_k/1$ queue, where the on periods are regulated by an exponential timer. To obtain the Laplace-Stieltjes transform of the distribution of the fluid content for the system under consideration, we take $H^*(s + \alpha) = (\frac{\mu k}{s + \alpha + \mu k})^k$, substitute in (2.6), leading to

$$F^*(s) = \frac{c_1 \alpha - \lambda + \lambda(1 + \frac{a_1 \alpha}{\mu k})^{-k}}{a_1 \alpha + \lambda - \lambda(1 + \frac{a_1 \alpha}{\mu k})^{-k}} \left[\frac{a_1(s + \alpha) + \lambda - \lambda(1 + \frac{a_1(s + \alpha)}{\mu k})^{-k}}{c_1(s + \alpha) - \lambda + \lambda(1 + \frac{a_1(s + \alpha)}{\mu k})^{-k}} \right] \quad (3.1)$$

For tractable analysis, we discuss the following special cases:

Case 1: $k = 1$

In this case, the active and silent periods are the service time and idle periods of $M/M/1$ queue. Then, (3.1) gives

$$F^*(s) = \frac{\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1}}{\alpha + \frac{\lambda + \mu}{a_1}} \frac{s + \alpha + \frac{\lambda + \mu}{a_1}}{s + \alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1}}. \quad (3.2)$$

On inversion, (3.2) yields the probability density function $f(x)$ of the buffer content for the fluid queue fed by single on-off source as

$$f(x) = \frac{\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1}}{\alpha + \frac{\lambda + \mu}{a_1}} \left[\delta(x) + \frac{c_0 \lambda}{a_1 c_1} e^{-(\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1})x} \right]. \quad (3.3)$$

The mean buffer content $E(W)$ is obtained as,

$$E(W) = \int_0^\infty x f(x) dx = \frac{c_0 \lambda}{a_1 c_1 (\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1})(\alpha + \frac{\lambda + \mu}{a_1})}$$

and the second moment

$$E(W^2) = \int_0^\infty x^2 f(x) dx = \frac{2c_0 \lambda}{a_1 c_1 (\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1})^2 (\alpha + \frac{\lambda + \mu}{a_1})}.$$

Further, from (3.3), the distribution function of the buffer content W is obtained as

$$P(W \leq x) = \int_0^x f(u) du$$

$$= \frac{\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1}}{\alpha + \frac{\lambda+\mu}{a_1}} \left[1 + \frac{\lambda c_0 [1 - e^{-(\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1})x}]}{\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1}} \right]$$

so that, the tail probability distribution function of W is

$$P(W > x) = \frac{c_0 \lambda e^{-(\alpha - \frac{\lambda}{c_1} + \frac{\mu}{a_1})x}}{a_1 c_1 (\alpha + \frac{\lambda+\mu}{a_1})}, \quad (3.4)$$

which yields the average outflow from the system under consideration as

$$c_1 P(W > 0) = \frac{c_0 \lambda}{a_1 \alpha + \lambda + \mu}$$

which equals $c_0 \cdot \frac{1}{\mu + a_1 \alpha} (\frac{1}{\mu + a_1 \alpha} + \frac{1}{\lambda})^{-1}$, the average inflow into the system.

Case 2: $k = 2$

In this case, the active and silent periods of the fluid queue are the service time and idle periods of $M/E_2/1$ queue. We further assume $c_0 = 2$ and $c_1 = 1$ for convenience. Now, substituting $k = 2, c_0 = 2$ and $c_1 = 1$ in (3.1), we get

$$F^*(s) = \frac{\alpha - \lambda + \lambda (\frac{2\mu}{2\mu + \alpha})^2}{\alpha + \lambda - \lambda (\frac{2\mu}{2\mu + \alpha})^2} \left[\frac{s + \alpha + \lambda - \lambda (\frac{2\mu}{s + 2\mu + \alpha})^2}{s + \alpha - \lambda + \lambda (\frac{2\mu}{s + 2\mu + \alpha})^2} \right]$$

which, after some mathematical manipulation simplifies to

$$F^*(s) = \frac{4\mu^2 + (\alpha - \lambda)(\alpha + 4\mu)}{4\mu^2 + (\alpha + \lambda)(\alpha + 4\mu)} \left[1 + \frac{2\lambda(s + d_1) + 4\lambda\mu + \lambda^2}{(s + d_1)^2 - d_2^2} \right],$$

where, $d_1 = \alpha + 2\mu - \frac{\lambda}{2}$ and $d_2 = \sqrt{(\frac{\lambda}{2})^2 + 2\lambda\mu}$. On inversion, we get the probability density function $f(x)$ of the buffer content as

$$\begin{aligned} f(x) &= \frac{4\mu^2 + (\alpha - \lambda)(\alpha + 4\mu)}{4\mu^2 + (\alpha + \lambda)(\alpha + 4\mu)} \left[\delta(x) + e^{-(d_1 - d_2)x} \left(\lambda + \frac{4\lambda\mu + \lambda^2}{2d_2} \right) \right. \\ &\quad \left. + e^{-(d_1 + d_2)x} \left(\lambda - \frac{4\lambda\mu + \lambda^2}{2d_2} \right) \right]. \end{aligned} \quad (3.5)$$

Consequently, the first two moments of the buffer content $E(W)$ and $E(W^2)$ are obtained as

$$E(W) = \frac{4\mu^2 + (\alpha - \lambda)(\alpha + 4\mu)}{4\mu^2 + (\alpha + \lambda)(\alpha + 4\mu)} \times \left[\frac{1}{2d_2} \left(\frac{2\lambda d_2 + 4\lambda\mu + \lambda^2}{(d_1 - d_2)^2} + \frac{2\lambda d_2 - 4\lambda\mu - \lambda^2}{(d_1 + d_2)^2} \right) \right]$$

so that

$$E(W) = \frac{4\mu^2 + (\alpha - \lambda)(\alpha + 4\mu)}{4\mu^2 + (\alpha + \lambda)(\alpha + 4\mu)} \left[\frac{2\lambda[d_1(\alpha + 6\mu + \frac{\lambda}{2}) + d_2^2]}{(d_1^2 - d_2^2)^2} \right].$$

and the second moment

$$E(W^2) = \frac{4\mu^2 + (\alpha - \lambda)(\alpha + 4\mu)}{4\mu^2 + (\alpha + \lambda)(\alpha + 4\mu)} \left[\frac{2}{(d_1 - d_2)^3} \left(\lambda + \frac{4\lambda\mu + \lambda^2}{2d_2} \right) + \frac{2}{(d_1 + d_2)^3} \left(\lambda - \frac{4\lambda\mu + \lambda^2}{2d_2} \right) \right].$$

From (3.5), the cumulative distribution function of the buffer content W of the fluid can be found as

$$P(W \leq x) = \frac{4\mu^2 + (\alpha - \lambda)(\alpha + 4\mu)}{4\mu^2 + (\alpha + \lambda)(\alpha + 4\mu)} \left[1 + \frac{[1 - e^{-(d_1 - d_2)x}]}{d_1 - d_2} \times \left(\lambda + \frac{4\lambda\mu + \lambda^2}{2d_2} \right) + \frac{[1 - e^{-(d_1 + d_2)x}]}{d_1 + d_2} \left(\lambda - \frac{4\lambda\mu + \lambda^2}{2d_2} \right) \right]$$

and the average outflow from the system can be found as before.

Case 3: $k \rightarrow \infty$

This model reduces to the fluid queue whose active and silent periods are the service time and idle periods of $M/D/1$ queue. For this system, with $c_0 = 2, c_1 = 1$ and taking the limit as $k \rightarrow \infty$ in (3.1), we get

$$F^*(s) = \frac{r_1}{r_2} \left[\frac{s + \alpha + \lambda - \lambda e^{-\frac{s+\alpha}{\mu}}}{s + \alpha - \lambda - \lambda e^{-\frac{s+\alpha}{\mu}}} \right]$$

where $r_1 = \alpha - \lambda + \lambda e^{-\frac{\alpha}{\mu}}$ and $r_2 = \alpha + \lambda - \lambda e^{-\frac{\alpha}{\mu}}$. After some algebra the above equation can be expressed as

$$F^*(s) = \frac{r_1}{r_2} \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{\lambda}{s + \alpha} \right)^n \sum_{m=0}^n (-1)^m \binom{n}{m} e^{-\frac{m(s+\alpha)}{\mu}} \right]$$

On inversion, the probability density function $f(x)$ is obtained as

$$f(x) = \frac{r_1}{r_2} \left[\delta(x) + 2 \sum_{n=1}^{\infty} \lambda^n \sum_{m=0}^n (-1)^m \binom{n}{m} e^{-\frac{m\alpha}{\mu}} \times \int_0^x e^{-\alpha(x-u)} \frac{(x-u)^{n-1}}{(n-1)!} \delta(u - \frac{m}{\mu}) du \right] \quad (3.6)$$

From (3.6) we obtain the mean and second moment of the buffer content as

$$E(W) = \frac{2\lambda[1 - (1 + \frac{\alpha}{\mu})e^{-\frac{\alpha}{\mu}}]}{r_1 r_2}$$

and

$$E(W^2) = \frac{2\lambda[2(\lambda\alpha e^{-\frac{2\alpha}{\mu}} + \mu^2(1 - e^{-\frac{\alpha}{\mu}})) - e^{-\frac{\alpha}{\mu}}[\alpha r_1 + 2\mu r_2]]}{r_1^2 r_2 \mu^2}.$$

4 NUMERICAL ILLUSTRATIONS

In this section, the effect of system parameters on the buffer content and the mean buffer content are discussed numerically. The graphs of the tail distribution of the buffer content are plotted for chosen parametric values of the queueing system. Figures 1a and 1b display the tail distribution of the buffer content for the models with active periods of the fluid queue as the exponential service time of $M/E_1/1$ queue and the service time of $M/E_2/1$ queue. It is observed that in all the cases, the tail probabilities $P(W > x)$ decrease towards zero for increasing value of the buffer content x , for $\lambda = 1.2, \mu = 3, c_0 = 2$ and $c_1 = 1$, under the influence of exponential timer. Further, the tail probabilities decrease for increasing values of the timer parameter α . Figures 2a, 2b, and 2c depict the mean buffer content $E(W)$ against the exponential timer parameter α for $\mu = 3.7, c_0 = 2, c_1 = 1$ and for various values of λ . The associated curves in the graphs show that the mean buffer content $E(W)$ decreases as α increases. However, the decrease is less rapid for higher values of α and decreasing values of λ , before converging to the same value. Figure 3 indicates that, for $\lambda = 0.2, \mu = 1, c_0 = 2$ and $c_1 = 1$, the average buffer content decreases more rapidly for increasing values of the timer parameter α , for the fluid queue with single on-off source ($M/E_1/1$) followed by single on-off source models such as $M/E_2/1$ and $M/D/1$.

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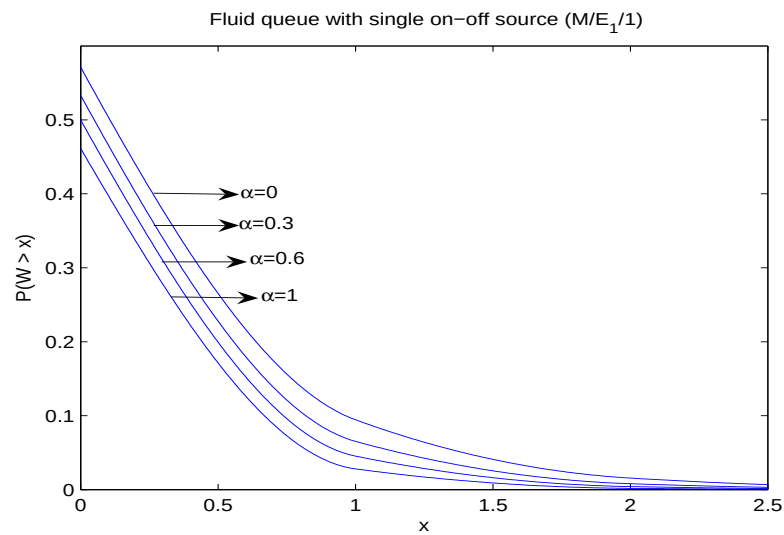


Figure 1a: $P(W > x)$ versus x for $\lambda = 1.2, \mu = 3, c_0 = 2$ and $c_1 = 1$

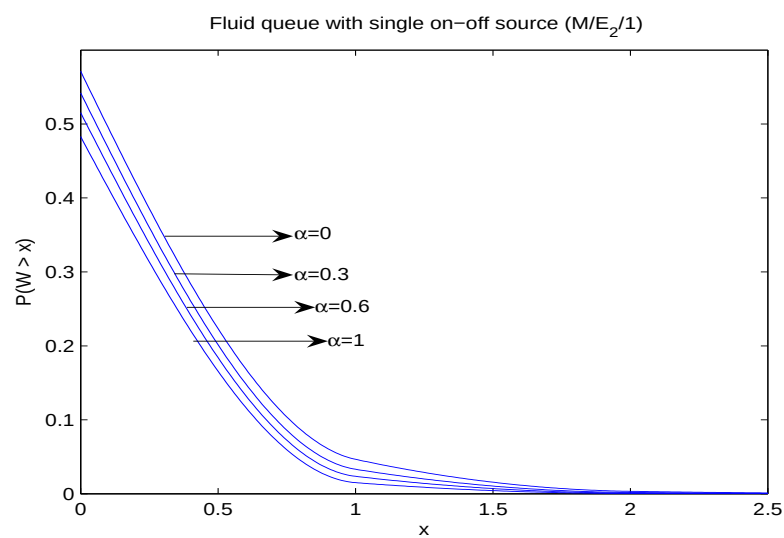


Figure 1b: $P(W > x)$ versus x for $\lambda = 1.2, \mu = 3, c_0 = 2$ and $c_1 = 1$

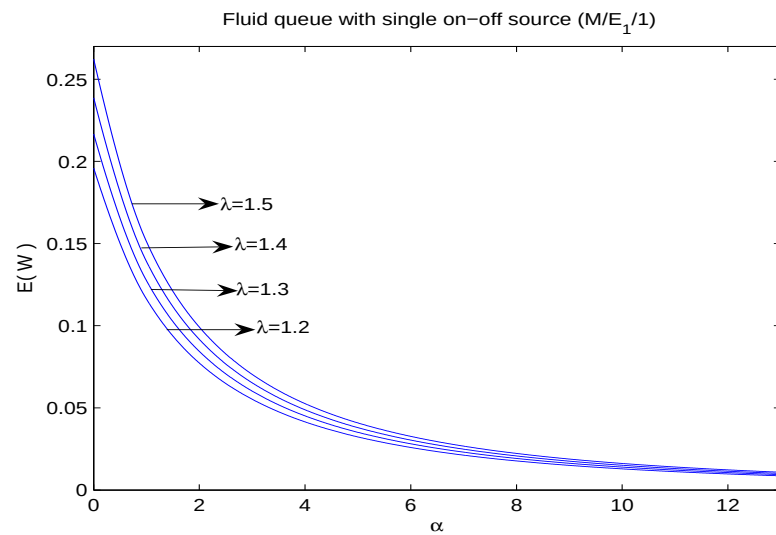


Figure 2a: $E(W)$ versus α for $\mu = 3.7$, $c_0 = 2$ and $c_1 = 1$

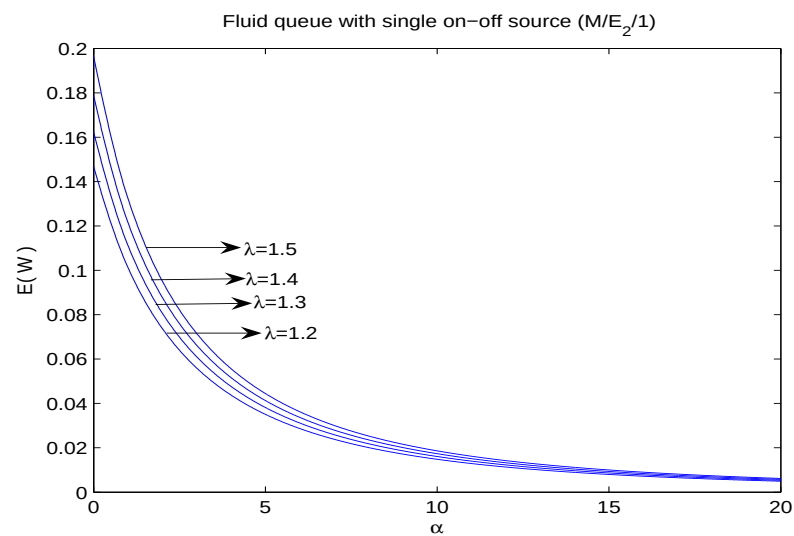


Figure 2b: $E(W)$ versus α for $\mu = 3.7$, $c_0 = 2$ and $c_1 = 1$

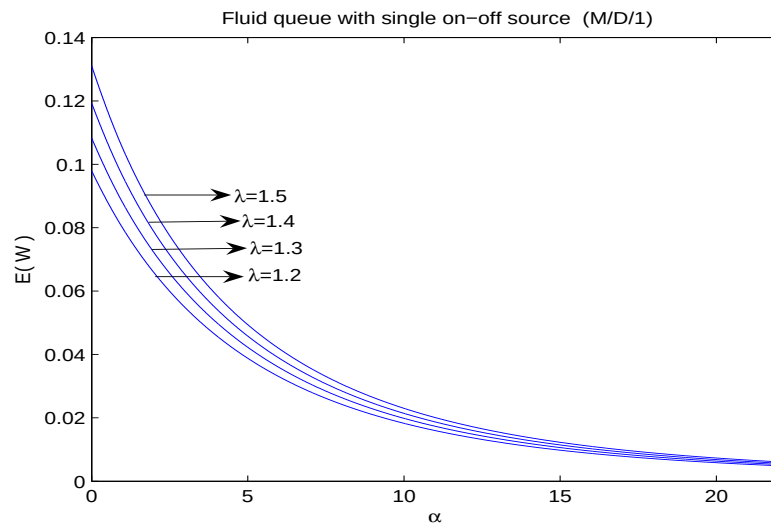


Figure 2c: $E(W)$ versus α for $\mu = 3.7$, $c_0 = 2$ and $c_1 = 1$

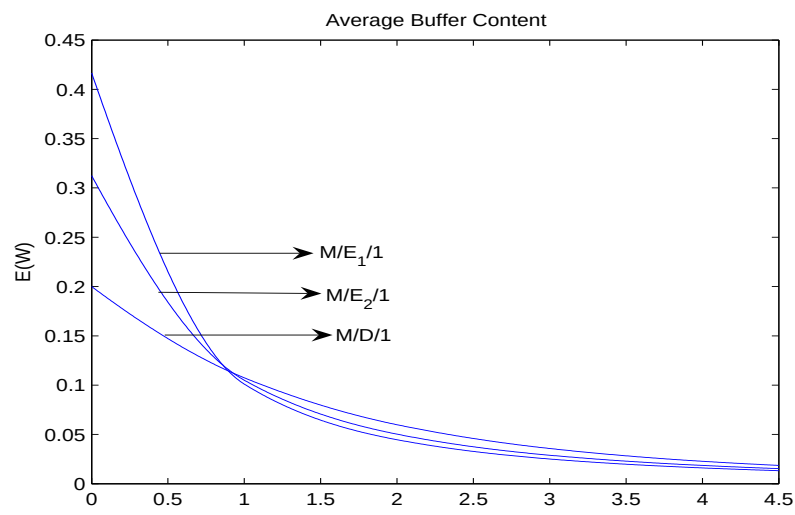


Figure 3: Comparison of $E(W)$ versus α for $\lambda = 0.2$, $\mu = 1$, $c_0 = 2$ and $c_1 = 1$

