



REGIONAL TRIANGULAR TILE REWRITING GRAMMARS

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Abstract: Triangular Tile Rewriting Grammars (TTRG) are the recent class of formal models for the generation of triangular picture languages. They combined the features of triangular tiling system (TTS) and array rewriting. Regional hexagonal tile rewriting grammars are the hexagonal array generating devices which employ replacement rules formalized by regional hexagonal tiling. In this work we introduce Regional Triangular Tile Rewriting Grammars (RTTRG) for the generation of triangular picture patterns. We compare this model with context free triangular array grammars, TTS and TTRG.

Key words: Triangular Picture language, regional Tiling, grammar, Triangular tiling system.

1. INTRODUCTION

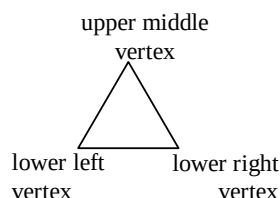
TRIANGULAR picture patterns occur in several application areas especially, in picture processing and image analysis [1], [4], [5]. To study the problem of triangular picture generation where pictures are considered as connected, finite arrays of symbols in a triangular grid, there has been sustained attention in adapting techniques of formal string language theory for developing new grammar models [4],[5]. Triangular tile rewriting grammars (TTRG) [5] are recently introduced grammar models on triangular picture generation. They combined the features of triangular tiling system (TTS) [5] and array rewriting grammars [2]. Regional hexagonal tile rewriting Grammars (RHTRG) [3] are the recent class

of grammars on hexagonal picture processing where the derivation rules involve a simple type of tiling named regional.

In the present work, we introduce Regional triangular tile rewriting grammars (RTTRG) as a restriction of TTRG with regional tiling. We study the generative power of RTTRG by comparing it with Context-free Triangular Array Grammars (CF-TAG), TTS and TTRG.

2. PRELIMINARIES

Let Σ be a finite alphabet of symbols. A triangular picture (array) p over Σ is an assembly of symbols taken from Σ in an equilateral triangular region of two-dimensional plane. A typical triangular picture is of the form



A sample triangular picture over $\Sigma = \{t\}$ is

$$\begin{array}{ccc} & t & \\ & t & t \end{array}.$$

Σ_T^{**} is the collection of all triangular pictures over the alphabet Σ . A triangular language L over Σ is hence any subset of Σ_T^{**} . The triad of axes  determines the coordinates of position of each element of a triangular picture [5].

Given $p \in \Sigma_T^{**}$, let $\hat{p}$ be the triangular array obtained by bordering p with a symbol $\#$ not in Σ .

For example,

$$\hat{p} = \begin{array}{ccccccc} & & \# & & & & \\ & & \# & & \# & & \\ \# & & \# & b & \# & & \\ & \# & c & c & \# & & \\ \# & \# & \# & \# & \# & & \end{array} \quad \text{for} \quad p = \begin{array}{ccc} & b & \\ c & & c \end{array}$$

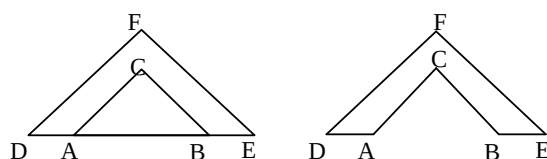
Given a triangular picture $p \in \Sigma_T^{**}$, each side of p has the same number of the elements as the shape of p is an equilateral triangle and thus size of the triangular picture is defined to be t , the number of elements in each side of p . For example

$$t = 3 \text{ for } p = \begin{array}{ccc} & a & \\ a & & a \\ a & a & a \end{array}$$

Given a triangular picture p of size t , for $2 \leq k \leq t$ we denote by $T_k(p)$, the set of all (equilateral) triangular subpictures of p of size k . Each member of $T_2(p)$ is called a triangular tile. We represent symbolically, the set of all triangular tiles contained in a picture p by $[[p]]$.

Definition 1.

Consider a triangle ABC . Extend A to D , B to E and C to F to get a triangle DEF with $|DA|=|BE|=2|CF|$ and $|DF|=|EF|=|DE|$.



ABC is the innermost boundary and DEF is the outermost boundary. Now remove the side AB , we get the upper cap (U) with vertex at F . Similarly we get cap for other sides. Depending on the direction of a vertex, a cap is classified as right cap (R), left cap (L) and upper cap (U) as in Figure.1 and the corresponding cap catenations [5] are denoted as $\rightarrow$, $\leftarrow$, $\uparrow$

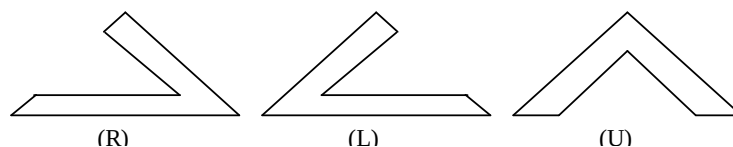


Figure.1 Types of Caps

Definition 2.

The triangular picture $q \in \Gamma_T^{**}$ of size t is called a projection of $p \in \Sigma_T^{**}$ of same size if there exist an injection $\mu: \Sigma \rightarrow \Gamma$ such that $q(i, j, k) = \mu(p(i, j, k))$ be a triangular picture language. The projection by mapping π of language $L \subseteq \Sigma_T^{**}$ is

$$M = \{q / q = \mu(p), \forall p \in L\} \subseteq \Gamma_T^{**}.$$

Definition 3.

A triangular picture language $L \subseteq \Sigma_T^{**}$ is called local if there exists a finite set Ω of triangular tiles over $\Sigma \cup \{\#\}$ such that $L = \{p \in \Sigma_T^{**} / T_2(\hat{p}) \subseteq \Omega\}$. If $M \subseteq \Gamma_T^{**}$ is a

projection of a local language L over Σ then M is known as a recognizable triangular language. The family of local (recognizable) triangular picture languages is symbolized as T_{LOC} (T_{REC}).

Definition 4.

A triangular tiling system (TTS) T is a 4-tuple $(\Sigma, \Gamma, \mu, \Omega)$, where Σ and Γ are two finite alphabets, $\mu: \Gamma \rightarrow \Sigma$ is a projection and Ω is a set of triangular tiles over $\Gamma \cup \{\#\}$. The class of all triangular picture languages recognizable by triangular tiling system is denoted by $\chi(TTS)$.

Definition 5.

A regular triangular array grammar (R-TAG)[5] is $G = (V, I, C, R, S, L)$ where V, I and C are finite non-empty sets of symbols known as variables, intermediates and constants respectively. $S \in V$ is the start symbol. $L = \{L_B / B \in I\}$ where L_A is an intermediate regular language of caps (refer definition.1). $R = R_1 \cup R_2$ is a finite non-empty set of productions where R_1 consists of starting rules of the forms:

$$(1) S \rightarrow T \begin{array}{c} \leftarrow \\ \rightarrow \end{array} U$$

$$(2) S \rightarrow T \begin{array}{c} \rightarrow \\ \leftarrow \end{array} U$$

$$(3) S \rightarrow T \begin{array}{c} \uparrow \\ \downarrow \end{array} U$$

where $U \in V$, $U \neq S$ and T is a triangular array over C . The rules of R_2 are of the forms:

$$(1) U_1 \rightarrow B \begin{array}{c} \leftarrow \\ \rightarrow \end{array} U_2 \quad (2) U_1 \rightarrow B \begin{array}{c} \rightarrow \\ \leftarrow \end{array} U_2$$

$$(3) U_1 \rightarrow B \begin{array}{c} \uparrow \\ \downarrow \end{array} U_2$$

where $U_1, U_2 \in V$, $U_1, U_2 \neq S$ and $B \in I$.

G is called CF-TAG (CS-TAG) if one of the intermediate languages in the above definition is context-free (context-sensitive).

3. REGIONAL TRIANGULAR TILE REWRITING GRAMMAR

In this segment, we introduce a triangular tile rewriting grammar with a specific set of tiling known as regional tiling. The following notions are similar to that of [3] for the instance of triangular pictures and the term picture represents a (equilateral) triangular array.

Definition 6.

An entry $p(i, j, k)$ of a picture p at the location (i, j, k) is a symbol of p present in that location. If all the entries of a picture p are indistinguishable to $r \in \Sigma$, then p is called r -uniform or r -picture. The field of a picture p of size t is the set $f(p)$ consists of all locations of entries in p . A subfield $f_s(p)$ of $f(p)$ is a collection of locations of entries in p characterizing a subpicture q of size t' if $1 \leq t' \leq t$. If a subfield is r -uniform then it is called the label of the subfield.

Definition 7.

A uniform division of a picture p is any separation $p = \{f_{s1}, f_{s2}, f_{s3}, \dots, f_{sn}\}$ of $f(p)$ into uniform subfields $f_{s1}, f_{s2}, \dots, f_{sn}$. The atomic separation of p , $\text{atom}(p)$, is the uniform division of $f(p)$ described by single entries. A uniform division is called concrete if adjacent subfields composed of different labels. The unique separations given by concrete uniform division of a picture p is $\alpha^*(p)$.

Definition 8.

A Triangular tile rewriting grammar, TTRG is a 4-tuple (Σ, V, S, P) , where Σ is the terminal alphabet, V is a set of variable symbols. $S \in V$ is the initial symbol. P is a set of rules of the following forms:

Even : $B \rightarrow c$, where $B \in V, c \in \Sigma$.

Uneven : $B \rightarrow v$, where $B \in V$ and v is a finite set of triangular tiles over $V \cup \{ \# \}$ and $T_{\text{LOC}}(v)$ allows a concrete uniform division for any p in it.

Definition 9.

A uniform division of a picture p is regional division if distinct subfields of p have different labels. A triangular picture language is regional if all its pictures are regional.

Definition 10.

A regional TTRG (RTTRG) is a 4-tuple (Σ, V, S, P) where Σ is the alphabet constant symbols, V is the set of variable symbols. $S \in V$ is the initials symbol and P is the set of productions of the kinds:

Even : $B \rightarrow c$, where $A \in V, c \in \Sigma$

Uneven : $B \rightarrow v$, where $B \in V$ and v is a finite set of triangular tiles over $V \cup \{ \# \}$ and $T_{\text{LOC}}(v)$ allows a regional division for any p in it

Picture derivation :

Consider a RTTRG, $G = (\Sigma, V, S, P)$. Let $p, q \in (\Sigma \cup V)^k$ be pictures of identical size (where k denotes size of array). Let α, α' be the uniform divisions of $f(p)$ with $\alpha = \{f_{s1}, f_{s2}, \dots, f_{sn}\}$. We say that (q, α') is derived in one step from (p, α) , written as $(p, \alpha) \xRightarrow{G} (q, \alpha')$ iff for some $B \in V$ and for some production rule $r \in P$ with left

part B there exists a subfield f_s in α called submission region such that

- (i) q is derived by replacing the sub picture x at f_s in p with a picture y of same size defined as follows:
 - a) If r is Even, then $y = c$
 - b) If r is of Uneven, then $y \in T_{\text{LOC}}(v)$.
- (ii) α' is a uniform division of $f(p)$ into the subfields $(\alpha \setminus \{f_s\}) \cup \text{dispf}_s \alpha^*(y)$ where $\text{dispf}_s \alpha^*(y)$ denotes the displacement of concrete uniform division of y to the position of f_s .

Definition 11.

The triangular picture language $L(G)$ defined by regional triangular tile rewriting grammar (RTTRG) G is a set of $p \in \Sigma_T^{**}$ such that $\langle s^{[p]}, f(p) \rangle \xRightarrow[G]{*} (p, \text{atom}(p))$ where $\xRightarrow[G]{*}$ is the transitive closure of $\xRightarrow[G]$. The family of triangular languages generated by RTTRG is denoted by $\chi(\text{RTTRG})$.

Example. 1

The language of triangular pictures whose elements along the circumradial lines are a's and other elements are b's is generated by the RTTRG grammar $G = (\Sigma, V, S, P)$ where $\Sigma = \{a, b\}$, $N = \{A, B, C, A', A'', B', B'', C', C'', V_1, V_2, \dots, V_6\}$ and P consists of the following production rules:

$$\begin{aligned}
 S \rightarrow & \left[\begin{array}{cccccccc} & & & \# & & & & \\ & & & \# & \# & & & \\ & & \# & B & \# & & & \\ & \# & B & B & \# & & & \\ & \# & B & S & B & \# & & \\ \# & A & S & S & C & \# & & \\ \# & A & S & S & S & C & \# & \\ \# & A & S & S & S & S & C & \# \\ \# & A & A & A & A & C & C & C & \# \\ \# & \# & \# & \# & \# & \# & \# & \# & \# \end{array} \right] / \\
 & \left[\begin{array}{cccccccc} & & & \# & & & & \\ & & & \# & \# & & & \\ & & \# & B & \# & & & \\ & \# & B & B & \# & & & \\ \# & A & S & S & C & \# & & \\ \# & A & S & S & C & \# & & \\ \# & A & A & A & C & C & \# & \\ \# & \# & \# & \# & \# & \# & \# & \# \end{array} \right] / \\
 & \left[\begin{array}{cccccccc} & & & \# & & & & \\ & & \# & & \# & & & \\ & \# & V_1 & & \# & & & \\ \# & V_2 & & V_3 & & \# & & \\ \# & V_4 & & V_5 & & V_6 & & \# \\ \# & \# & \# & \# & \# & \# & \# & \# \end{array} \right] / \left[\begin{array}{cccccccc} & & & \# & & & & \\ & & \# & & \# & & & \\ & \# & V_1 & & \# & & & \\ \# & V_2 & & V_3 & & \# & & \\ \# & \# & \# & \# & \# & \# & \# & \# \end{array} \right] \\
 A \rightarrow & \left[\begin{array}{cccccccc} & & & \# & & & & \\ & & \# & & \# & & & \\ & \# & A' & & \# & & & \\ \# & A & & \# & & \# & & \\ \# & A & \# & \# & \# & \# & & \\ \# & A & A & A & A & A'' & \# & \\ \# & \# & \# & \# & \# & \# & \# & \# \end{array} \right],
 \end{aligned}$$

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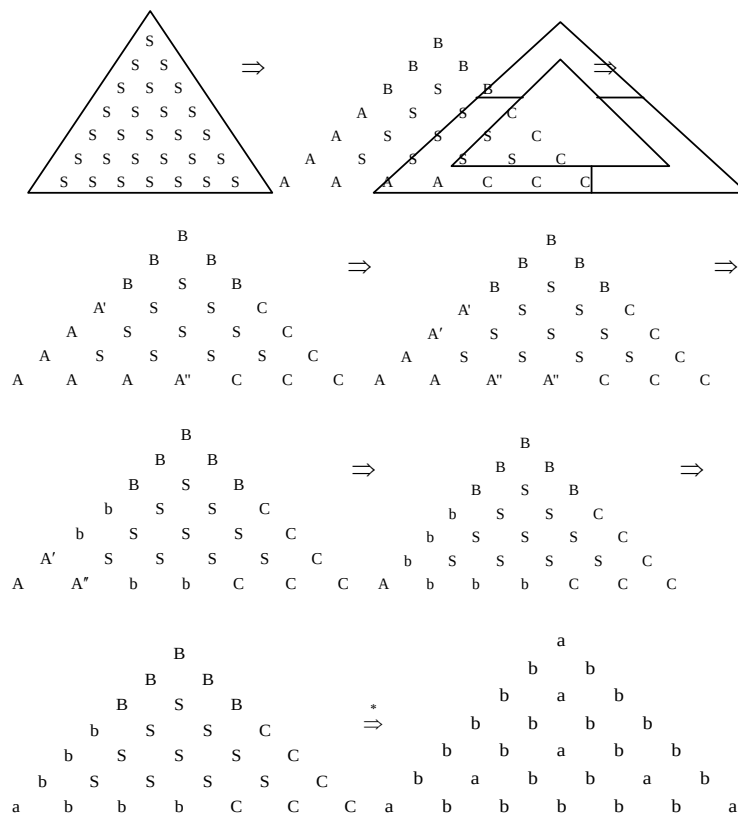
$$B \rightarrow \left[\begin{array}{ccccccc} & & & \# & & & \\ & & & \# & \# & & \\ & & \# & B & \# & & \\ & \# & B & B & \# & & \\ \# & B & \# & B & \# & & \\ \# & B' & \# & \# & B'' & \# & \\ \# & \# & \# & \# & \# & \# & \end{array} \right] C \rightarrow \left[\begin{array}{ccccccc} & & & \# & & & \\ & & & \# & \# & & \\ & & \# & C' & \# & & \\ & \# & C & \# & \# & & \\ \# & \# & \# & \# & C & \# & \\ \# & C'' & C & C & C & \# & \\ \# & \# & \# & \# & \# & \# & \end{array} \right]$$

$A, B, C, V_1, V_2, V_3, V_4, V_5, V_6 \rightarrow a$

$A', A'', B', B'', C', C'' \rightarrow b$

This grammar is applicable to only pictures of odd size.

The derivations of the picture is given in the following sequel.



The language $L =$

[illegible]

$$\Sigma = \{+, -\}, V = \{S, A, A', B, B', C, C', D, D', D'', E, E', E''\}$$

Figure 1 consists of three triangular lattice diagrams, each representing a different system: S, A, and B. Each diagram is a triangle of nodes, with arrows indicating transitions between states. The top diagram (S) shows a transition from state A to state #. The middle diagram (A) shows a transition from state B to state #. The bottom diagram (B) shows a transition from state E to state #. The diagrams are arranged vertically, with arrows pointing from the top diagram to the middle, and from the middle to the bottom.

$C', C'', D, D', D'' \rightarrow -$.

4.COMPARISON RESULTS

Theorem 1.

Proof :

Consider CF-TAG, G in chomsky normal form. Then the right part of the rules of G may involve three kinds of forms XY , $\bar{X}Y$ and $X\bar{Y}$. Then the production rules of G , $Z \rightarrow XY$ are comparable to RTTRG rules of the form.

$Z \rightarrow XY$ are comparable to RTTRG rules of the form

$Z \rightarrow XY$ are corresponding to RTTRG rules of the form

This production rules $Z \rightarrow c$, $c \in \Sigma$ are identical in both the grammars. The language of the Example2 cannot be generated by any CFTAG.

Suppose there exists a CFTAG grammar $G = (V, I, C, R, S, L_{A_1})$ where $V = \{S, S_1\}$, $I = \{A_1\}$, $C = \{1, 2\}$, $R = R_1 \cup R_2$ where $R_1 = \{S \rightarrow T \uparrow S_1\}$ and $P_2 = \{S_1 \rightarrow A_1 \uparrow S_2, S_1 \rightarrow A_1\}$ with $T = \begin{Bmatrix} a & \\ a & a \end{Bmatrix}$ and L_{A_1} is an intermediate language corresponding to A_1 .

The other elements of the language of CF-TAG are constructed through the upper cap catenation of first triangular array and hence first element of the language must lie under the upper cap of the second element of the language and so on but not yielding the pictures of Example 2. Similarly the proof can be made if we use the left and right cap catenation in the grammar.

The family of Triangular Tiling System languages and the family of RTTRG languages are incomparable.

First we prove that both the families of languages are not disjoint.

The language $L_1 = \left\{ \begin{array}{ccccccc} & & & & b & & \\ & & & & b & b & \\ b & & & & b & b & \\ & b & & & b & b & \\ & & b & b & b & b & \end{array} , \dots \right\}$ is present in both the families.

If we consider the triangular tile set

$$\Delta = \left[\begin{array}{ccccccc} & & & & \# & & \\ & & & & \# & \# & \\ & & & & \# & 2 & \# \\ & & \# & 2 & 2 & \# & \\ & \# & 2 & 1 & 2 & \# & \\ \# & 2 & 1 & 1 & 2 & \# & \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right] \text{ then}$$

$$L_2 = \left\{ \begin{array}{ccccccc} & & & & 2 & & \\ & & & & 1 & 2 & 2 \\ 1 & & & & 2 & 1 & 2 \\ & 1 & 1' & & 2 & 1 & 2 \\ & & 2 & 1 & 1 & 2 & \end{array} , \dots \right\} = L(\Delta)$$

This implies $L_2 \in T_{\text{LOC}}$. If we use the projection μ as $\mu(1) = \mu(2) = b$. Then we get $L_1 = \mu(L_2)$. Thus the tiling system $(\Sigma, \Gamma, \mu, \Delta)$ where $\Sigma = \{b\}$, $\Gamma = \{1, 2\}$ accepts the language L_1 . Hence L_1 is TTS recognizable.

Consider the RTTRG grammar, $G = (\Sigma, V, S, P)$, where $\Sigma = \{b\}$, $V = \{S, X, X', Y, Y'\}$

$$S \rightarrow \left[\begin{array}{ccccccc} & & & & \# & & \\ & & & & \# & \# & \\ & & & & \# & X & \# \\ & & \# & X & X & \# & \\ \# & X & Y & X & \# & & \\ \# & X & Y & Y & X & \# & \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right] \quad X \rightarrow \left[\begin{array}{ccccccc} & & & & \# & & \\ & & & & \# & \# & \\ & & & & \# & X & \# \\ & & \# & X & X & \# & \\ \# & X & \# & X & \# & & \\ \# & X' & \# & \# & X'' & \# & \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right]$$

$$Y \rightarrow \left[\begin{array}{ccccccc} & & & & \# & & \\ & & & & \# & \# & \\ & & & & \# & Y & \# \\ & & \# & Y & Y & \# & \\ \# & Y & \# & Y & \# & & \\ \# & Y' & \# & \# & Y'' & \# & \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right], \quad X, X', Y, Y' \rightarrow b$$

Clearly this RTTRG grammar generates all elements of L_1 .

Thus $\chi(\text{RTTRG}) \cap \chi(\text{TTS}) \neq \emptyset$

The set of all L-shaped palindromic left caps over the alphabet $\{a, b\}$ is generated by RTTRG, having the rules.

$$\begin{array}{l}
 S \rightarrow \left[\begin{array}{ccccccc} & & & \# & & & \\ & & \# & & \# & & \\ & \# & X' & \# & & & \\ & \# & X & \# & & & \\ \# & X & \# & \# & \# & & \\ \# & X & X & X & X & X'' & \# \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right] & X \rightarrow \left[\begin{array}{ccccccc} & & & \# & & & \\ & & \# & & \# & & \\ & \# & R & \# & & & \\ & \# & R & \# & & & \\ \# & R & \# & \# & \# & & \\ \# & R & R & R & R & R & \# \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right] \\
 R \rightarrow \left[\begin{array}{ccccccc} & & & \# & & & \\ & & \# & & \# & & \\ & \# & A' & \# & & & \\ & \# & R & \# & & & \\ \# & R & \# & \# & \# & & \\ \# & R & R & R & A'' & \# & \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right] & / & \left[\begin{array}{ccccccc} & & & \# & & & \\ & & \# & & \# & & \\ & \# & B' & \# & & & \\ & \# & R & \# & & & \\ \# & R & \# & \# & \# & & \\ \# & R & R & R & B'' & \# & \\ \# & \# & \# & \# & \# & \# & \# \end{array} \right]
 \end{array}$$

$A', A'' \rightarrow a, B', B'' \rightarrow b, R \rightarrow a, b$

This language is clearly not generated by any TTS as there is no native scheme available for the corresponding palindromic alphabet locations.

Theorem 3.

$$\chi(\text{RTTRG}) \subsetneq \chi(\text{TTRG})$$

Proof:

Since RTTRG grammar rules are inherited from TTRG rules with regional restriction, every RTTRG rule is a rule in TTRG also. Therefore $\chi(\text{RTTRG}) \subseteq \chi(\text{TTRG})$. But $\chi(\text{RTTRG}) \neq \chi(\text{TTRG})$ as TTRG can generate triangular pictures of odd or even size comprised of elements along the circumradial lines as a's and other elements as b's but RTTRG cannot derive it.

5.CONCLUSION

Regional Triangular Tile Rewriting Grammars are the variant form of triangular tiling based array rewriting models. They are proved to be having higher generative capacity than Context-free Triangular Array Grammar but weaker than Triangular Tile Rewriting Grammar. We can further explore some closure and decidability properties of this model and also the practical applications.

REFERENCES

- [1] Milan Sonka and J. Michael Fitzpatrick, *Hand book of Medical Imaging*, Vol.2., Medical image processing and analysis, SPIE press (2000)
- [2] G. Siromoney, R. Siromoney, and K. Krithivasan. Picture languages with array rewriting rules. *Information and Control*, 23(5) (1973), pp. 447–470.
- [3] T. Kamaraj and D.G. Thomas, Regional hexagonal tile rewriting grammars, In : R.P. Barneva et al (eds.), *IWCIA 2012, LNCS*, Vol. 7655, Springer, Heidelberg, (2012) pp. 181–195.
- [4] S. Kuberal, T. Kalyani, D.G. Thomas, and T. Kamaraj, Petri net generating Triangular arrays, *Procedia Computer science*, Elsevier, Vol 57 (2015), pp. 642–649.
- [5] S. Kuberal, T. Kalyani, D.G. Thomas, Triangular Tile Rewriting Grammars and Triangular Picture Languages, *Global Journal of Pure and Applied Mathematics* 12(3) (2016), pp. 1965–1978

