



On Intuitionistic Preopen Sets

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Abstract

The main focus of this paper is to introduce the concept of preopen set and preclosed sets in intuitionistic topological spaces. We studied and established more relations between the pre interior and pre closure in the intuitionistic topological concepts.

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1. Introduction

Intuitionistic fuzzy set was first introduced by K.T. Atanasav [1] in 1983. After that he introduced in [2] and [3], the concept of intuitionistic sets as generalisation of fuzzy sets. D. Coker [4] introduced generalised topological structures in fuzzy topological spaces using intuitionistic fuzzy sets. The concept of intuitionistic fuzzy sets, intuitionistic fuzzy topological spaces, intuitionistic topological spaces and intuitionistic fuzzy points were introduced by D. Coker [5,7,8,10]. In this paper, we focus on the main concept of intuitionistic preopen sets in an intuitionistic topological spaces where intuitionistic semiopen sets are introduced in [11] and intuitionistic α -open sets are introduced in [12].

2. Preliminaries

The following definitions and lemmas are essential for the development of the concepts.

Definition 2.1. [4] Let X be a nonempty fixed set. An *intuitionistic set* (IS in short) $\tilde{A}$ is an object having the form $\tilde{A} = \langle X, A^1, A^2 \rangle$ where A^1 and A^2 are subsets of X such that $A^1 \cap A^2 = \emptyset$. The set A^1 is called the set of *member* of $\tilde{A}$, while A^2 is called the set of *non member* of $\tilde{A}$. Every subset A of a nonempty set X is obviously an IS having the form $\langle X, A, A^c \rangle$. Several relations and operations between IS's are given below.

Definition 2.2. [4] Let X be a non empty set, $\tilde{A} = \langle X, A^1, A^2 \rangle$ and $\tilde{B} = \langle X, B^1, B^2 \rangle$ be an IS sets and let $\{\tilde{A}_i : i \in J\}$ be an arbitrary family of IS, where $\tilde{A}_i = \langle X, A_i^1, A_i^2 \rangle$. Then the following hold.

- (a) $\tilde{A} \subseteq \tilde{B}$ if and only if $A^1 \subseteq B^1$ and $B^2 \subseteq A^2$.
- (b) $\tilde{A} = \tilde{B}$ if and only if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$.
- (c) $\tilde{A} \subset \tilde{B}$ if and only if $A^1 \cup A^2 \supsetneq B^1 \cup B^2$.
- (d) $\tilde{\bar{A}} = \langle X, A^2, A^1 \rangle$ is called the complement of $\tilde{A}$. It is also denoted by $\tilde{X} - \tilde{A}$.
- (e) $\cup \tilde{A}_i = \langle X, \cup A_i^1, \cap A_i^2 \rangle$.
- (f) $\cap \tilde{A}_i = \langle X, \cap A_i^1, \cup A_i^2 \rangle$.
- (g) $\tilde{A} - \tilde{B} = \tilde{A} \cap \tilde{\bar{B}}$.
- (h) $[] \tilde{A} = \langle X, A^1, (A^1)^c \rangle$.
- (i) $\langle \rangle \tilde{A} = \langle X, (A^2)^c, A^2 \rangle$.
- (j) $\tilde{\emptyset} = \langle X, \emptyset, X \rangle$ and $\tilde{X} = \langle X, X, \emptyset \rangle$. Clearly for every $\tilde{A} = \langle X, A^1, A^2 \rangle$, $\tilde{A} \subset \tilde{X}$.

Definition 2.3. [4] An *intuitionistic topology* (IT in short) by subsets of a nonempty set X is a family τ of IS's satisfying the following axioms.

- (a) $\tilde{\emptyset}, \tilde{X} \in \tau$,
- (b) $\tilde{G}_1 \cap \tilde{G}_2 \in \tau$ for every $\tilde{G}_1, \tilde{G}_2 \in \tau$, and
- (c) $\cup \tilde{G}_i \in \tau$ for any arbitrary family $\{\tilde{G}_i : i \in J\} \subseteq \tau$.

The pair (X, τ) is called an *intuitionistic topological space* (ITS in short) and any IS G in τ is called an *intuitionistic open set* (IOS) in $\tilde{X}$. The complement $\tilde{\bar{A}}$ of an IO set $\tilde{A}$ in

an ITS (X, τ) is called an *intuitionistic closed set* (ICS).

Definition 2.4. [4] Let (X, τ) be an ITS and $\tilde{A} = \langle X, A^1, A^2 \rangle$ be an IS in X . Then the interior and the closure of $\tilde{A}$ are denoted by $\text{int}(\tilde{A})$ and $\text{cl}(\tilde{A})$, and are defined as follows.

$$\text{cl}(\tilde{A}) = \cap \{ \tilde{K} \mid \tilde{K} \text{ is an ICS and } \tilde{A} \subseteq \tilde{K} \} \text{ and}$$

$$\text{int}(\tilde{A}) = \cup \{ \tilde{G} \mid \tilde{G} \text{ is an IOS and } \tilde{G} \subseteq \tilde{A} \}.$$

Also, it can be established that $\text{cl}(\tilde{A})$ is an ICS and $\text{int}(\tilde{A})$ is an IOS. $\tilde{A}$ is an ICS if and only if $\text{cl}(\tilde{A}) = \tilde{A}$ and $\tilde{A}$ is an IOS if and only if $\text{int}(\tilde{A}) = \tilde{A}$. We say that $\tilde{A}$ is I-dense if $\text{cl}(\tilde{A}) = \tilde{X}$.

Lemma 2.5. [4] Let (X, τ) be an ITS and let $\tilde{A}$ and $\tilde{B}$ be an intuitionistic subset of $\tilde{X}$. Then the following hold.

- (i) $\text{cl}(\tilde{\emptyset}) = \tilde{\emptyset}$ and $\text{cl}(\tilde{X}) = \tilde{X}$.
- (ii) $\tilde{A}$ is an IC set if and only if $\tilde{A} = \text{cl}(\tilde{A})$.
- (iii) $\text{cl}(\text{cl}(\tilde{A})) = \text{cl}(\tilde{A})$.
- (iv) $\tilde{A} \subseteq \tilde{B}$ implies that $\text{cl}(\tilde{A}) \subseteq \text{cl}(\tilde{B})$.
- (v) $\text{cl}(\tilde{A} \cap \tilde{B}) \subseteq \text{cl}(\tilde{A}) \cap \text{cl}(\tilde{B})$.
- (vi) $\text{cl}(\tilde{A} \cup \tilde{B}) = \text{cl}(\tilde{A}) \cup \text{cl}(\tilde{B})$.

Lemma 2.6. [4] Let (X, τ) be an ITS and let $\tilde{A}$ and $\tilde{B}$ be intuitionistic sets. Then the following hold.

- (i) $\text{int}(\tilde{\emptyset}) = \tilde{\emptyset}$ and $\text{int}(\tilde{X}) = \tilde{X}$.
- (ii) $\tilde{A}$ is an IO set if and only if $\tilde{A} = \text{int}(\tilde{A})$.
- (iii) $\text{int}(\text{int}(\tilde{A})) = \text{int}(\tilde{A})$.
- (iv) $\tilde{A} \subseteq \tilde{B}$ implies that $\text{int}(\tilde{A}) \subseteq \text{int}(\tilde{B})$.
- (v) $\text{int}(\tilde{A} \cap \tilde{B}) = \text{int}(\tilde{A}) \cap \text{int}(\tilde{B})$.
- (vi) $\text{int}(\tilde{A} \cup \tilde{B}) \supseteq \text{int}(\tilde{A}) \cup \text{int}(\tilde{B})$.

Lemma 2.7. [12] Let (X, τ) be an ITS, and $\tilde{A}$ and $\tilde{B}$ be two IS in (X, τ) . If $\tilde{B}$ is an IO set, then $\text{cl}(\tilde{A}) \cap \tilde{B} \subseteq \text{cl}(\tilde{A} \cap \tilde{B})$. So if $\tilde{G}$ is IC set and $\tilde{H}$ is any IS, then $\text{int}(\tilde{H} \cup \tilde{G}) \subseteq \text{int}(\tilde{H}) \cup \tilde{G}$.

Lemma 2.8. [4] [Proposition 3.15] For any IS $\tilde{A}$ in (X, τ) , We have $\tilde{X} - \text{int}(\tilde{A}) = \text{cl}(\tilde{X} - \tilde{A})$ and $\tilde{X} - \text{cl}(\tilde{A}) = \text{int}(\tilde{X} - \tilde{A})$.

Definition 2.9. [11] Let (X, τ) be an ITS and $\tilde{A} = \langle X, A^1, A^2 \rangle$. $\tilde{A}$ is an *intuitionistic semiopen set* (in short, ISO set) in an intuitionistic topological space (X, τ) if there is an intuitionistic open set (IO set) $\tilde{G} \neq \langle X, \emptyset, X \rangle$ such that $\tilde{G} \subset \tilde{A} \subset \text{cl}(\tilde{G})$. Clearly, every IO set is an ISO set and $\tilde{\emptyset}$ and $\tilde{X}$ are ISO sets. Also, from the definition, it follows that the closure of every IO set is an intuitionistic semiopen set. The complement of every intuitionistic semiopen is said to be an *intuitionistic semiclosed* (ISC) set.

The semiinterior and the semiclosure of an IS, $\tilde{A}$ are denoted by $\text{Isint}(\tilde{A})$ and $\text{Iscl}(\tilde{A})$

and are defined as follows.

$$Iscl(\tilde{A}) = \cap \{\tilde{K} \mid \tilde{K} \text{ is an ISC and } \tilde{A} \subseteq \tilde{K}\} \text{ and}$$

$Isint(\tilde{A}) = \cup \{\tilde{G} \mid \tilde{G} \text{ is an ISO and } \tilde{G} \subseteq \tilde{A}\}$. It can be established that $Iscl(\tilde{A})$ is the smallest ISC set contained in all ISC sets containing $\tilde{A}$ and $Isint(\tilde{A})$ is the largest ISO set contained in $\tilde{A}$. $\tilde{A}$ is an ISC set if and only if $Iscl(\tilde{A}) = \tilde{A}$ and $\tilde{A}$ is an ISO set if and only if $Isint(\tilde{A}) = \tilde{A}$.

Definition 2.10. [12] Let (X, τ) be an ITS and $\tilde{A}$ be an IS. $\tilde{A}$ is said to be an intuitionistic α – open set (in short, $I\alpha O$ set) if $\tilde{A} \subseteq int(cl(int(\tilde{A})))$. The family of all intuitionistic α – open set is denoted by $I\alpha O$. The complement of an $I\alpha O$ set is called an Intuitionistic α – closed set (in short, $I\alpha C$ set).

Definition 2.11. [6] Let X be a nonempty set and $p \in X$, a fixed element. Then the IS $\tilde{p}$ defined by $\tilde{p} = \langle X, \{p\}, \{p\}^c \rangle$ is called an intuitionistic point (in short, IP).

3. Intuitionistic Preopen Sets

In this section, we defined intuitionistic preopen sets in an intuitionistic topological space (X, τ) and studied its characterizations and properties.

Definition 3.1. Let (X, τ) be an intuitionistic topological space and $\tilde{A}$ be an intuitionistic set. $\tilde{A}$ is said to be intuitionistic preopen if $\tilde{A} \subset int(cl(\tilde{A}))$.

Example 3.2. Let $\tilde{X} = \{1, 2, 3, 4, 5\}$ and consider the family $\tau = \{\tilde{\emptyset}, \tilde{X}, A_1, A_2, A_3, A_4\}$ where $A_1 = \langle X, \{1, 2, 3\}, \{4\} \rangle$, $A_2 = \langle X, \{3, 4\}, \{5\} \rangle$, $A_3 = \langle X, \{3\}, \{4, 5\} \rangle$, $A_4 = \langle X, \{1, 2, 3, 4\}, \{5\} \rangle$, $\tilde{X} = \langle X, X, \emptyset \rangle$ and $\tilde{\emptyset} = \langle X, \emptyset, X \rangle$. Let $\tilde{B} = \langle X, \{1, 2, 3, 4\}, \{5\} \rangle$. Then $cl(\tilde{B}) = \tilde{X}$ and $int(\tilde{X}) = \tilde{X}$. So $\tilde{B} \subset int(cl(\tilde{B}))$. Therefore, $\tilde{B}$ is an intuitionistic preopen set in (X, τ) . If $\tilde{C} = \langle X, \{5\}, \{2, 3, 4\} \rangle$, $cl(\tilde{C}) = \langle X, \{5\}, \{3, 4\} \rangle$, and $int(cl(\tilde{C})) = \tilde{\emptyset}$. Therefore $\tilde{C}$ is not an intuitionistic preopen set in (X, τ) .

The following Theorem 3.3 shows that the arbitrary union of intuitionistic preopen sets is an intuitionistic preopen set. Example 3.5 below shows that the intersection of two intuitionistic preopen sets is not an intuitionistic preopen set but if one of the set is an IO set, then the intersection of two intuitionistic preopen sets is an intuitionistic preopen set.

Theorem 3.3. Let (X, τ) be an ITS and $\{\tilde{A}_\alpha : \alpha \in I\}$ be a family of IPO sets in (X, τ) . Then $\cup \{\tilde{A}_\alpha \mid \alpha \in I\}$ is an IPO set.

Proof. Let $\{\tilde{A}_\alpha : \alpha \in I\}$ be a family of IPO sets in (X, τ) . Then for each α , $\tilde{A}_\alpha \subset int(cl(\tilde{A}_\alpha))$. Since $\tilde{A}_\alpha \subset \cup \tilde{A}_\alpha$, $int(cl(\tilde{A}_\alpha)) \subset int(cl(\cup \tilde{A}_\alpha))$ and also $\tilde{A}_\alpha \subset int(cl(\cup \tilde{A}_\alpha))$. Hence $\cup \tilde{A}_\alpha \subset int(cl(\cup \tilde{A}_\alpha))$ which shows that $\cup \tilde{A}_\alpha$ is an IPO set.

Definition 3.4. Let (X, τ) be an ITS. If $\tilde{A}$ and $\tilde{B}$ are any two IPO sets in (X, τ) , one of which is an IO set, then $\tilde{A} \cap \tilde{B}$ is an IPO set.

Proof. Let $\tilde{A}$ and $\tilde{B}$ be two Intuitionistic sets in (X, τ) . Suppose $\tilde{A}$ is an IPO set and $\tilde{B}$ is an IO set. Then $\tilde{A} \subseteq int(cl(\tilde{A}))$. Now $\tilde{A} \cap \tilde{B} \subset int(cl(\tilde{A}) \cap \tilde{B})$.

$$= int(cl(\tilde{A})) \cap int(\tilde{B}).$$

$$= int(cl(\tilde{A}) \cap \tilde{B}).$$

$$\subset int(cl(\tilde{A} \cap \tilde{B})) \text{ , by Lemma 2.7}$$

Therefore, $\tilde{A} \cap \tilde{B}$ is an IPO set.

The following Example 3.5, shows that the intersection of two IPO sets need not be an IPO set.

Example 3.5. Consider the ITS of Example 3.2. If $\tilde{A} = \langle X, \{1,2\}, \{3,4\} \rangle$ and $\tilde{B} = \langle X, \{3\}, \{2,4,5\} \rangle$, then $cl(\tilde{A}) = \tilde{X} = cl(\tilde{B})$. It follows that $\tilde{A}$ and $\tilde{B}$ are IPO sets. Now $\tilde{A} \cap \tilde{B} = \langle X, \{\emptyset\}, \{2,3,4,5\} \rangle$ and $cl(\tilde{A} \cap \tilde{B}) = \langle X, \{\emptyset\}, \{3,4\} \rangle$. Also, $int(cl(\tilde{A} \cap \tilde{B})) = \emptyset$. Therefore, $\tilde{A} \cap \tilde{B}$ is not an IPO set.

Theorem 3.6. Let (X, τ) be an ITS, $\tilde{A}$ be an IPO set and $\tilde{B}$ be an intuitionistic set such that $\tilde{B} \subseteq \tilde{A} \subseteq cl(\tilde{B})$. Then $\tilde{B}$ is also an IPO set.

Definition 3.7. Let (X, τ) be an ITS and $\tilde{A} = \langle X, A^1, A^2 \rangle$ be an IS. $\tilde{A}$ is said to be an *intuitionistic preclosed* (in short, IPC) if it is the complement of an intuitionistic preopen set.

Lemma 3.8. Let (X, τ) be an ITS and $\tilde{A} = \langle X, A^1, A^2 \rangle$ be an IS. Then $\tilde{A}$ is IPC if and only if $cl(int(\tilde{A})) \subseteq \tilde{A}$.

Proof. $\tilde{A}$ is IPC if and only if $\tilde{A}$ is IPO if and only if $\tilde{A} \subset int(cl(\tilde{A}))$ if and only if $int(cl(\tilde{A})) \subset \tilde{A}$ if and only if $cl[cl(\tilde{A})] \subset \tilde{A}$ if and only if $cl(int(\tilde{A})) \subset \tilde{A}$ if and only if $cl(int(\tilde{A})) \subset \tilde{A}$, by Lemma 2.8.

Definition 3.9. Let (X, τ) be an intuitionistic topological space and $\tilde{A}$ be an intuitionistic set. Then $\tilde{A}$ is said to be an *intuitionistic regular open set* (in short, IRO set) if $\tilde{A} = int(cl(\tilde{A}))$. $\tilde{A}$ is said to be an *intuitionistic regular closed set* (in short, IRC set) if $\tilde{A} = cl(int(\tilde{A}))$. By Lemma 2.8, it follows that $\tilde{A}$ is an IRO set if and only if $\tilde{A}$ is an IRC set. The following Theorem 3.10 gives some properties of IRO and IRC sets.

Theorem 3.10. Let (X, τ) be an ITS and $\tilde{A}$ be an IS. Then the following hold.

- $cl(\tilde{A})$ is an IRC if $\tilde{A}$ is an IO set.
- If $\tilde{F}$ is an IC set, then $int(\tilde{F})$ is an IRO set.
- $int(cl(\tilde{A}))$ is an IRO set.
- $cl(int(\tilde{A}))$ is an IRC set.

Proof. (a). Let $\tilde{A}$ be an IO set. Then $\tilde{A} = int(\tilde{A})$. Now $cl(\tilde{A}) = cl(int(\tilde{A})) \subset cl(int(cl(\tilde{A})))$. Thus $cl(\tilde{A}) \subset cl(int(cl(\tilde{A})))$. Again $cl(int(cl(\tilde{A}))) \subset cl(\tilde{A})$. Thus $cl(int(cl(\tilde{A}))) = cl(\tilde{A})$.

(b). If $\tilde{F}$ is IC, then $\tilde{X} - \tilde{F}$ is IO and so $cl(\tilde{X} - \tilde{F})$ is intuitionistic regular closed by (a). Therefore $\tilde{X} - cl(\tilde{X} - \tilde{F}) = int(\tilde{F})$ is an IRO set.

(c). By (b), since $cl(\tilde{A})$ is an IC set, $int(cl(\tilde{A}))$ is an IRO set.

(d). By (a), since $int(\tilde{A})$ is an IO set, $cl(int(\tilde{A}))$ is an IRC set.

The following Theorem 3.11 gives a property of IPO sets and Theorem 3.11 below gives characterizations of IPO sets in terms of IRO sets.

Theorem 3.11. Let (X, τ) be an ITS and $\tilde{A}$ be an IPO set. Then $cl(\tilde{A})$ is an IRC set.

Proof. $\tilde{A}$ is an IPO set implies that $\tilde{A} \subseteq int(cl(\tilde{A}))$ which implies that $cl(\tilde{A}) \subseteq$

$$cl(int(cl(\tilde{A}))) \subseteq cl(\tilde{A}) \text{ and so } cl(\tilde{A}) = cl(int(cl(\tilde{A})))$$

which implies that $cl(\tilde{A})$ is an IRC set.

Theorem 3.12. Let (X, τ) be an ITS and $\tilde{A}$ be an IS. Then the following statements are equivalent.

- (a). $\tilde{A}$ is an IPO set.
- (b). There is an IRO set $\tilde{G}$ such that $\tilde{A} \subset \tilde{G}$ and $cl(\tilde{A}) = cl(\tilde{G})$.
- (c). $\tilde{A}$ is the intersection of an IRO set and an I-dense set.
- (d). $\tilde{A}$ is the intersection of an IO set and an I-dense set.

Proof. (a) $\Rightarrow$ (b). Since $\tilde{A}$ is an IPO set, $\tilde{A} \subseteq int(cl(\tilde{A}))$. Let $\tilde{G} = int(cl(\tilde{A}))$. Then $\tilde{G}$ is an IRO set containing $\tilde{A}$. Clearly $cl(\tilde{G}) = cl(int(cl(\tilde{A}))) = cl(\tilde{A})$.

(b) $\Rightarrow$ (c). Suppose (b) holds. Let $\tilde{D} = \tilde{A} \cup (\tilde{X} - \tilde{G})$. Then $cl(\tilde{D}) = cl(\tilde{A} \cup (\tilde{X} - \tilde{G})) = cl(\tilde{A}) \cup cl(\tilde{X} - \tilde{G}) = \tilde{G} \cup cl(\tilde{X} - \tilde{G}) = \tilde{X}$. Therefore, $\tilde{D}$ is an I-dense. Again $\tilde{D} \cap \tilde{G} = (\tilde{A} \cup (\tilde{X} - \tilde{G})) \cap \tilde{G} = \tilde{A} \cap \tilde{G} = \tilde{A}$.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a). If $\tilde{D} \cap \tilde{G} = \tilde{A}$ where $\tilde{D}$ is an I-dense set and $\tilde{G}$ is IO set, then $cl(\tilde{A}) = cl(\tilde{D} \cap \tilde{G}) \supseteq cl(\tilde{D}) \cap \tilde{G} = \tilde{G}$, and so $cl(\tilde{A}) \supseteq cl(\tilde{G})$, by Lemma 2.7. Again, $\tilde{A} \subseteq \tilde{G}$ gives $cl(\tilde{A}) \subseteq cl(\tilde{G})$ and so $cl(\tilde{A}) = cl(\tilde{G})$. Now $\tilde{A} \subseteq \tilde{G} \subseteq cl(\tilde{G}) = cl(\tilde{A})$ and so $cl(\tilde{A}) = int(cl(\tilde{G})) \supseteq int(\tilde{G}) = \tilde{G} \supseteq \tilde{A}$. Therefore, $\tilde{A}$ is an IPO set.

4. Intuitionistic Preclosure and Intuitionistic Preinterior Operator

Let (X, τ) be an ITS and $\tilde{A} = \langle X, A^1, A^2 \rangle$ be an IS. Then the intuitionistic preinterior and the intuitionistic preclosure of a subset $\tilde{A}$ are denoted by $Ipint(\tilde{A})$ and $Ipcl(\tilde{A})$, and are defined as follows. $Ipcl(\tilde{A}) = \cap \{\tilde{K} | \tilde{K} \text{ is an IPC and } \tilde{A} \subseteq \tilde{K}\}$ and $Ipint(\tilde{A}) = \cup \{\tilde{G} | \tilde{G} \text{ is an IPO and } \tilde{G} \subseteq \tilde{A}\}$.

It can be established that $Ipcl(\tilde{A})$ is the smallest IPC set containing $\tilde{A}$ and $Ipint(\tilde{A})$ is the largest IPO set contained in $\tilde{A}$, $\tilde{A}$ is an IPC if and only if $Ipcl(\tilde{A}) = \tilde{A}$ and $\tilde{A}$ is an IPO if and only if $Ipint(\tilde{A}) = \tilde{A}$. The following Theorems 4.1, 4.2, 4.3 and 4.4 give some properties of the preclosure and preinterior operators, the proof of which are omitted.

Theorem 4.1. Let (X, τ) be a ITS and let $\tilde{A}$ and $\tilde{B}$ be intuitionistic sets. Then the following hold.

- (i) $Ipcl(\tilde{\emptyset}) = \tilde{\emptyset}$ and $Ipcl(\tilde{X}) = \tilde{X}$.
- (ii) $\tilde{A}$ is an IPC set if and only if $\tilde{A} = Ipcl(\tilde{A})$.
- (iii) $Ipcl(Ipcl(\tilde{A})) = Ipcl(\tilde{A})$.
- (iv) $\tilde{A} \subseteq \tilde{B}$ implies that $Ipcl(\tilde{A}) \subseteq Ipcl(\tilde{B})$.
- (v) $Ipcl(\tilde{A} \cap \tilde{B}) \subseteq Ipcl(\tilde{A}) \cap Ipcl(\tilde{B})$.

$$(vi) \text{ } Ipcl(\tilde{A} \cup \tilde{B}) = Ipcl(\tilde{A}) \cup Ipcl(\tilde{B}).$$

Theorem 4.2. Let (X, τ) be an ITS and let $\tilde{A}$ and $\tilde{B}$ be intuitionistic sets. Then the following hold.

$$(i) \text{ } Ipint(\tilde{\emptyset}) = \tilde{\emptyset} \text{ and } Ipint(\tilde{X}) = \tilde{X}.$$

$$(ii) \text{ } \tilde{A} \text{ is an IPO set if and only if } \tilde{A} = Ipint(\tilde{A}).$$

$$(iii) \text{ } Ipint(Ipint(\tilde{A})) = Ipint(\tilde{A}).$$

$$(iv) \text{ } \tilde{A} \subseteq \tilde{B} \text{ implies that } Ipint(\tilde{A}) \subseteq Ipint(\tilde{B}).$$

$$(v) \text{ } Ipint(\tilde{A} \cap \tilde{B}) = Ipint(\tilde{A}) \cap Ipint(\tilde{B}).$$

(vi) $Ipint(\tilde{A} \cup \tilde{B}) \supseteq Ipint(\tilde{A}) \cup Ipint(\tilde{B})$. The following Lemma 4.3 is essential to prove the remaining results of the section.

Lemma 4.3. Let (X, τ) be an ITS and $\tilde{A}$ be an IS. Then $\tilde{x} \in Ipcl(\tilde{A})$ if and only if every IPO set $\tilde{U}$ containing $\tilde{x}$, intersects $\tilde{A}$.

Proof. The proof follows from [[1]Lemma 2.3].

Theorem 4.4. Let (X, τ) be an ITS and let $\tilde{A}$ be an intuitionistic set. Then the following hold.

$$(i) \text{ } Ipint(\tilde{A}) = \tilde{A} \cap int(cl(\tilde{A}))$$

$$(ii) \text{ } Ipcl(\tilde{A}) = \tilde{A} \cup cl(int(\tilde{A}))$$

$$(iii) \text{ } Iacl(\tilde{A}) = \tilde{A} \cup cl(int(cl(\tilde{A})))$$

$$(iv) \text{ } Iaint(\tilde{A}) = \tilde{A} \cap int(cl(int(\tilde{A}))).$$

Proof. (i) Since $Ipint(\tilde{A})$ is an IPO set, $Ipint(\tilde{A}) \subset int(cl(\tilde{A}))$ and so $Ipint(\tilde{A}) \subset \tilde{A} \cap int(cl(\tilde{A}))$. Conversely, let $\tilde{x} \in int(cl(\tilde{A}))$. If U is any IO set containing $\tilde{x}$, then $U \cap int(cl(\tilde{A}))$ is an IO set containing $\tilde{x}$. Since $\tilde{x} \in cl(\tilde{A})$, $U \cap int(cl(\tilde{A})) \cap \tilde{A} \neq \emptyset$ [9] which implies that $\tilde{x} \in cl(int(cl(\tilde{A})) \cap \tilde{A})$. Therefore, $int(cl(\tilde{A})) \subset cl(int(cl(\tilde{A})) \cap \tilde{A})$ and so, $int(cl(\tilde{A})) \subset int(cl(int(cl(\tilde{A})) \cap \tilde{A}))$.

Since $\tilde{A} \cap int(cl(\tilde{A})) \subset int(cl(\tilde{A})) \subset int(cl(int(cl(\tilde{A})) \cap \tilde{A}))$, $\tilde{A} \cap int(cl(\tilde{A}))$ is an IPO set contained in $\tilde{A}$ it follows that $\tilde{A} \cap int(cl(\tilde{A})) \subset Ipint(\tilde{A})$. Hence $Ipint(\tilde{A}) = \tilde{A} \cap int(cl(\tilde{A}))$.

(ii) Now $cl(int(\tilde{A} \cup cl(int(\tilde{A})))) \subseteq cl(int(\tilde{A})) \cup cl(int(\tilde{A})) = cl(int(\tilde{A})) \subseteq \tilde{A} \cup cl(int(\tilde{A}))$, by Lemma 2.7, $\tilde{A} \cup cl(int(\tilde{A}))$ is an IPC containing $\tilde{A}$. Therefore, $Ipcl(\tilde{A}) \subset \tilde{A} \cup cl(int(\tilde{A}))$. Conversely, since $Ipcl(\tilde{A})$ is IPC, we have $cl(int(\tilde{A})) \subset cl(int(Ipcl(\tilde{A}))) \subset Ipcl(\tilde{A})$ and hence $\tilde{A} \cup cl(int(\tilde{A})) \subset Ipcl(\tilde{A})$. Therefore, $Ipcl(\tilde{A}) = \tilde{A} \cup cl(int(\tilde{A}))$.

The proof of (iii) and (iv) are similar to the proof of (i) and (ii) respectively.

Theorem 4.5. Let (X, τ) be an ITS and $\tilde{A}$ is an IS. Then,

$$(a) \text{ } Ipcl(\tilde{\tilde{A}}) = \overline{Ipint(\tilde{A})}.$$

$$(b) \text{ } Ipint(\tilde{\tilde{A}}) = \overline{Ipcl(\tilde{A})}.$$

Proof. (a) $Ipcl(\tilde{A}) = (\tilde{X} - \tilde{A}) \cup cl(int(\tilde{X} - \tilde{A})) = (\tilde{X} - \tilde{A}) \cup (cl(int(\tilde{X} - \tilde{A}))) = (\tilde{X} - \tilde{A}) \cup (\tilde{A} - int(cl(\tilde{A}))) = \tilde{X} - (\tilde{A} \cap int(cl(\tilde{A}))) = Ipint(\tilde{A})$.

(b). The proof is similar to (a).

The following Theorem 4.6 give characterizations of I-dense sets.

Theorem 4.6. Let (X, τ) be an ITS and let $\tilde{A}$ is an IS. Then the following are equivalent.

- (a) $\tilde{A}$ is an I - dense.
- (b) $Ipcl(\tilde{A}) = \tilde{X}$.
- (c) If $\tilde{B}$ is any intuitionistic preclosed set such that $\tilde{A} \subset \tilde{B}$, then $\tilde{B} = \tilde{X}$.
- (d) Every nonempty intuitionistic preopen set has a nonempty intersection with $\tilde{A}$.
- (e) $Ipint(\tilde{X} - \tilde{A}) = \tilde{\emptyset}$.

Proof. (a) $\Rightarrow$ (b). Suppose $\tilde{x} \notin Ipcl(\tilde{A})$. Then there exists an intuitionistic preopen set $\tilde{G}$ containing $\tilde{x}$ such that $\tilde{G} \cap \tilde{A} = \tilde{\emptyset}$. Since $\tilde{G}$ is a nonempty intuitionistic preopen set, there is a nonempty intuitionistic open set $\tilde{H}$ such that $\tilde{H} \subset \tilde{G}$ and so $\tilde{H} \cap \tilde{A} = \tilde{\emptyset}$, a contradiction. Therefore, $Ipcl(\tilde{A}) = \tilde{X}$.

(b) $\Rightarrow$ (c). If $\tilde{B}$ is any intuitionistic preclosed set such that $\tilde{A} \subset \tilde{B}$, then $\tilde{X} = Ipcl(\tilde{A}) \subset Ipcl(\tilde{B}) = \tilde{B}$ which implies that $\tilde{B} = \tilde{X}$.

(c) $\Rightarrow$ (d). If $\tilde{G}$ is any nonempty intuitionistic preopen set such that $\tilde{G} \cap \tilde{A} = \tilde{\emptyset}$, then $\tilde{A} \subset \tilde{X} - \tilde{G}$ and $\tilde{X} - \tilde{G}$ is intuitionistic preclosed. By (c), it follows that $\tilde{G} = \tilde{\emptyset}$, a contradiction.

(d) $\Rightarrow$ (e). Suppose that $Ipint(\tilde{X} - \tilde{A}) \neq \tilde{\emptyset}$. Then $Ipint(\tilde{X} - \tilde{A})$ is a nonempty intuitionistic preopen set such that $Ipint(\tilde{X} - \tilde{A}) \cap \tilde{A} \neq \tilde{\emptyset}$, a contradiction to the hypothesis.

(e) $\Rightarrow$ (a). $Ipint(\tilde{X} - \tilde{A}) = \tilde{\emptyset}$ implies that $\tilde{X} - Ipint(\tilde{X} - \tilde{A}) = \tilde{X}$. Thus $Ipcl(\tilde{A}) = \tilde{X}$. Hence $cl(\tilde{A}) = \tilde{X}$ which shows that $\tilde{A}$ is I-dense.

The following Theorem 4.7 give characterizations of IPC sets.

Theorem 4.7. Let (X, τ) be an ITS and let $\tilde{A}$ be an intuitionistic set. Then the following are equivalent.

- (a) $\tilde{A}$ is an IPC set.
- (b) $int(cl(\tilde{A})) \subseteq \tilde{A}$.
- (c) $cl(int(\tilde{A})) \supseteq \tilde{A}$.
- (d) $\tilde{A}$ is an intuitionistic preopen set.

Proof. (a) $\Rightarrow$ (b). If $\tilde{A}$ is an intuitionistic preclosed set, then $\tilde{X} - \tilde{A}$ is an IPO set and so $\tilde{X} - \tilde{A} \subset int(cl(\tilde{X} - \tilde{A}))$. Therefore, $\tilde{X} - int(cl(\tilde{X} - \tilde{A})) \subset \tilde{A}$ which implies that $cl(int(\tilde{A})) \subseteq \tilde{A}$.

(b) $\Rightarrow$ (c). By (b), $int(cl(\tilde{A})) \subseteq \tilde{A}$ and so $\tilde{X} - \tilde{A} \subset \tilde{X} - int(cl(\tilde{A}))$ which implies that $cl(int(\tilde{A})) \supseteq \tilde{A}$.

(c) $\Rightarrow$ (d). By definition, $(\tilde{A})$ is an intuitionistic preopen set.

(d) $\Rightarrow$ (a). By definition, $\tilde{A}$ is an IPC.

The following Theorems 4.8 to 4.10 give some properties of the operators $Ipint$ and $Ipcl$.

Theorem 4.8. For every IO set U in an ITS (X, τ) and every $\tilde{A} \subset \tilde{X}$, we have $int(cl(U \cap \tilde{A})) = int(cl(U)) \cap int(cl(\tilde{A}))$.

Proof. If $\tilde{U}$ is an IO set, $\tilde{U} \cap int(cl(\tilde{A})) = int(\tilde{U} \cap cl(\tilde{A})) \subset \tilde{U} \cap cl(\tilde{A})$. By Lemma 2.7, we have $\tilde{U} \cap cl(\tilde{A}) \subset cl(\tilde{U} \cap \tilde{A})$. Hence $cl(\tilde{U} \cap int(cl(\tilde{A}))) \subset cl(\tilde{U} \cap \tilde{A})$. Since $int(cl(\tilde{A}))$ is an IPO set, we have $int(cl(\tilde{U})) \cap int(cl(\tilde{A})) \subset cl(\tilde{U}) \cap int(cl(\tilde{A})) \subset cl(\tilde{U} \cap int(cl(\tilde{A})))$. Therefore, $int(cl(\tilde{U}) \cap int(cl(\tilde{A}))) \subset cl(\tilde{U}) \cap cl(\tilde{A}) \subset cl(\tilde{U} \cap \tilde{A})$ which implies that $int(cl(\tilde{U}) \cap int(cl(\tilde{A}))) \subset int(cl(\tilde{U} \cap \tilde{A}))$.

On the other hand, $int(cl(\tilde{U} \cap \tilde{A})) \subset int(cl(\tilde{U}) \cap cl(\tilde{A})) = int(cl(\tilde{U})) \cap int(cl(\tilde{A}))$.

Hence $int(cl(\tilde{U} \cap \tilde{A})) = int(cl(\tilde{U})) \cap int(cl(\tilde{A}))$.

Theorem 4.9. Let $\tilde{A}$ and $\tilde{B}$ be subsets of an ITS (X, τ) . Then $cl(int(cl(\tilde{A} \cup \tilde{B}))) = cl(int(cl(\tilde{A}))) \cup cl(int(cl(\tilde{B})))$.

Proof. We know that $int(cl(\tilde{A} \cup \tilde{B})) = int(cl(\tilde{A}) \cup cl(\tilde{B})) \subset cl(\tilde{A}) \cup int(cl(\tilde{B})) \subset cl(\tilde{A}) \cup cl(int(cl(\tilde{B})))$. Since $cl(int(cl(\tilde{B})))$ is an IC set, we get $int(cl(\tilde{A} \cup \tilde{B})) \subset int(cl(\tilde{A}) \cup cl(int(cl(\tilde{B})))) \subset int(cl(\tilde{A})) \cup cl(int(cl(\tilde{B})))$, by Lemma 2.7 and so $int(cl(\tilde{A} \cup \tilde{B})) \subset cl(int(cl(\tilde{A}))) \cup cl(int(cl(\tilde{B})))$. On the other hand, we have $cl(int(cl(\tilde{A} \cup \tilde{B}))) = cl(int(cl(\tilde{A}) \cup cl(\tilde{B}))) \supset cl(int(cl(\tilde{A}))) \cup int(cl(\tilde{B})) = cl(int(cl(\tilde{A}))) \cup cl(int(cl(\tilde{B})))$. Hence $cl(int(cl(\tilde{A} \cup \tilde{B}))) = cl(int(cl(\tilde{A}))) \cup cl(int(cl(\tilde{B})))$. The following Theorem 4.10 shows that the family of all α -open sets are exactly the sets which are both semiopen and preopen.

Theorem 4.10. For any subset $\tilde{A}$ of an intuitionistic topological space (X, τ) , the following conditions are equivalent.

(a) $\tilde{A} \in \tau_\alpha(\tilde{X})$

(b) $\tilde{A} \in ISO(\tilde{X}) \cap IPO(\tilde{X})$

(c) There exists an IO set $\tilde{U}$ such that $\tilde{U} \subset \tilde{A} \subset int(cl(\tilde{U}))$.

Proof. (a) $\Rightarrow$ (b). Suppose $\tilde{A} \in \tau_\alpha$. Then $\tilde{A} \subset int(cl(int(\tilde{A})))$ and so $\tilde{A} \subset cl(int(\tilde{A}))$.

Also, $\tilde{A} \subset int(cl(int(\tilde{A})))$ implies that $\tilde{A} \subset int(cl(\tilde{A}))$. Therefore, $\tilde{A} \in ISO(\tilde{X}) \cap IPO(\tilde{X})$.

(b) $\Rightarrow$ (c). Since $\tilde{A} \in ISO(\tilde{X})$, there exists an IO set $\tilde{U}$ such that $\tilde{U} \subset \tilde{A} \subset cl(\tilde{U})$.

Therefore, $cl(\tilde{A}) = cl(\tilde{U})$ and $int(cl(\tilde{A})) = int(cl(\tilde{U}))$.

Since $\tilde{A} \in IPO(\tilde{X})$, it follows that $\tilde{A} \subset int(cl(\tilde{A}))$. Hence $\tilde{U} \subset \tilde{A} \subset int(cl(\tilde{U}))$.

(c) $\Rightarrow$ (a). Suppose $\tilde{U} \subset \tilde{A} \subset int(cl(\tilde{U}))$ where U is an IO set. Thus $\tilde{U} \subset int(\tilde{A}) \subset int(cl(\tilde{U}))$ and $cl(\tilde{U}) \subset cl(int(\tilde{A})) \subset cl(int(cl(\tilde{U}))) \subset cl(\tilde{U})$. Here $cl(\tilde{U}) = cl(int(\tilde{A}))$ and $int(cl(\tilde{U})) = int(cl(int(\tilde{A})))$ which implies that $\tilde{A} \subset int(cl(int(\tilde{A})))$. Therefore, $\tilde{A} \in \tau_\alpha(\tilde{X})$. The following Theorem 4.11, gives the relation between the

closure and interior operators.

Theorem 4.11. For any subset $\tilde{A}$ of an intuitionistic topological space (X, τ) , then the following hold.

$$(a) \text{Iacl}(\text{int}(\tilde{A})) = \text{cl}(\text{int}(\tilde{A})).$$

$$(b) \text{cl}(\text{Iaint}(\tilde{A})) = \text{cl}(\text{int}(\tilde{A})).$$

$$(c) \text{Int}(\text{Iacl}(\tilde{A})) = \text{int}(\text{cl}(\tilde{A})).$$

$$(d) \text{Iaint}(\text{cl}(\tilde{A})) = \text{int}(\text{cl}(\tilde{A})).$$

Proof. (a) By Theorem 4.4, $\text{Iacl}(\text{int}(\tilde{A})) = \text{int}(\tilde{A}) \cup \text{cl}(\text{int}(\text{cl}(\text{int}(\tilde{A})))) = \text{int}(\tilde{A}) \cup \text{cl}(\text{int}(\tilde{A})) = \text{cl}(\text{int}(\tilde{A}))$.

(b) Since $\tau \subset \tau_\alpha$, $\text{int}(\tilde{A}) \subset \text{Iaint}(\tilde{A})$ for all $\tilde{A} \subset \tilde{X}$ and so $\text{cl}(\text{int}(\tilde{A})) \subset \text{cl}(\text{Iaint}(\tilde{A}))$. On the other hand, $\text{Iaint}(\tilde{A}) = \tilde{A} \cap \text{int}(\text{cl}(\text{int}(\tilde{A})))$ by Theorem 4.4 and so $\text{cl}(\text{Iaint}(\tilde{A})) = \text{cl}(\tilde{A} \cap \text{int}(\text{cl}(\text{int}(\tilde{A})))) \subset \text{cl}(\tilde{A}) \cap \text{cl}(\text{int}(\text{cl}(\text{int}(\tilde{A})))) \subset \text{cl}(\tilde{A}) \cap \text{cl}(\text{int}(\tilde{A}))$. Therefore, $\text{cl}(\text{Iaint}(\tilde{A})) \subset \text{cl}(\text{int}(\tilde{A}))$. Hence $\text{cl}(\text{Iaint}(\tilde{A})) = \text{cl}(\text{int}(\tilde{A}))$.

(c) $\text{int}(\text{Iacl}(\tilde{A})) = \text{int}(\tilde{A} \cup \text{cl}(\text{int}(\text{cl}(\tilde{A})))) = \text{int}(\tilde{A}) \cup \text{int}(\text{cl}(\text{int}(\text{cl}(\tilde{A})))) = \text{int}(\tilde{A}) \cup \text{int}(\text{cl}(\tilde{A})) = \text{int}(\text{cl}(\tilde{A}))$.

(d) By Theorem 4.4, $\text{Iaint}(\tilde{A}) = \tilde{A} \cap \text{int}(\text{cl}(\text{int}(\tilde{A})))$ and so $\text{Iaint}(\text{cl}(\tilde{A})) = \text{cl}(\tilde{A}) \cap \text{int}(\text{cl}(\text{int}(\text{cl}(\tilde{A})))) = \text{cl}(\tilde{A}) \cap \text{int}(\text{cl}(\tilde{A})) = \text{int}(\text{cl}(\tilde{A}))$.

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