



A Machine Replacement Policy based on Markov Chains

¹S. Kopperundevi and ²S. Aarthi

¹Department of Mathematics,

Dr.M.G.R. Educational and Research Institute University,
Chennai.

²Department of Mathematics,

Dr.M.G.R. Educational and Research Institute University,
Chennai.

Abstract

A different optimal single machine replacement policy using a single stage decision making process is proposed based on the quality of items produced. This quality is determined using the attribute acceptance sampling plans to accept or reject a production lot. In a stage of this policy, if the number of defective items in a sample of produced items is more than an upper threshold, the machine is replaced. However, the machine is not replaced if the number of defective items is less than a lower threshold. The primary objective of the work is to determine the optimal values of both the upper and the lower thresholds using a Markov process to minimize the total cost associated with a machine replacement policy.

Key Words: Machine replacement, inspection, defective items, markov process, testing of hypothesis.

1. Introduction

As the optimal management of equipment replacement is continuously becoming more important nationally and locally (i.e., at a firm level), it is attracting a significant interest in operations research and management science.

The traditional equipment replacement analysis focus on the evaluation of the keep or replace alternative for each period within the planning horizon for single, two, or multiple machines. These machines work in series or parallel, where several motivations such as machine deterioration, technological change, and capacity expansion have been focused. Solutions generally involve the use of dynamic programming under various assumptions concerning costs, planning horizon, number of machines and their types.

Two approaches have been applied to the machine replacement problem.

- 1) The optimal replacement depends on the intensity of technological change, the rate of capacity expansion, and the deteriorating rate.
- 2) The maintenance policy that is being followed in practice is a combination of preventive and corrective maintenance.

Machine replacement policy uses single stage decision making process based on the quality of items produced.

- If the number of defective items in a sample of produced items is more than an upper threshold, the machine is replaced.
- If the number of defective items is less than a lower threshold, the machine is not replaced.
- If the number of defective item falls between the upper and the lower thresholds, the decision making process continues inspecting and possibly repairing the machine.

The optimal policy for two-machine replacement problem using Bayesian inference in the context of the finite mixture model was determined by Fallahnezhad, M.S., Niaki S.T.A. and Eshragh-Jahromi, A. (2007). They discussed the analysis of time-to-failure data and proposed an optimal decision-making procedure for machine replacement strategy.

Moreover, Fallahnezhad, M.S. and Niaki, S.T.A. (2010), proposed a dynamic programming model of the two-machine replacement problem.

A two state partially observable Markov decision process for machine replacement problem was proposed by Grosfeld-Nir, A. (2007). He proved that the “dominance in expectation” (the expected profit is larger in the good state than in the bad state) suffices for the optimal policy to be of a control limit type (CLT). In other words, continue if and only if the good state probability exceeds the CLT.

2. Assumptions

- (1) The machine produces the items continuously.
- (2) The quality of items produced by the machine is inspected periodically.
- (3) The machine replacement strategy is employed in every period of item inspection.

Notations:

$E(TC)$: The expected total system cost

$E(AC)$: The expected total cost of accepting (not replacing) the machine

$E(RP)$: The expected total cost of replacing the machine

$E(I)$: The expected total cost of inspecting and repairing the machine

N : The number of produced items in a period

p : The probability of producing a defective item

c_1 : The lower threshold for the number of defective items in the first stage

c_2 : The upper threshold for the number of defective items in the first stage

I : The cost of inspecting and repairing a possible defect

c : The cost of producing a defective item

R : The cost of machine replacement

P : The transition probability matrix

Q : The square matrix containing the transition probabilities of going from a non-absorbing state to another non-absorbing state

R : The matrix containing all probabilities of going from a non-absorbing state to an absorbing state (i.e., finished or scrapped product)

A : An identity matrix representing the probability of staying in a state

O : The matrix representing the probabilities of escaping an absorbing state (always zero)

M : The fundamental matrix containing the expected number of transitions from any non-absorbing state to another non-absorbing state before absorption occurs

F : The absorption probability matrix containing the long-run probabilities of the transition from any non-absorbing state to an absorbing state.

P_{ij} : The probability of going from state i to state j in a single stage

m_{ij} : The expected long-run number of times that the transient state j is occupied before absorption occurs, given that the initial state is i

f_{ij} : The long-run probability of going from a non-absorbing state (i) to an absorbing state (j)

3. Model Development

The purpose of this research is to develop a Markovian model to determine the optimum values of the thresholds in each inspection stage.

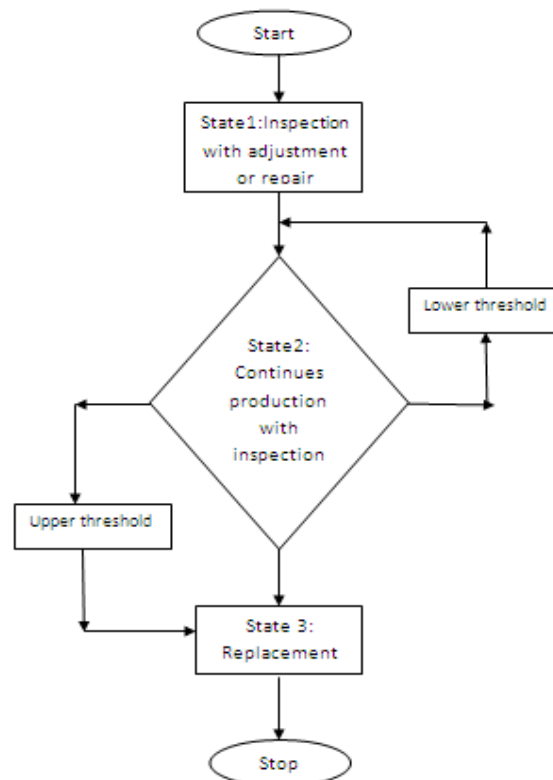
Based on the notations, the expected total system cost can be expressed as:

$$E(TC) = E(AC) + E(RP) + E(I)$$

A Single-Stage Single-Machine Replacement Policy

- A lot containing n items is first inspected. If the number of defective items in the lot is less than or equal to c_1 , then the machine is in a good state and continues production.
- If the number of defective items is more than c_2 , then the machine is in a bad state and needs replacement.
- If the number of defective items is greater than c_1 but less than or equal c_2 , then the machine is inspected for malfunctioning, is adjusted or repaired.

The following states are used in Markov chain process:



State 2 is known as an acceptable machine, the decision-making process stops.

State 3 the performance of the machine is not satisfactory and causes to stop the

decision-making process.

Where states 2 and 3 are absorbing.

$$\text{Probability of inspecting the machine} = p_{11} = \sum_{i=c_1+1}^{c_2} \binom{n}{i} p^i (1-p)^{n-i} = F(c_2) - F(c_1)$$

$$\text{Probability of non-replacement} = p_{12} = \sum_{i=0}^{c_1} \binom{n}{i} p^i (1-p)^{n-i} = F(c_1)$$

$$\text{Probability of replacing the machine} = p_{13} = \sum_{i=c_2+1}^n \binom{n}{i} p^i (1-p)^{n-i} = 1 - F(c_2)$$

Where $F(\cdot)$ represents the cumulative binomial distribution function with parameters n and p .

The transition probability matrix of this process is:

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}.$$

The $\mathbf{P}$ matrix is an absorbing Markov chain with states 2 and 3 being absorbing and state 1 being transient.

Analyzing this absorbing Markov chain requires the rearrangement of the single-step probability matrix as:

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}.$$

Rearranging the $\mathbf{P}$ matrix in the latter form yields the following matrix:

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 2 & 3 & 1 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_{12} & p_{13} & p_{11} \end{bmatrix} \end{matrix}.$$

The fundamental matrix $\mathbf{M}$ is obtained as,

$$\mathbf{M} = \mathbf{m}_{11} = (\mathbf{I} - \mathbf{Q})^{-1} = \frac{1}{1 - p_{11}}.$$

Where $\mathbf{I}$ is identity matrix. The value m_{11} represents the expected long-run number of times that the transient state 1 is occupied before absorption occurs (i.e., replaced or not replaced), given that the initial state is 1.

Hence, the long-run absorption probability matrix, $\mathbf{F}$, can be calculated as,

$$\mathbf{F} = \mathbf{M} \times \mathbf{R} = \begin{bmatrix} f_{12} = \frac{p_{12}}{1-p_{11}} & f_{13} = \frac{p_{13}}{1-p_{11}} \end{bmatrix},$$

f_{12} and f_{13} , represent the probabilities of a machine not being replaced and replaced.

The expected total system cost can be derived by,

- The expected acceptance cost, $E(AC)$, is determined by the expected cost of the defective items, $c(Np)$, multiplied by the absorption probability of accepting the machine (i.e., f_{12}).
- The expected replacement cost, $E(RP)$, is obtained by the replacement cost, R , multiplied by the absorption probability of replacing machine (i.e., f_{13}). Also, once a machine goes into one of these two absorbing states (i.e., states 2 and 3), it cannot go back to state 1. Hence, the number of visits to the absorbing states is equal to 1, and the expected inspection cost is given by $I(m_{11}-1)$.

Therefore, the expected cost for machine replacement policy can be expressed as a function of f_{12} , f_{13} , and m_{11} as follows:

$$E(TC) = cNpf_{12} + Rf_{13} + I(m_{11}-1)$$

Substituting the values,

$$E(TC) = cNp \left(\frac{p_{12}}{1-p_{11}} \right) + R \left(\frac{p_{13}}{1-p_{11}} \right) + I \left(\frac{1}{1-p_{11}} - 1 \right)$$

$$E(TC) = cNp \left(\frac{p_{12}}{1-p_{11}} \right) + R \left(1 - \frac{p_{12}}{1-p_{11}} \right) + I \left(\frac{p_{11}}{1-p_{11}} \right)$$

In terms of cumulative binomial distribution, we get

$$E(TC) = cNp \left(\frac{F(c_1)}{1-F(c_2)+F(c_1)} \right) + R \left(1 - \frac{F(c_1)}{1-F(c_2)+F(c_1)} \right) + I \left(\frac{F(c_2)-F(c_1)}{1-F(c_2)+F(c_1)} \right)$$

$$E(TC) = cNp \left(\frac{F(c_1)}{1-F(c_2)+F(c_1)} \right) + R \left(\frac{1-F(c_2)+F(c_1)-F(c_1)}{1-F(c_2)+F(c_1)} \right) + I \left(\frac{F(c_2)-F(c_1)}{1-F(c_2)+F(c_1)} \right)$$

Therefore we get,

$$E(TC) = \frac{1}{1-F(c_2)+F(c_1)} \{ [cNp \times F(c_1)] + R[1-F(c_2)] + I[F(c_2)-F(c_1)] \} \rightarrow (1)$$

Where the terms $F(c_1)$ and $F(c_2)$ are functions of the probability of producing a defective item p .

The optimal replacement policy turns into determining the values of c_1 and c_2 that minimize the expected total system cost numerically.

To determine the boundary limits of c_1 and c_2 , the concepts of type-I and type-II error probabilities are utilized.

The type-I error, ε_1 , shows the probability of rejecting the machine when the defective production percent of the machine is acceptable.

The type-II error, ε_2 , is the probability of accepting the machine when the defective percentage is not acceptable.

When $p = \delta_1$, the probability of rejecting the machine is less than ε_1 and if $p = \delta_2$, the probability of accepting the machine is less than ε_1 . Hence, we have

$$p = \delta_1 \rightarrow f_{13} \leq \varepsilon_1 \rightarrow f_{12} \geq 1 - \varepsilon_1,$$

$$p = \delta_2 \rightarrow f_{12} \leq \varepsilon_2 \rightarrow f_{13} \geq 1 - \varepsilon_2.$$

Example

Consider a single-stage system with the number of produced items in a period, $N = 1000$, the probability of producing a defective item $p = 0.1$, the cost of producing a defective item, $c = 6$, the cost of machine replacement, $R = 600$, the cost of machine inspection and repair, $I = 300$, the number of items in an inspected lot, $n = 50$, $\delta_1 = 0.05$, $\delta_2 = 0.2$, $\varepsilon_1 = 0.05$, and $\varepsilon_2 = 0.1$.

The feasible values of c_1 and c_2 are obtained using,

$$P = 0.05 \rightarrow f_{13} \leq 0.05 \rightarrow f_{12} \geq 0.95 \rightarrow (2)$$

$$p = 0.2 \rightarrow f_{12} \leq 0.1 \rightarrow f_{13} \geq 0.9 \rightarrow (3)$$

i.e., the probability of accepting the machine when $\delta_1 = 0.05$ should be more than 0.95 and the probability of rejecting the machine when $\delta_2 = 0.2$ should be more than 0.9.

Based on above inequalities, the feasible values of c_1 and c_2 among a set of alternative values are first determined.

Using the alternative values, optimal replacement policy is derived using equation (1).

Let $c_1 = 1$ and $c_2 = 7$,

Binomial distribution formula $= n c_x p^x q^{n-x}$

Giving $x = 1$ to 10 accordingly as feasible values. Substituting the values $n = 50$, $p = 0.1$, $q = 0.9$

Such that binomial distribution $= 50 c_1 (0.1)^1 (0.9)^{50-1} = 50 \times 0.1 \times 0.00573 = 0.02863$

Similarly solving for $x = 1, 2, \dots$

n=50	Binomial distribution	Cumulative binomial distribution
When x=1	0.02863	0.02863
x=2	0.07795	0.10658
x=3	0.13857	0.24515
x=4	0.18090	0.42605
x=5	0.18493	0.61098
x=6	0.15409	0.76507
x=7	0.10763	0.87270
x=8	0.06428	0.93698

When $c_1=1$, $F(c_1)=0.02863$

$c_2=7$, $F(c_2)=0.87270$

$$E(TC) = \frac{1}{1 - 0.87270 + 0.02863} \left\{ \begin{array}{l} [6 \times 1000 \times 0.1 \times 0.02863] \\ + 600[1 - 0.87270] \\ + 300[0.87270 - 0.02863] \end{array} \right\}$$

$$= \frac{1}{0.15593} [17.1780 + 76.3800 + 253.2210] = \frac{346.779}{0.15593}$$

$$E(TC) = 2223.940$$

When $c_1=4$, $F(c_1)=0.4260$

$c_2=6$, $F(c_2)=0.7650$

Substituting the values in (1),

$$E(TC) = \frac{1}{1 - 0.7650 + 0.4260} \left\{ \begin{array}{l} [6 \times 1000 \times 0.1 \times 0.4260] \\ + 600[1 - 0.7650] \\ + 300[0.7650 - 0.4260] \end{array} \right\}$$

$$= \frac{1}{0.6610} [255.60 + 141 + 101.70] = \frac{498.300}{0.6610}$$

$$E(TC) = 753.857$$

When $c_1=4$, $F(c_1)=0.42605$

$c_2=8$, $F(c_2)=0.93698$

Substituting the values in (1),

$$E(TC) = \frac{1}{1 - 0.93698 + 0.42605} \left\{ \begin{array}{l} [6 \times 1000 \times 0.1 \times 0.42605] \\ + 600[1 - 0.93698] \\ + 300[0.93698 - 0.42605] \end{array} \right\}$$

$$= \frac{1}{0.48907} [255.63 + 37.812 + 153.279] = \frac{446.721}{0.48907}$$

$$E(TC) = 913.41$$

The expected total cost for different feasible combination of c_1 and c_2 ,

c_1	c_2	$E(TC)$
1	7	2223.93
2	6	1178.47
2	8	2068.91
4	6	753.88
4	8	913.41
4	10	980.96

Eqn (1) is solved for all feasible sets. The best combination value is $c_1=4$ and $c_2=6$ with minimum value of the expected total cost being 753.88.

4. Conclusion

The optimum value of control thresholds in an acceptance sampling design for a machine replacement problem was determined numerically using a Markovian approach. We developed a general model for the expected cost by considering acceptance, replacement, and inspection costs. The model was then used to determine the optimum value of control thresholds for a single-stage strategy.

References

- [1] Bowling S.R., Khasawneh M.T., Kaewkuekool S., Cho B.R., A Markovian approach to determining optimum process target levels for a multi-stage serial production system, *European Journal of Operational Research* 159 (2004), 636-650.
- [2] Fallahnezhad M.S., Niaki S.T.A., Eshragh-Jahromi A., A one-stage two-machines replacement strategy based on the Bayesian inference method, *Journal of Industrial and Systems Engineering* 1 (2007), 235-250.
- [3] Fallahnezhad M.S., Niaki S.T.A., A multi-stage two-machines replacement strategy using mixture models, Bayesian inference and stochastic dynamic programming, *Communications in Statistics-Theory and Methods* 40 (2010), 702-725.
- [4] Grosfeld-Nir A., Control limits for two-state partially observable Markov decision processes, *European Journal of Operational Research* 182 (2007), 300-304.
- [5] Niaki S.T.A., Fallahnezhad M.S., A decision making framework in production processes using Bayesian inference and stochastic dynamic programming, *Journal of Applied science* 7 (2007), 3618-3627.

