SOLUTION TO SOME OPEN PROBLEM ON CYCLE SUPER MAGIC LABELING OF GRAPHS

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Abstract

A simple graph G admits an H-covering if every edge in E(G) belongs to a subgraph of G isomorphic to H. The graph G is said to be H-magic if there exists bijection f: V(G)∪E(G) → { 1,2,3,…,|V(G)∪E(G)|} such that for every subgraph H’ of G isomorphic to H, \( \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) \) is constant. G is said to be H-supermagic if f(V(G)) = {1, 2, 3 … |V(G)| }. In this paper, we generalize the results found in the article “A.A.G.Ngurah, A.N.M. Salman, L.Susilowati, H-supermagic labelings of graphs, Discrete Math. 310 (2010) 1293-1300”. Also we provide a partial solution to an open problem found in the same article.

Keywords: generalized-supermagic labeling, generalized-supermagic graph.

1. Introduction

We consider finite and simple graphs. The vertex and edge sets of a graph G are denoted by V(G) and E(G), respectively. Let H be a graph. An edge-covering of G is a family of subgraphs H₁, H₂,…,Hₖ such that each edge of E(G) belongs to at least one of the subgraph Hᵢ, 1 ≤ i ≤ k. Then it is said that G admits an (H₁, H₂,…, Hₖ )-(edge) covering. If every Hᵢ is isomorphic to a given graph H, then G admits an H-covering. Suppose G admits an H-covering. A total labeling f: V(G)∪E(G) → { 1,2,3,…, |V(G)| + |E(G)|} is called an H-magic labeling of G if there exists a positive integer kᵢ (called the magic constant) such that for every subgraph H’ of G isomorphic to H, \( \sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) \) = kᵢ. A graph that admits such a labeling is called H-magic. An H-magic labeling f is called an H-supermagic labeling if f(V(G)) = {1, 2, 3,…, |V(G)|}. A graph that admits an H-supermagic labeling is called an H-supermagic graph. The sum of all vertex and edge labels on H (under a labeling f) and is denoted by \( \sum f(H) \).
The notion of H- magic labeling was introduced by Gutierrez and Llado [3] in 2005. They proved that the star graph $K_{1,n}$ and the complete bipartite graphs $K_{m,n}$ are $K_{1,h}$- supermagic for some h. They also proved that the paths $P_n$ and the cycles $C_n$ are $P_h$- supermagic for some h.

In 2007, Llado and Moragas [6] studied some $C_n$- supermagic graphs. They proved that the wheel $W_n$, the windmill $W(r,k)$, the subdivided wheel $W_n(r,k)$, and the graph obtained by joining two end vertices of any number of internally disjoint paths of length $p \geq 2$ are $C_n$- supermagic for some h. Maryati et al. [7] studied some $P_h$- supermagic trees. They proved that shrubs, balanced subdivision of shrubs, and banana trees are $P_h$- supermagic for some h.

For $H \cong K_2$, an $H$- supermagic graph is also called a super edge- magic graph. The notion of a super-edge-magic graph was introduced by Enomoto et al. [1] as a particular type of edge-magic graph given by Kotzig and Rosa [4]. The usage of the word ‘super’ was introduced in [1]. There are many graphs that have been proved to be (super) edge- magic graphs; see for instance [8,9,11,12]. For further information about (super) edge – magic graphs, see [2]. The H- magic labeling of a plane graph was introduced by Lih [5].

In this paper, we study $C_m$–supermagic labeling of some classes of connected graphs such as some generalized fans, generalized friendship graphs and also provide a partial solution to the open problem found in [10]. This open problem is proved by Toru Kojima [13] in which $P_m \times G$ is $C_4$ – supermagic for $m \geq 2$ if G be a caterpillar. But we give the solution to the open problem in which $G \cong P_m \times P_n$ as in the open problem 2.1 [10].

\section*{2.C_m–supermagic graphs}

To generalize the ladder $P_m \times P_2$, we can always substitute $P_2$ with any path $P_n, \ n \geq 3$. The resulting graph is the grid $P_m \times P_n$ which is a graph with mn vertices and $(m-1) \ n + (n-1) \ m$ edges.

Now we define grid graph $G \cong P_m \times P_n$ as $V(G) = \{ u_{i,j} : 1 \leq i \leq m, \ 1 \leq j \leq n \}$ and $E(G) = \{ u_{i,j} u_{i,j+1} : 1 \leq i \leq m-1, \ 1 \leq j \leq n \} \cup \{ u_{i,j} u_{i+1,j} : 1 \leq i \leq m, \ 1 \leq j \leq n-1 \}$.

Ngurah et.al.[10] proved that for any integer $m \geq 3$ and $n = 3, 4, 5$ the grid $G \cong P_m \times P_n$ is $C_4$ – supermagic. They left the following as an open problem.

\textbf{Open Problem 2.1 [10].} Determine whether there is a $C_4$ – supermagic labeling of $P_m \times P_n$ for the remaining cases of m and n.

The following theorem gives a partial solution to the above open problem.

\textbf{Theorem 2.2.} For any positive integer $m \geq 3$ and $n = 6, 7$ the grid $G \cong P_m \times P_n$ is $C_4$ – supermagic.

\textbf{Proof.} We define a total labeling $h : V(G) \cup E(G) \rightarrow \{1,2,3,\ldots,3mn-n-m\}$.

Label the edges of $G$ in the following way:

$$h(u_{i,j} u_{i+1,j}) = 2mn + 1 - (m+n) + i - (j-1)(m-1), \quad \text{for} \ 1 \leq i \leq m-1 \ \text{and} \ 1 \leq j \leq n$$

For even $m$ and $1 \leq i \leq m$, 2
h(u_{ij}, u_{i,j+1}) = \begin{cases} 
 3mn + 1 - (m + n) - j - \frac{1}{2} (i - 1) (n - 1), & \text{for odd i and } 1 \leq j \leq n - 1, \\
 1 \cdot (5mn + 2 - m - 2n) - j - \frac{1}{2} (i - 1) (n - 1), & \text{for even i and } 1 \leq j \leq n - 1.
\end{cases}

For odd m and \( 1 \leq i \leq m \),
\[
\begin{align*}
\frac{1}{2} (5mn + 2 - m - 2n) - j - \frac{1}{2} (i - 1) (n - 1), & \quad \text{for odd } i \text{ and } 1 \leq j \leq n - 1, \\
3mn + 1 - (m + n) - j - \frac{1}{2} (i - 2) (n - 1), & \quad \text{for even } i \text{ and } 1 \leq j \leq n - 1.
\end{align*}
\]

To label the vertices of G, we consider two cases depending on the values of n.

The case of n = 6:

For even m and \( 1 \leq i \leq m \),
\[
\begin{align*}
\frac{1}{2} [ j + (n+1) ] m + \frac{1}{2} [ i - (m-1) ], & \quad \text{for odd } i \text{ and } j = 1,3,5, \\
\frac{1}{2} [ j + n ] m + \frac{1}{2} ( i - m ), & \quad \text{for even } i \text{ and } j = 1,3,5.
\end{align*}
\]

For odd m and \( 1 \leq i \leq m \),
\[
\begin{align*}
\frac{1}{2} ( j+2n ) m + \frac{1}{2} [ i - (m-1) ], & \quad \text{for odd } i \text{ and } j = 1,3,5, \\
\frac{1}{2} [ j + (n+1) ] m + \frac{1}{2} [ i - (m-1) ], & \quad \text{for even } i \text{ and } j = 1,3,5.
\end{align*}
\]

Let \( C_4^{(i,j)} \), \( 1 \leq i \leq m-1 \) and \( 1 \leq j \leq n-1 \) be the subcycle of G with \( V(C_4^{(i,j)}) = \{ u_{ij}, u_{i,j+1}, u_{i+1,j}, u_{i+1,j+1} \} \),
\( E(C_4^{(i,j)}) = \{ u_{ij}, u_{i,j+1}, u_{i+1,j}, u_{i+1,j+1} \} \).

It can be checked that for each \( 1 \leq i \leq m-1 \) and \( 1 \leq j \leq n-1 \), \( \Sigma h(C_4^{(i,j)}) = \}
\[
\begin{cases} 
10nm - m - 14, & \text{for even m}, \\
10nm - m - 11, & \text{for odd m}.
\end{cases}
\]

The case of n = 7:

\[
\begin{align*}
h(u_{ij}) = \begin{cases} 
i + \frac{1}{2} (j-2) m, & \text{for } 1 \leq i \leq m \text{ and } j = 2,4,6, \\
i + \frac{1}{2} (j+5) m, & \text{for } 1 \leq i \leq m \text{ and } j = 1,3,5,7.
\end{cases}
\end{align*}
\]

Let \( C_4^{(i,j)} \), \( 1 \leq i \leq m-1 \) and \( 1 \leq j \leq n-1 \) be the subcycle of G with \( V(C_4^{(i,j)}) = \{ u_{i,j}, u_{i+1,j}, u_{i+1,j+1} \} \),
\( E(C_4^{(i,j)}) = \{ u_{i,j}, u_{i+1,j}, u_{i+1,j+1} \} \).

It can be verified that for each \( 1 \leq i \leq m-1 \) and \( 1 \leq j \leq n-1 \), \( \Sigma h(C_4^{(i,j)}) = \}
\[
\begin{cases} 
10nm - 2m - 17, & \text{for even m}, \\
10nm - 2m - 14, & \text{for odd m}.
\end{cases}
\]

\( \). Hence G is \( C_4 \)-supermagic. \( \Box \)
**Open Problem 2.3.** Determine whether there is a $C_4$ – supermagic labeling of $P_m \times P_n$ for the remaining cases of $m$ and $n$.

The fan $F_n \cong P_n + K_1$ is a graph with $V(F_n) = \{c, x_i : 1 \leq i \leq n\}$ and $E(F_n) = \{x_i x_{i+1} : \text{for } 1 \leq i \leq n-1\} \cup \{c x_i : 1 \leq i \leq n\}$.

The generalized fan graph denoted by $F_{n,m}$ is a graph with $V(F_{n,m}) = \{c, x_i : 1 \leq i \leq n\} \cup \{v_{ij} : 1 \leq i \leq m-3, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor\}$ and $E(F_{n,m}) = \{x_i x_{i+1} : \text{for } 1 \leq i \leq n-1\} \cup \{v_{ij} x_k : i = 1, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor, k = 2j\} \cup \{v_{i+1} v_{ij} : 1 \leq i \leq m-4, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor\} \cup \{c v_{ij} : i = m-3, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor\} \cup \{c x_i : i = 1, 3, 5, \ldots, n\}$.

Note that the vertices $v_{ij}$ are introduced between the vertices $x_k$’s ($k = 2j, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$) and $c$.

Our next result shows that the generalized fan $F_{n,m}$ is $C_m –$ supermagic for $n = 3$ and $m \geq 3$, $m \neq 4$.

**Theorem 2.4.** For any integer $m \geq 3$, $m \neq 4$, the generalized fan $F_{3,m}$ is $C_m –$ supermagic.

**Proof.** Define a total labeling $h : V(F_{3,m}) \cup E(F_{3,m}) \rightarrow \{1, 2, 3, \ldots, 2m+3\}$ as follows:

We label the vertices of $F_{3,m}$ in the following way:

$$h(u) = \begin{cases} 
4, & \text{if } u = c, \\
\frac{1}{2}(i+1), & \text{if } u = x_i, \text{ for } i = 1, 3, \\
i + 1, & \text{if } u = x_i, \text{ for } i = 2.
\end{cases}$$

For the cases $m \geq 5$, we introduce additional vertices $v_i$ between $x_2$ and $c$.

The vertex labeling of $v_i$ and edge labeling are defined as follows:

$$h(u) = \begin{cases} 
4 + i, & \text{if } u = v_i, \text{ for } 1 \leq i \leq m-3, \\
m + 5, & \text{if } u = v_{1}x_2, \\
m + 6 + i, & \text{if } u = v_{i+1}v_i, \text{ for } 1 \leq i \leq m-4, \\
3m - i, & \text{if } u = cv_i, \text{ for } i = m-3, \\
m + 1 + i, & \text{if } u = x_{i}x_{i+1}, \text{ for } i = 1, 2, \\
m + 7 - i, & \text{if } u = cx_i, \text{ for } i = 1, 3.
\end{cases}$$

For $i = 1, 2$, let $C^{(i)}_m$ be the subcycle of $F_{3,m}$ with

$$V(C^{(i)}_m) = \{c, x_i, x_{i+1}\} \cup \{v_i : 1 \leq i \leq m-3\}$$ and

$$E(C^{(i)}_m) = \{x_i x_{i+1}\} \cup \{v_{1}x_2\} \cup \{v_k v_{k+1} : 1 \leq k \leq m-4\} \cup \{cv_i : r = m-3\} \cup \{cx_j : j = 2i - 1\}.$$
It can be checked that for $i = 1,2$, $r = m-3$, and $j = 2i - 1$,

$$\sum (C_m(i)) = h(c) + h(x_i) + h(x_{i+1}) + \sum_{i=1}^{i-1} h(v_k) + h(x_ix_{i+1}) + h(\sum_{i=1}^{r} (v_kv_{k+1})) + h(cv_i) + h(v_1x_2)$$

$$= 30 + (m-3) (11+2m).$$

Hence $F_{3,m}$ is $C_m$–supermagic. □

An illustration is given in Figure 1.

![Diagram of a C₅–supermagic labeling of $F_{3,5}$](image)

**Fig.1.** A C₅–supermagic labeling of $F_{3,5}$

**Open Problem 2.5.** Determine whether there are $C_m$–supermagic labeling of generalized fan graphs $F_{n,m}$ for the remaining cases of $m$ and $n$.

Let us define friendship graph as follows:

The friendship graph $F_n$ is a set of $n$ triangles having a common centre vertex, and otherwise disjoint.

Let $c$ denote the centre vertex. For the $i^{th}$ triangle, let $x_i$ and $y_i$ denote the other two vertices.

We define the generalized friendship graph $GF_n^m$ with vertex and edge sets by

- $V(GF_n^m) = \{x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{c\}$ and
- $E(GF_n^m) = \{x_{ij}x_{ij+1} : 1 \leq i \leq n, 1 \leq j \leq m-2\} \cup \{x_{ij}c : 1 \leq i \leq n, j = m-2\}$.

The following result is interesting because it characterizes $C_m$–supermagicness of generalized friendship graph $GF_n^m$.

**Theorem 2.6.** For any integer $m \geq 3$, the generalized friendship graph $GF_n^m$ is $C_m$–supermagic.

**Proof.** Suppose that a bijection $h : V(GF_n^m) \cup E(GF_n^m) \rightarrow \{1,2,3,\ldots,2mn+1\}$ is $C_m$–supermagic total labeling.
We label the vertices of $G_{F_n^m}$ in the following way:

The case of odd $n$.

For $i = 1, 2, 3, \ldots n$ and $j = 1, 2, 3, \ldots m-1$,

$$h(u) = \begin{cases} 
  i + \frac{1}{2}(j-1)(2n+1), & \text{if } u = x_{ij}, \text{ for } j = 1, 3 \\
  n(j-1) + i + 1, & \text{if } u = x_{ij}, \text{ for } j = 5, 7, \ldots, m-1, \\
  nj + 2i, & \text{if } u = x_{ij}, \text{ for } j = 2, 4, \ldots, m-1. \\
  n+1, & \text{if } u = c 
\end{cases}$$

The edge labeling for $x_{ij}, x_{ij+1}$, is given by,

For odd $m$, and for $i = 1, 2, 3, \ldots, n$,

$$h(u) = \begin{cases} 
  n(m + j - 2) + i + 1, & \text{if } u = x_{ij}x_{ij+1}, \text{ for } j = 1, 3, 5, \ldots, m-2. \\
  n(m + j - 1) - i + 2, & \text{if } u = x_{ij}x_{ij+1}, \text{ for } j = 2, 4, 6, \ldots, m-2. 
\end{cases}$$

For even $m$, and for $i = 1, 2, 3, \ldots, n$,

$$h(u) = \begin{cases} 
  n(m + j - 1) - i + 2, & \text{if } u = x_{ij}x_{ij+1}, \text{ for } j = 1, 3, 5, \ldots, m-2. \\
  n(m + j - 2) + i + 1, & \text{if } u = x_{ij}x_{ij+1}, \text{ for } j = 2, 4, 6, \ldots, m-2. 
\end{cases}$$

The edge labeling for $x_{ij}, c$, is given by,

$$h(u) = \begin{cases} 
  \frac{1}{2} [4mn - 5n + 2i + 3], & \text{if } u = x_{ij}c, \text{ for } 1 \leq i \leq \frac{n-1}{2}, \text{ for } j = 1, \\
  \frac{1}{2} [4mn - 7n + 2i + 3], & \text{if } u = x_{ij}c, \text{ for } \frac{n+1}{2} \leq i \leq n, \text{ for } j = 1, 
\end{cases}$$

The edge labeling for $x_{ij+1}, c$, is given by,

$$h(u) = \begin{cases} 
  2mn + 1 - i \text{, } n, & \text{if } u = x_{ij+1}c, \text{ for } 1 \leq i \leq \frac{n-1}{2}, \text{ for } j = m-2, \\
  2mn + 1 - i \text{, } n, & \text{if } u = x_{ij+1}c, \text{ for } \frac{n+1}{2} \leq i \leq n, \text{ for } j = m-2. 
\end{cases}$$

The case of even $n$.

We label the vertices of $G_{F_n^m}$ in the following way:

$$h(u) = \begin{cases} 
  i, & \text{if } u = x_{ij}, \text{ for } 1 \leq i \leq \frac{n}{2}, \text{ and } j = 1, \\
  i + 1, & \text{if } u = x_{ij}, \text{ for } \frac{n+1}{2} \leq i \leq n, \text{ and } j = 1, \\
  n(j-1) + i + 1, & \text{if } u = x_{ij}, \text{ for } j = 3, 5, 7, \ldots, m-1 \text{ and } 1 \leq i \leq n, \\
  nj + 2i, & \text{if } u = x_{ij}, \text{ for } j = 2, 4, 6, \ldots, m-1 \text{ and } 1 \leq i \leq n, \\
  n, & \text{if } u = c, \text{ if } n = 2, \\
  n + 2, & \text{if } u = c, \text{ if } n > 2. 
\end{cases}$$
The edge labeling for $x_{i,j}, x_{i,j+1}$ is given by,
For odd $m$, and for $i = 1, 2, 3, \ldots, n$,
$$h(u) = \left\{ \begin{array}{ll}
n (m + j - 2) + i + 1, & \text{if } u = x_{i,j} x_{i,j+1}, \text{ for } j = 1, 3, 5, \ldots, m-2. \\
n (m + j - 1) - i + 2, & \text{if } u = x_{i,j} x_{i,j+1}, \text{ for } j = 2, 4, 6, \ldots, m-3. \\
\end{array} \right.$$ 

For even $m$, and for $i = 1, 2, 3, \ldots, n$,
$$h(u) = \left\{ \begin{array}{ll}
n (m + j - 1) - i + 2, & \text{if } u = x_{i,j} x_{i,j+1}, \text{ for } j = 1, 3, 5, \ldots, m-3. \\
n (m + j - 2) + i + 1, & \text{if } u = x_{i,j} x_{i,j+1}, \text{ for } j = 2, 4, 6, \ldots, m-2. \\
\end{array} \right.$$ 

The edge labeling for $x_{i,j}c$ is given by,
$$h(u) = \left\{ \begin{array}{ll}
\frac{1}{2} [4mn - 5n + 2i + 2], & \text{if } u = x_{i,j}c, \text{ for } 1 \leq i \leq \frac{n}{2}, \text{ for } j = 1, \\
\frac{1}{2} [4mn - 7n + 2i + 2], & \text{if } u = x_{i,j}c, \text{ for } \frac{n}{2} + 1 \leq i \leq n, \text{ for } j = 1, \\
\end{array} \right.$$

The edge labeling for $x_{i,j+1}c$ is given by,
$$h(u) = \left\{ \begin{array}{ll}
2mn - n + 3 - 2i, & \text{if } u = x_{i,j+1}c, \text{ for } 1 \leq i \leq \frac{n}{2}, \text{ for } j = m-2, \\
2mn - 2i + 2, & \text{if } u = x_{i,j+1}c, \text{ for } \frac{n}{2} + 1 \leq i \leq n, \text{ for } j = m-2, \\
\end{array} \right.$$

For each $i$, $1 \leq i \leq n$, $k = m-1$, we have
$$\left( \sum_{j=1}^{i} h(x_{i,j}) \right) + h(c) + \left( \sum_{j=1}^{i} h(x_{i,j} x_{i,j+1}) \right) + h(x_{i,k}c) + h(x_{i,l}c) = \left\{ \begin{array}{ll}
\frac{1}{2} (27n + 15) + (m-3) [12n + 3 + (m-4) 2n] \text{ for odd } n, \\
(13n+8) + (m-3) [12n + 3 + (m-4)2n] \text{ for even } n. \\
\end{array} \right.$$ 

This completes the proof. □

References


