



Cube Graceful Labeling of Graphs

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Abstract

In this paper, the cube graceful labeling of the path, the star, bistar, coconut tree, comb, the subdivision of the edges of stars, the subdivision of the central edge of the bistar, the subdivision of the edges of stars of the bistar and the subdivision of the edges of the bistar graphs are studied.

Key Words: Cube graceful labeling, cube graceful graph.

1. Introduction

The graphs considered in this paper are finite, undirected and without loops or multiple edges. Let $G = (V, E)$ be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [3].

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions.

There are several types of graph labeling and a detailed survey is found in [1].

Rosa [5] introduced β -valuation of a graph and Golomb [2] called it as graceful labeling. Several authors worked on graceful labeling, odd graceful labeling, even graceful labeling, super graceful labeling and skolem-graceful labeling.

Recently the concept of square graceful labeling was introduced by Tharmaraj *et al.* [6], [7]. They studied the square graceful labeling of various graphs. In [4] square graceful labeling of some other graphs is studied.

In this paper, cube graceful labeling is introduced. Cube graceful labeling of P_n , $K_{1,n}$, $B_{m,n}$, $mK_{1,n}$, coconut tree, $P_n \circ K_1$, the subdivision of the edges of stars $K_{1,n}$, the

subdivision of the central edge of the bistar $B_{m,n}$, the subdivision of the edges of stars of the bistar $B_{m,n}$ and the subdivision of the edges of the bistar $B_{m,n}$ graphs are studied.

The following definitions are necessary for the present study.

Definition 1.1: The coconut tree graph is obtained by identifying the central vertex of $K_{1,n}$ with a pendant vertex of the path P_m .

Definition 1.2: A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions.

2. Square Graceful Graphs

Definition 1.3: A (p, q) graph $G = (V, E)$ is said to be a cube graceful graph if there exists an injective function $f : V(G) \rightarrow \{0, 1, 2, 3, \dots, q^3\}$ such that the induced mapping $f_p : E(G) \rightarrow \{1^3, 2^3, 3^3, \dots, q^3\}$ defined by $f_p(uv) = |f(u) - f(v)|$ is an injection. The function f is called a cube graceful labeling of G .

3. Main Results

Observation 1.4: In any cube graceful graph G , the vertices with labels 0 and q^3 are always adjacent.

Proof: To get the edge label q^3 , 0 and q^3 must be the labels of the adjacent vertices.

Theorem 1.5: P_n is cube graceful for all $n > 1$.

Proof: Let $V(P_n) = \{u_i, 1 \leq i \leq n\}$ and

$$E(P_n) = \{u_i u_{i+1}, 1 \leq i \leq n-1\}$$

Then P_n has n vertices and $(n-1)$ edges.

Let $f : V(P_n) \rightarrow \{0, 1, 2, \dots, (n-1)^3\}$ be defined as follows

$$f(u_1) = 0$$

$$f(u_i) = \begin{cases} f(u_{i-1}) + [n+1-i]^3 & \text{when } i \text{ is even} \\ f(u_{i-1}) - [n+1-i]^3 & \text{when } i \text{ is odd} \end{cases} \quad 2 \leq i \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(u_i u_{i+1}) = (n-i)^3, 1 \leq i \leq n-1$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (n-1)^3$, which are distinct. Hence, the path P_n is cube graceful for all $n > 1$.

Theorem 1.6: $K_{1,n}$ is cube graceful for all n .

Proof: Let $V(K_{1,n}) = \{v, v_i, 1 \leq i \leq n\}$ and

$$E(K_{1,n}) = \{vv_i; 1 \leq i \leq n\}$$

Then $K_{1,n}$ has $(n+1)$ vertices and n edges.

Let $f : V(K_{1,n}) \rightarrow \{0, 1, 2, \dots, n^3\}$ be defined as follows

$$f(v) = 0$$

$$f(v_i) = i^3; 1 \leq i \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(vv_i) = i^3; 1 \leq i \leq n$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, n^3$, which are distinct. Hence, the star $K_{1,n}$ is cube graceful for all n .

Theorem 1.7: $B_{m,n}$ is cube graceful for all $m, n \geq 1$.

Proof: Let $V(B_{m,n}) = \{u, u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$E(B_{m,n}) = \{uu_i, uv, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then $B_{m,n}$ has $(m + n + 2)$ vertices and $(m + n + 1)$ edges.

Let $f : V(B_{m,n}) \rightarrow \{0, 1, 2, \dots, (m + n + 1)^3\}$ be defined as follows

$$f(u) = 0$$

$$f(u_i) = [m + n + 1 - (i - 1)]^3; 1 \leq i \leq m$$

$$f(v) = (n + 1)^3$$

$$f(v_j) = f(v) - (n + 1 - j)^3; 1 \leq j \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(uu_i) = [m + n + 1 - (i - 1)]^3; 1 \leq i \leq m$$

$$f^*(uv) = (n + 1)^3$$

$$f^*(vv_j) = [n - (j - 1)]^3; 1 \leq j \leq n$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (m + n + 1)^3$, which are distinct. Hence, the bistar $B_{m,n}$ is cube graceful for all $m, n \geq 1$.

Theorem 1.8: $mK_{1,n}$ is cube graceful for all m and n .

Proof: Let $V(mK_{1,n}) = \{v_i, v_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$E(mK_{1,n}) = \{v_i v_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then G has $m(n + 1)$ vertices and mn edges.

Let $f : V(G) \rightarrow \{0, 1, 2, \dots, (mn)^3\}$ be defined as follows

$$f(v_i) = i - 1, 1 \leq i \leq m$$

$$f(v_{ij}) = \{[m - (i - 1)]n - (j - 1)\}^3 + (i - 1); 1 \leq i \leq m, 1 \leq j \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(v_i v_{ij}) = \{[m - (i - 1)]n - (j - 1)\}^3; 1 \leq i \leq m, 1 \leq j \leq n$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (mn)^3$, which are distinct. Hence, the graph $mK_{1,n}$ is a cube graceful graph for all values of m and n .

Theorem 1.9: Coconut tree is cube graceful for all $n \geq 1$ and $m > 1$.

Proof: Let G be the coconut tree.

Let $V(G) = \{v, v_i, u_j; 1 \leq i \leq n, 2 \leq j \leq m\}$ and

$$E(G) = \{vv_i, vu_2, u_j u_{j+1}; 1 \leq i \leq n, 2 \leq j \leq m-1\}$$

Then G has $(n+m)$ vertices and $(n+m-1)$ edges.

Let $f: V(G) \rightarrow \{0, 1, 2, \dots, (n+m-1)^3\}$ be defined as follows

$$f(v) = 0$$

$$f(v_i) = (n+m-i)^3, 1 \leq i \leq n$$

$$f(u_2) = (m-1)^3$$

$$f(u_j) = \begin{cases} f(u_{j-1}) + [m-1-(j-2)]^3 & \text{when } j \text{ is even} \\ f(u_{j-1}) - [m-1-(j-2)]^3 & \text{when } j \text{ is odd} \end{cases} \quad 3 \leq j \leq m$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(vv_i) = (n+m-i)^3, 1 \leq i \leq n$$

$$f^*(vu_2) = (m-1)^3$$

$$f^*(u_j u_{j+1}) = (m-j)^3, 2 \leq j \leq m-1$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (n+m-1)^3$, which are distinct. Hence, the coconut tree is cube graceful for all $n \geq 1$ and $m > 1$.

Example: The Cube Graceful Labeling of the Coconut Tree Graph $T(3, 5)$ is Given in Figure 1

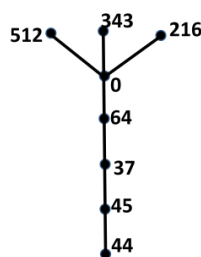


Figure 1: Coconut Tree Graph $T(3, 5)$

Theorem 1.10: $P_n \odot K_1$ is cube graceful for all $n \geq 2$.

Proof: Let $V(P_n \odot K_1) = \{u_i, v_i; 1 \leq i \leq n\}$ and

$$E(P_n \odot K_1) = \begin{cases} u_i u_{i+1}, & 1 \leq i \leq n-1 \\ u_i v_i, & 1 \leq i \leq n \end{cases}$$

Then $P_n \odot K_1$ has $2n$ vertices and $(2n - 1)$ edges.

Let $f : V(P_n \odot K_1) \rightarrow \{0, 1, 2, \dots, (2n - 1)^3\}$ be defined as follows

$$f(u_i) = \left[\frac{i(i-1)}{2} \right]^2, \quad 1 \leq i \leq n$$

$$f(v_i) = f(u_i) + [2n - 1 - (i - 1)]^3, \quad 1 \leq i \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(u_i u_{i+1}) = i^3, \quad 1 \leq i \leq n - 1$$

$$f^*(u_i v_i) = [2n - 1 - (i - 1)]^3, \quad 1 \leq i \leq n$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (2n - 1)^3$, which are distinct. Hence, $P_n \odot K_1$ is cube graceful for all $n \geq 2$.

Theorem 1.11: The graph obtained by the subdivision of the edges of the star $K_{1,n}$ is cube graceful for all n .

Proof: Let $V(G) = \{v, v_i, u_i; 1 \leq i \leq n\}$ and

$$E(G) = \{vv_i, v_i u_i; 1 \leq i \leq n\}$$

Then G has $(2n + 1)$ vertices and $2n$ edges.

Let $f : V(G) \rightarrow \{0, 1, 2, \dots, (2n)^3\}$ be defined as follows

$$f(v) = 0$$

$$f(v_i) = (2n + 1 - i)^3; \quad 1 \leq i \leq n$$

$$f(u_i) = n(7n^2 - 9ni + 9n + 3i^2 - 6i + 3), \quad 1 \leq i \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(vv_i) = [2n - (i - 1)]^3, \quad 1 \leq i \leq n$$

$$f^*(v_i u_i) = [n - (i - 1)]^3, \quad 1 \leq i \leq n$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (2n)^3$, which are distinct. Hence, the graph obtained by the subdivision of the edges of the star $K_{1,n}$ is cube graceful for all n .

Theorem 1.12: The graph obtained by the subdivision of the central edge of the bistar $B_{m,n}$ is a cube graceful graph.

Proof: Let $V(G) = \{u_i, u, w, v, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$E(G) = \{uu_i, uw, wv, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then G has $(m + n + 3)$ vertices and $(m + n + 2)$ edges.

Let $f : V(G) \rightarrow \{0, 1, 2, \dots, (m + n + 2)^3\}$ be defined as follows

$$f(w) = 0$$

$$f(u) = (m + n + 2)^3$$

$$f(v) = (m + n + 1)^3$$

$$f(u_i) = f(u) - [m + n - (i - 1)]^3, 1 \leq i \leq m$$

$$f(v_j) = f(v) - [n - (j - 1)]^3, 1 \leq j \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(uw) = (m + n + 2)^3$$

$$f^*(wv) = (m + n + 1)^3$$

$$f^*(uu_i) = [m + n - (i - 1)]^3, 1 \leq i \leq m$$

$$f^*(vv_j) = [n - (j - 1)]^3, 1 \leq j \leq n$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (m + n + 2)^3$, which are distinct. Hence, the graph obtained by the subdivision of the central edge of the bistar $B_{m,n}$ is a cube graceful graph.

Example: The cube graceful labeling of the subdivision of the central edge of the bistar $B_{m,n}$ is given in Figure 2

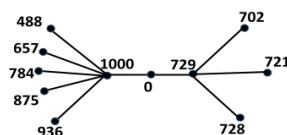


Figure 2: Subdivision of the Central Edge of the Bistar $B_{5,3}$

Theorem 1.13: The graph obtained by the subdivision of the edges of the star of the bistar $B_{m,n}$ is a cube graceful graph.

Proof: Let $V(G) = \{x_i, u_i, u, v, v_j, w_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and

$$E(G) = \{uu_i, u_i x_i, uv, vv_j, v_j w_j; 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then G has $(2m + 2n + 2)$ vertices and $(2m + 2n + 1)$ edges.

Let $f : V(G) \rightarrow \{0, 1, 2, \dots, (2m + 2n + 1)^3\}$ be defined as follows

$$f(u) = 0$$

$$f(v) = 1$$

$$f(u_i) = [2m + 2n + 1 - (i - 1)]^3, 1 \leq i \leq m$$

$$f(v_j) = [m + 2n + 1 - (j - 1)]^3 + 1, 1 \leq j \leq n$$

$$f(w_j) = f(v_j) - [m + n + 1 - (j - 1)]^3, 1 \leq j \leq n$$

$$f(x_i) = f(u_i) - [m + 1 - (i - 1)]^3, 1 \leq i \leq m$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(uv) = 1$$

$$f^*(uu_i) = [2m + 2n + 1 - (i - 1)]^3, 1 \leq i \leq m$$

$$f^*(vv_j) = [m + 2n + 1 - (j - 1)]^3, 1 \leq j \leq n$$

$$f^*(v_j w_j) = [m + n + 1 - (j - 1)]^3, 1 \leq j \leq n$$

$$f^*(u_i x_i) = [m + 1 - (i - 1)]^3, 1 \leq i \leq m$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (2m + 2n + 1)^3$, which are distinct. Hence, the graph obtained by the subdivision of the edges of the star of the bistar $B_{m,n}$ is a cube graceful graph.

Theorem 1.14: The graph obtained by the subdivision of the edges of the bistar $B_{m,n}$ is a cube graceful graph.

Proof: Let $V(G) = \{x, v, z_j, u, w_i, u_i, v_j; 1 \leq j \leq n, 1 \leq i \leq m\}$ and

$$E(G) = \{xv, vz_j, uw_i, xu, u_i w_i, z_j v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$$

Then G has $(2m + 2n + 3)$ vertices and $(2m + 2n + 2)$ edges.

Let $f : V(G) \rightarrow \{0, 1, 2, \dots, (2m + 2n + 2)^3\}$ be defined as follows

$$f(x) = 1$$

$$f(v) = 0$$

$$f(z_j) = [2m + 2n + 2 - (j - 1)]^3; 1 \leq j \leq n$$

$$f(u) = 8$$

$$f(w_i) = f(u) + [2m + n + 2 - (i - 1)]^3, 1 \leq i \leq m$$

$$f(u_i) = f(w_i) - [m + n + 2 - (i - 1)]^3, 1 \leq i \leq m$$

$$f(v_j) = f(z_j) - [n + 2 - (j - 1)]^3, 1 \leq j \leq n$$

Let f^* be the induced edge labeling of f . Then,

$$f^*(xv) = 1$$

$$f^*(xu) = 8$$

$$f^*(vz_j) = [2m + 2n + 2 - (j - 1)]^3, 1 \leq j \leq n$$

$$f^*(uw_i) = [2m + n + 2 - (i - 1)]^3, 1 \leq i \leq m$$

$$f^*(w_iu_i) = [m + n + 2 - (i - 1)]^3, 1 \leq i \leq m$$

$$f^*(z_jv_j) = [n + 2 - (j - 1)]^3, 1 \leq j \leq n$$

The induced edge labels are $1^3, 2^3, 3^3, \dots, (2m + 2n + 2)^3$, which are distinct. Hence, the graph obtained by the subdivision of the edges of the bistar $B_{m,n}$ is a cube graceful graph.

4. Conclusion

In this paper, cube graceful labeling is studied. Some non-cube graceful graphs are to be observed. Union of cube graceful graph labeling of various graphs may be studied.

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References

- [1] J.A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, 15, #DS6, 2008.N.Y and Dunod Paris, (1967) 349-355.

- [2] S.W. Golomb, How to number a graph in Graph theory and Computing, R.CRead. ed., Academic Press Newyork, (1972) 23-37.
- [3] F. Harary, Graph Theory, Narosa Publishing House, New Delhi, 2001.
- [4] K. Murugan, Square graceful labeling of some graphs, International Journal of Innovative Research in Science, Engineering and Technology, Vol. 4, Issue 2, (2015), 511-520.
- [5] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (International Symposium, Rome, July 1966), Gordon & Breach, N.Y and Dounod Parus (1967), 349-355.
- [6] Tharmaraj and P.B.Sarasija, Square graceful graphs, International Journal of Mathematics and Soft Computing. Vol 4, No.1, (2014) 129-137.
- [7] T.Tharmaraj and P.B.Sarasija, Some square graceful graphs, International Journal of Mathematics and Soft Computing. Vol.5, No.1, (2015) 119-127.

