



Some Results On Strong Efficient Open Domination

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Abstract: Let $G = (V, E)$ be a simple, undirected, finite graph without isolated vertices. A subset D of $V(G)$ is a *Strong(Weak) efficient open dominating set* of G if $|N_s(v) \cap D| = 1$ ($|N_w(v) \cap D| = 1$), for every $v \in V(G)$ where $N_s(v)$ and $N_w(v)$ are strong and weak neighborhood respectively. A graph G is *strong(weak) efficient open dominating graph* if it contains a strong(weak) efficient open dominating set. In this paper we determine the Nordhaus - Gaddum type results, strong efficient open domatic partition, and some operations on strong efficient open domination.

Key words: Strong(weak) Efficient Open Domination, Nordhaus-Gaddum type results, Domatic Partition.

1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph without isolated vertices. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and *closed neighborhood* of $v \in V$ is $N[v] = N(v) \cup \{v\}$. If v is a vertex of $V(G)$, then the degree of v is defined by the cardinality of $N(v)$ and is denoted by $degv$. The minimum and maximum degree of the vertices of G is denoted by δ and Δ respectively. For graph theoretic terminology, we refer to [3] and [5]. The strong neighborhood and weak neighborhood of a vertex $v \in V(G)$ are defined by $N_s(v) = \{u \in V(G) | uv \in E(G) \text{ and } degu \geq degv\}$ and $N_w(v) = \{u \in V(G) | uv \in E(G) \text{ and } degu \leq degv\}$ respectively. A subset D of V is a *dominating set* of G if every vertex in $V - D$ is adjacent to atleast one vertex in D . The *domination number*, $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . In [10], Prof. E. Sampathkumar and L. Pushpalatha have defined strong(weak) domination in graphs. A subset D of V is called a *strong(weak) dominating set* of G if for every vertex, v in $V - D$ there exists $u \in D$ such that $uv \in E(G)$ and $degv \geq degu$ ($degv \leq degu$). The *strong(weak) domination number*, $\gamma_s(G)$ ($\gamma_w(G)$) of G is the minimum cardinality of a strong(weak) dominating set of G . D.W.Bange et al [4] defined an efficient dominating set D as a set of vertices of a graph G such that $|N[v] \cap D| = 1$, for every $v \in V(G)$ and N. Meena et al [9] extend this into strong(weak) efficient dominating set D as a set of vertices of a graph G such that $|N_s[v] \cap D| = 1$ ($|N_w[v] \cap D| = 1$), for every $v \in V(G)$. The *strong(weak) efficient domination number*, $\gamma_{se}(G)$ ($\gamma_{we}(G)$) of G is the minimum cardinality of a strong(weak) efficient dominating set of G .

A subset D of V is a *total dominating set* of G if every vertex in V is adjacent to atleast one vertex in D . The *total domination number*, $\gamma_t(G)$ of G is the minimum

cardinality of a total dominating set of G . Gavlas and Schultz are defined an efficient open dominating set D as a set of vertices of a graph G such that $|N(v) \cap D| = 1$, for every $v \in V(G)$ in [7]. The *efficient open domination number*, $\gamma_{te}(G)$ of G is the cardinality of a efficient open dominating set of G . A study of domination, total domination and efficient open domination in graphs and its advanced topics are given in [2, 6, 8, 11]. We introduced a strong(weak) efficient open domination in graphs [1]. In this paper we determine the Nordhaus - Gaddum type results, strong efficient open domatic partition, and some operations on strong efficient open domination.

2 Strong(Weak) Efficient Open Domination

Definition 2.1 [1]

A subset D of $V(G)$ is called a *strong efficient open dominating set* (or *SEOD set*, for short) of G if $|N_s(v) \cap D| = 1$, for every $v \in V(G)$. A subset D of $V(G)$ is called a *weak efficient open dominating set* (or *WEOD set*, for short) of G if $|N_w(v) \cap D| = 1$, for every $v \in V(G)$.

The strong (weak) efficient open domination number, denoted by $\gamma_{ste}(G)$ ($\gamma_{wte}(G)$), is the minimum cardinality of a strong (weak) efficient open dominating set of G . We also call the corresponding set that γ_{ste} (γ_{wte}) - set of G .

A graph G is called a strong (weak) efficient open dominating graph or SEOD (WEOD) graph if it contains a strong(weak) efficient open dominating set. Also says that G is strong (weak) efficient open dominatable.

3 Nordhaus - Gaddum Type Results

Lemma 3.1

Let G be a strong efficient open dominating graph with non increasing degree sequence $(d_1, d_2, \dots, d_n)$ and the complement $\bar{G}$ of G is strong efficient open dominatable then G is either regular or

$$d_i = \begin{cases} \frac{n}{2} & \text{if } i \leq t \\ \frac{n}{2} - 1 & \text{if } i > t \end{cases} \quad \text{where both } n \text{ and } t > 0 \text{ are even.}$$

Proof:

Let G and its complement $\bar{G}$ be strong efficient open dominating graph. Then $\Delta(G) \leq \frac{n}{2}$ and $\Delta(\bar{G}) \leq \frac{n}{2}$. Hence $\Delta(G) \leq \frac{n}{2}$ and $\delta(\bar{G}) \geq n - 1 - \frac{n}{2} = \frac{n}{2} - 1$. Therefore, $\frac{n}{2} - 1 \leq \deg v \leq \frac{n}{2}$, for all $v \in V$. If n is odd, then G is $\frac{n-1}{2}$ - regular graph. If n is even, $\deg v$ is either $\frac{n}{2}$ or $\frac{n}{2} - 1$, for all $v \in V$. Hence G may be regular.

Otherwise, let t be an positive integer such that t number of vertices have a degree $\frac{n}{2}$ and $n - t$ vertices have a degree $\frac{n}{2} - 1$ in G . Since n is even, either $\frac{n}{2}$ or $\frac{n}{2} - 1$ is odd. Suppose if $\frac{n}{2}$ is odd, then t is even and also $n - t$ is even. Suppose if $\frac{n}{2}$ is even, then $\frac{n}{2} - 1$ is odd. Therefore $n - t$ is even implies t is even.

$$\therefore d_i = \begin{cases} \frac{n}{2} & \text{if } i \leq t \\ \frac{n}{2} - 1 & \text{if } i > t \end{cases}$$

Theorem 3.2

A graph G and its complement $\bar{G}$ are connected strong efficient open dominating graphs then both G and $\bar{G}$ are efficient open dominatable. Moreover, $\gamma_{ste}(G) = \gamma_{ste}(\bar{G}) = 2$.

Proof:

Let a graph G and its complement, $\bar{G}$ be strong efficient open dominating graph. Then if G is regular, then clearly, both G and $\bar{G}$ are efficient open dominatable. otherwise if

$$d_i = \begin{cases} \frac{n}{2} & \text{if } i \leq t \\ \frac{n}{2} - 1 & \text{if } i > t \end{cases}, \text{ where both positive integer } n \text{ and } t \text{ are even. Since } \Delta = \frac{n}{2}, \text{ then}$$

there exists adjacent maximum degree vertices u, v in $V(G)$ such that $\{u, v\}$ is a strong efficient open dominating set of G . Hence $\gamma_{ste}(G) = 2$. Similarly, $\gamma_{ste}(\bar{G}) = 2$.

Hence both G and $\bar{G}$ are efficient open dominatable.

Theorem 3.3

For a connected graph G , $4 \leq \gamma_{ste}(G) + \gamma_{ste}(\bar{G}) \leq \frac{3n}{2}$;

$4 \leq \gamma_{ste}(G) \cdot \gamma_{ste}(\bar{G}) \leq \frac{n^2}{2}$. The upper limit is true for C_4 and the lower limit is sharp for a connected graph, $\bar{G}$.

Proof:

Since both G and $\bar{G}$ are strong efficient open dominating graph,

$$\therefore \gamma_{ste}(G) \geq 2 \text{ and } \gamma_{ste}(\bar{G}) \geq 2$$

$$\Rightarrow \gamma_{ste}(G) + \gamma_{ste}(\bar{G}) \geq 4 \text{ and } \gamma_{ste}(G) \cdot \gamma_{ste}(\bar{G}) \geq 4.$$

Also, for upper limit, $\gamma_{ste}(G) \leq \frac{n}{2}$ and $\gamma_{ste}(\bar{G}) \leq \frac{n}{2}$, for a connected $\bar{G}$.

Suppose if $\bar{G}$ is not connected and contains K_2 as a component then, $\gamma_{ste}(\bar{G}) \leq n \therefore$

$$\gamma_{ste}(G) + \gamma_{ste}(\bar{G}) \leq \frac{n}{2} + n = \frac{3n}{2} \text{ and } \gamma_{ste}(G) \cdot \gamma_{ste}(\bar{G}) \leq \frac{n}{2} \cdot n = \frac{n^2}{2}.$$

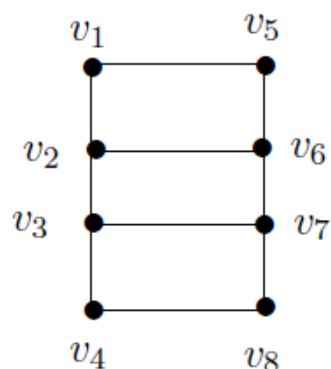
Further, $\gamma_{ste}(C_4) = 2$ and $\gamma_{ste}(\bar{C}_4) = 4$. Hence the upper limit is sharp for C_4 . By theorem (3.2), the lower limit is sharp for a connected graph, $\bar{G}$.

4 Domatic Number in Strong Efficient Open Dominating Graphs

Definition 4.1

A **strong efficient open domatic partition** is a partition of the vertices of a graph into disjoint strong efficient open dominating sets. The maximum number of disjoint strong efficient open dominating sets in a domatic partition of a graph G is called its **strong efficient open domatic number** and is denoted by $d_{ste}(G)$.

Example 4.2 Consider the following graph G



There exists two strong efficient open dominating sets D_1 and D_2 , where $D_1 = \{v_1, v_2, v_3, v_4\}$ and $D_2 = \{v_5, v_6, v_7, v_8\}$. Also, $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = V(G)$. Hence $d_{ste}(G) = 2$.

Proposition 4.3

For a cycle C_{4n} , ($n \geq 1$), $d_{ste}(C_{4n}) = 2$.

Proof:

Let $V(C_{4n}) = \{x_1, x_2, \dots, x_{4n}\}$. It is clear that the γ_{ste} -set of C_{4n} are $\{x_1, x_2, x_5, x_6, \dots, x_{4n-3}, x_{4n-2}\}$ and $\{x_3, x_4, x_7, x_8, \dots, x_{4n-1}, x_{4n}\}$. Hence, clearly $d_{ste}(C_{4n}) = 2, n \geq 1$.

Proposition 4.4

For a cycle $K_{m,m}$, ($m \geq 1$), $d_{ste}(K_{m,m}) = m$.

Proof:

Let (X, Y) be the partition of the vertex set of $K_{m,m}$, where $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ the distinct γ_{ste} -sets of $K_{m,m}$ are $\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_m, y_m\}$. Hence, $d_{ste}(K_{m,m}) = m$.

Theorem 4.5

For any strong efficient open dominating graph G with $\delta > 1$, $d_{ste}(G) = 1$.

Proof:

Let G be a strong efficient open dominating graph with $\delta = 1$. Then all the strong efficient open dominating set of G must contains the supporting vertices of G . Hence, $d_{ste}(G) = 1$.

Theorem 4.6

For any strong efficient open dominating graph G with exactly two maximum degree vertices,

$$d_{ste}(G) = 1.$$

Proof:

Let G be a strong efficient open dominating graph with exactly two maximum degree vertices v_1 and v_2 . Then v_1 and v_2 must be adjacent. Also, $N_s(v_1) = \{v_2\}$ and $N_s(v_2) = \{v_1\}$. Then every strong efficient open dominating set of G must contain the vertices $\{v_1, v_2\}$. Hence $d_{ste}(G) = 1$

5 Operations on Strong Efficient Open Domination

In this section, we discuss about the strong efficient open domination in the union, $G_1 \cup G_2$, cartesian product $G_1 \blacksquare G_2$, and direct product $G_1 \times G_2$ of the graphs G_1 and G_2 .

Proposition 5.1

A graph $G_1 \cup G_2$ is need not to be a strong efficient open dominating graph if G_1 and G_2 is strong efficient open dominatable.

Example 5.2

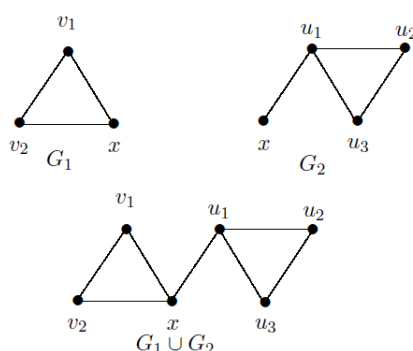
- P_2 and P_4 are strong efficient open dominating graph with a common unique end vertex. Then clearly, $P_2 \cup P_4 = P_5$, is not a strong efficient open dominating graph.
- $D_{r,r}$ and P_2 are strong efficient open dominating graph with a common unique end vertex. Then clearly, $D_{r,r} \cup P_2$, is not a strong efficient open dominating graph.

Remark 5.3

A graph $G_1 \cup G_2$ is a strong efficient open dominating graph for a non strong efficient open dominating graph G_1 and G_2 .

Example 5.4

Consider the following graphs G_1 and G_2 .



Clearly, both G_1 and G_2 are not a strong efficient open dominatable whereas $G_1 \cup G_2$ is a strong efficient open dominating graph.

Theorem 5.5

Let G_1 and G_2 be any two strong efficient open dominating graph with a γ_{ste} - set D_1 and D_2 respectively. Then $G_1 \cup G_2$ is a strong efficient open dominatable if $V(G_1) \cap V(G_2)$ contains the adjacent pair of maximum degree vertices in both $\langle D_1 \rangle$ and $\langle D_2 \rangle$.

Proof:

Let G_i be a strong efficient open dominating graph with γ_{ste} - set D_i , ($i = 1, 2$) and let x, y be the adjacent maximum degree vertices of both G_1 and G_2 with an edge. Then $x, y, xy \in \langle D_1 \rangle \cap \langle D_2 \rangle$. Then $deg_{G_1 \cup G_2} x = deg_{G_1 \cup G_2} y = \Delta(G_1 \cup G_2) = \Delta(G_1) + \Delta(G_2)$. Hence, $G_1 \cup G_2$ is a strong efficient open dominating graph with $\gamma_{ste}(G_1 \cup G_2) = \gamma_{ste}(G_1) + \gamma_{ste}(G_2) - 2$.

Note 5.6

A graph $G_1 \cup G_2$ is strong efficient open dominating graph if either $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$

Remark 5.7

A join sum of two complete graph, G has exactly two maximum degree vertices u, v (say) with a bipartition $N(u)$ and $N(v)$ of $V(G)$. Hence G is strong efficient open dominatable with $\gamma_{ste}(G) = 2$.

Proposition 5.8

A graph $G_1 \blacksquare G_2$ is need not to be a strong efficient open dominating graph if G_1 and G_2 is strong efficient open dominatable.

Example 5.9

- $P_4 \blacksquare P_8$ and $P_4 \blacksquare P_4$ are strong efficient open dominatable.
- $P_4 \blacksquare P_{12}$ and $P_8 \blacksquare P_8$ are not a strong efficient open dominatable

Remark 5.10

A graph $G_1 \times G_2$ is need not to be a strong efficient open dominating graph if G_1 and G_2 is strong efficient open dominatable.

Theorem 5.11

For a strong efficient open dominating paths P_{n_1} and P_{n_2} , $P_{n_1} \times P_{n_2}$ is a strong efficient open dominating graph.

Proof:

Let P_{n_1} and P_{n_2} be the distinct strong efficient open dominating paths with the vertex set $V(P_{n_1}) = \{u_1, u_2, \dots, u_{n_1}\}$ and $V(P_{n_2}) = \{v_1, v_2, \dots, v_{n_2}\}$ respectively. Then $V(P_{n_1} \times P_{n_2}) = \{(u_i, v_j) | 1 \leq i \leq n_1; 1 \leq j \leq n_2\}$

Let $Y_i = \{2, 3, 6, 7, \dots, 4k - 2, 4k - 3, \dots, n_i - 2, n_i - 1\}, (i = 1, 2)$.

Then $D = \{(u_{y_1}, v_{y_2}) | y_1 \in Y_1 \text{ and } y_2 \in Y_2\}$ form a strong efficient open dominating set

of $P_{n_1} \times P_{n_2}$. Hence proved.

Remark 5.12

Clearly all the vertices of D , defined in above theorem, are equal in degree ($\deg v = 4$, for all $v \in D$) and $\langle D \rangle = mK_2$, ($m > 1$). Also the strong efficient open dominating set with less than $|D|$ can not be found.

$$\therefore \gamma_{ste}(P_{n_1} \times P_{n_2}) = \gamma_{ste}(P_{n_1}) \cdot \gamma_{ste}(P_{n_2}).$$

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