



COMMON FIXED POINT THEOREMS IN C^* -ALGEBRA-VALUED S -METRIC SPACES

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ABSTRACT. In this paper, we establish common fixed point theorems for two mappings in C^* -algebra-valued S -metric spaces which satisfy new contractive conditions. We prove a common fixed point theorem for Banach contraction principle in this space.

1. INTRODUCTION

Fixed point theory has attracted many researchers since 1922 with the admired Banach fixed point theorem. This theorem supplies a method for solving a variety of applied dilemma in mathematical sciences and engineering. The study of common fixed points of compatible mappings has emerged as an area of vigorous research activity ever since Jungck [4] introduced the notion of compatible mappings. Compatibility is useful mainly in the study of common fixed points of contractive type mapping pairs and then requires the assumption of continuity and completeness.

Gähler [3] and Dhage [2] introduced the concepts of 2-metric spaces and D -metric spaces, respectively, Mustafa and Sims [9] introduced a new structure of generalized metric spaces which are called G -metric spaces as a generalization of metric spaces. Further, Sedghi, Shobe and Zhou [10] introduced the notion of a D^* -metric space as an improved version of a Dhage D -metric space. Later, he examined the shortcomings of both G -metric and D^* -metric spaces and gave the concept of a new generalized metric space called a S -metric space [11] and investigated some of their properties. Also discussed a fixed point theorem for self mapping on a complete S -metric space.

Now, we present some necessary concepts and results in C^* -algebra [1, 8]. Suppose that $\mathcal{A}$ is an unital algebra. An involution on $\mathcal{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathcal{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. The pair $(\mathcal{A}, *)$ is called a $*$ -algebra. A Banach $*$ -algebra is a $*$ -algebra $\mathcal{A}$ together with a complete sub-multiplicative norm such that $\|a^*\| = \|a\|$ ($\forall a \in \mathcal{A}$). A C^* -algebra is a Banach $*$ -algebra such that $\|a^*a\| = \|a\|^2$ [8]. It is clear that under the norm topology, $L(H)$, the set of all bounded linear operators on a Hilbert space H , is a C^* -algebra. In [6], Ma established the notion of

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C^* - algebra-valued metric spaces and proved some fixed point theorems for self maps with contractive or expansive mappings. The main idea consists in using the set of all positive elements of a unital C^* - algebra instead of the set of real numbers. Further in [7], introduced a concept of C^* - algebra-valued b-metric spaces which generalizes the concept of C^* -algebra valued metric spaces. In [5], the authors introduced the concept of C^* -algebra valued S -metric spaces and presented some fixed point results for mappings under contractive conditions in these spaces.

In this Paper, we establish coincidence fixed point theorems for two mappings in C^* -algebra-valued S -metric spaces which satisfy new contractive conditions. We prove a common fixed point theorem for Banach contraction principle in this space.

2. PRELIMINARIES

Throughout this paper, $\mathcal{A}$ will denote an unital C^* -algebra. Set $\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$. We call an element $a \in \mathcal{A}$ a positive element, denote it by $0_{\mathcal{A}} \preceq a$, if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$, where $0_{\mathcal{A}}$ is zero element in $\mathcal{A}$ and $\sigma(a)$ is the spectrum of a . There is a natural partial ordering on $\mathcal{A}_h$ given by $a \preceq b$ if and only if $0_{\mathcal{A}} \preceq b - a$. From now on, $\mathcal{A}_+$ and $\mathcal{A}'$ will denote the set $\{a \in \mathcal{A} : 0_{\mathcal{A}} \preceq a\}$ and the set $\{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$, respectively.

Lemma 2.1. [8] Suppose that $\mathcal{A}$ is a unital C^* -algebra with a unit $1_{\mathcal{A}}$.

- (1) For any $x \in \mathcal{A}_+$, we have $x \preceq 1_{\mathcal{A}} \Leftrightarrow \|x\| \leq 1$.
- (2) If $a \in \mathcal{A}_+$ with $\|a\| < \frac{1}{2}$, then $1_{\mathcal{A}} - a$ is invertible and $\|a(1_{\mathcal{A}} - a)^{-1}\| < 1$.
- (3) Suppose that $a, b \in \mathcal{A}$ with $a, b \succeq 0_{\mathcal{A}}$ and $ab = ba$, then $ab \succeq 0_{\mathcal{A}}$.
- (4) Let $a \in \mathcal{A}'$, if $b, c \in \mathcal{A}$ with $b \succeq c \succeq 0_{\mathcal{A}}$, and $1_{\mathcal{A}} - a \in \mathcal{A}'_+$ is an invertible operator, then $(1_{\mathcal{A}} - a)^{-1}b \succeq (1_{\mathcal{A}} - a)^{-1}c$.

With the help of positive elements in $\mathcal{A}$, one can give the definition of C^* -algebra-valued metric space.

Definition 2.2. [6] Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow \mathcal{A}$ Satisfies:

- (1) $0_{\mathcal{A}} \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_{\mathcal{A}}$ iff $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a C^* - algebra-valued metric on X and $(X, \mathcal{A}, d)$ is said to be a C^* -algebra-valued metric space.

Definition 2.3. [5] Let X be a nonempty set. Suppose the mapping $S : X \times X \times X \rightarrow \mathcal{A}$ Satisfies:

- (S1) $0_{\mathcal{A}} \preceq S(x, y, z)$ for all $x, y, z \in X$;
- (S2) $S(x, y, z) = 0_{\mathcal{A}}$ if and only if $x = y = z$;
- (S3) $S(x, y, z) \preceq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

Then S is said to be C^* - algebra-valued S -metric on X and $(X, \mathcal{A}, S)$ is said to be a C^* -algebra-valued S -metric space.

Definition 2.4. [5] Suppose that $(X, \mathcal{A}, S)$ be a C^* -algebra-valued S -metric space. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X . If $S(x_n, x_n, x) \xrightarrow{\|\cdot\|_{\mathcal{A}}} 0_{\mathcal{A}}$ ($n \rightarrow \infty$) [i.e. $\|S(x_n, x_n, x)\| \rightarrow 0$ ($n \rightarrow \infty$)], then it is said that $\{x_n\}$ converges to x , and we denote it by $\lim_{n \rightarrow \infty} \{x_n\} = x$.

If for any $p \in \mathbb{N}$, $S(x_{n+p}, x_{n+p}, x_n) \xrightarrow{\|\cdot\|_{\mathcal{A}}} 0_{\mathcal{A}}$ ($n \rightarrow \infty$), then $\{x_n\}$ is called a Cauchy sequence in X .

If every Cauchy sequence is convergent in X , then $(X, \mathcal{A}, S)$ is called a complete C^* -algebra-valued S -metric space.

Example 2.5.

Let $X = \mathbb{R}$ and $\mathcal{A} = M_2(\mathbb{C})$, the set of bounded linear operators on a Hilbert space $\mathbb{C}^2$. Define $S : X \times X \times X \rightarrow \mathcal{A}$ by

$$S(x, y, z) = \begin{bmatrix} |x - z| + |y - z| & 0 \\ 0 & k|x - z| + |y - z| \end{bmatrix},$$

where $k > 0$ is a constant. Then $(X, \mathcal{A}, S)$ is a complete C^* -algebra-valued S -metric space.

Lemma 2.6. (1) If $\{b_n\}_{n=1}^\infty \subseteq \mathcal{A}$ and $\lim_{n \rightarrow \infty} b_n = 0_{\mathcal{A}}$, then for any $a \in \mathcal{A}$, $\lim_{n \rightarrow \infty} a^* b_n a = 0_{\mathcal{A}}$.

(2) If $a, b \in \mathcal{A}_h$ and $c \in \mathcal{A}'_+$, then $a \preceq b$ deduces $ca \preceq cb$, where $\mathcal{A}'_+ = \mathcal{A}_+ \cap \mathcal{A}'$.

Lemma 2.7. [5] Let $(X, \mathcal{A}, S)$ be a complete C^* -algebra-valued S -metric space. Then, $S(x, x, y) = S(y, y, x)$.

Proof. By (S3), we get,

$$\begin{aligned} S(x, x, y) &\preceq S(x, x, x) + S(x, x, x) + S(y, y, x) \\ (1) \quad &= S(y, y, x) \\ S(y, y, x) &\preceq S(y, y, y) + S(y, y, y) + S(x, x, y) \\ (2) \quad &= S(x, x, y). \end{aligned}$$

From (1) and (2), $S(x, x, y) = S(y, y, x)$. □

Lemma 2.8. [5] Let $\{x_n\}$ be a sequence in X . If $\{x_n\}$ converge to x and y , respectively, then $x = y$.

Proof. Let $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} x_n = y$. By (S3) and by lemma 2.7, we have

$$\begin{aligned} S(x, x, y) &\preceq S(x, x, x_n) + S(x, x, x_n) + S(y, y, x_n) \\ &= S(x_n, x_n, x) + S(x_n, x_n, x) + S(x_n, x_n, y) \\ &= 2S(x_n, x_n, x) + S(x_n, x_n, y) \\ &\longrightarrow 0_A \quad (n \rightarrow \infty). \end{aligned}$$

Therefore, $\|S(x, x, y)\| = 0 \Leftrightarrow x = y$. □

The following definition extends the concept of compatible mappings of Jungck [4], from metric spaces to C^* -algebra-valued S metric space.

Definition 2.9. Let $(X, \mathcal{A}, S)$ be a C^* -algebra-valued S metric space. A Pair $\{f, g\}$ is said to be compatible if and only if $S(fgx_n, fgx_n, gfx_n) \rightarrow 0_A$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$, for some $z \in X$.

Definition 2.10. A point x in X is called a coincidence point of f and g if and only if $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g . If f and g commute at all of their coincidence points, then they are weakly compatible but the converse is not true.

Theorem 2.11. If the mappings f and g on the C^* -algebra-valued S metric space $(X, \mathcal{A}, S)$ are compatible, then they are weakly compatible.

Proof. Let $fx = gx$ for some $x \in X$. It suffices to show that $fgx = gfx$, putting $x_n \equiv x$ for every $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n$. Since f and g are compatible, we get $\lim_{n \rightarrow \infty} S(fgx_n, fgx_n, gfx_n) \rightarrow 0_A$, as $(n \rightarrow \infty)$, that is, $\|S(fgx_n, fgx_n, gfx_n)\| \rightarrow 0$, $(n \rightarrow \infty)$. Hence $S(fgx_n, fgx_n, gfx_n) = 0_A$, which means $fgx = gfx$. □

Lemma 2.12. [12] Let f and g be a weakly compatible mappings of a set X . If f and g have a unique point of coincidence, then it is the unique common fixed point of f and g .

3. MAIN RESULT

Theorem 3.1. Let $(X, \mathcal{A}, S)$ be a complete C^* -algebra-valued S -metric space. Suppose that two mappings $f, g : X \rightarrow X$ satisfy

$$(3) \quad S(fx, fx, gy) \preceq a^* S(x, x, y) a$$

where $a \in \mathcal{A}$ with $\|a\| < 1$, for all $x, y \in X$. Then f and g have a unique common fixed point in X .

Proof. Without loss of generality, one can suppose that $a \neq 0_A$. Let $x_0 \in X$. Construct a sequences $\{x_n\} \subseteq X$ such that $x_{2n+1} = fx_{2n}$; $x_{2n+2} = gx_{2n+1}$

By using (3), we have

$$\begin{aligned}
 S(x_{2n+2}, x_{2n+2}, x_{2n+1}) &= S(gx_{2n+1}, gx_{2n+1}, fx_{2n}) \\
 &\preceq a^* S(x_{2n+1}, x_{2n+1}, x_{2n}) a \\
 &\preceq (a^*)^2 S(x_{2n}, x_{2n}, x_{n-1}) a^2 \\
 &\vdots \\
 &\preceq (a^*)^{2n+1} S(x_1, x_1, x_0) (a)^{2n+1},
 \end{aligned}$$

where we use the property, if $b, c \in \mathcal{A}_h$, then $b \preceq c$ implies $a^*ba \preceq a^*ca$ [8].

Similarly

$$S(x_{2n+1}, x_{2n+1}, x_{2n}) \preceq (a^*)^{2n} S(x_1, x_1, x_0) (a)^{2n}.$$

For any $n \in N$ and by Lemma 2.7,

$$S(x_{n+1}, x_{n+1}, x_n) \preceq (a^*)^n S(x_1, x_1, x_0) (a)^n.$$

Let $S(x_1, x_1, x_0) = c^2$ for some $c \in A'_+$, for any $p \in N$, we have

$$\begin{aligned}
 S(x_{n+p}, x_{n+p}, x_n) &\preceq S(x_{n+p}, x_{n+p}, x_{n+p-1}) + S(x_{n+p}, x_{n+p}, x_{n+p-1}) + S(x_n, x_n, x_{n+p-1}) \\
 &= 2S(x_{n+p}, x_{n+p}, x_{n+p-1}) + S(x_n, x_n, x_{n+p-1}) \\
 &\preceq 2S(x_{n+p}, x_{n+p}, x_{n+p-1}) + S(x_{n+p-1}, x_{n+p-1}, x_n) \\
 &\preceq 2(a^*)^{n+p-1} S(x_1, x_1, x_0) (a)^{n+p-1} + 2(a^*)^{n+p-2} S(x_1, x_1, x_0) (a)^{n+p-2} \\
 &\quad \dots + 2(a^*)^n S(x_1, x_1, x_0) (a)^n \\
 &\preceq 2 \sum_{k=n}^{n+p-1} (a^*)^k S(x_1, x_1, x_0) a^k \\
 &= 2 \sum_{k=n}^{n+p-1} (ca^k)^* (ca^k) \\
 &\preceq 2 \sum_{k=n}^{n+p-1} |ca^k|^2 \\
 &\preceq 2 \sum_{k=n}^{n+p-1} \|ca^k\|^2 1_A \\
 &\preceq 2\|c\|^2 1_A \sum_{k=n}^{n+p-1} \|a\|^{2k} \\
 &\preceq 2\|c\|^2 1_A \frac{\|a\|^{2m}}{1 - \|a\|^2} \\
 &\longrightarrow 0_{\mathcal{A}} \quad (m \rightarrow \infty).
 \end{aligned}$$

From Definition 2.4, we get that $\{x_n\}$ is a Cauchy sequence with respect to $\mathcal{A}$. The completion of X implies that there exist $x \in X$ such that $\lim_{n \rightarrow \infty} x_n = x$.

Using the condition (S3) and by Lemma 2.7,

$$\begin{aligned}
 S(x, x, gx) &\preceq S(x, x, x_{2n+1}) + S(x, x, x_{2n+1}) + S(x_{2n+1}, x_{2n+1}, gx) \\
 &= 2S(x, x, x_{2n+1}) + S(x_{2n+1}, x_{2n+1}, gx) \\
 &= 2S(x, x, x_{2n+1}) + S(fx_{2n}, fx_{2n}, gx) \\
 &\preceq 2S(x, x, x_{2n+1}) + a^*S(x_n, x_n, x)a \\
 &\longrightarrow 0_A \quad (n \rightarrow \infty).
 \end{aligned}$$

Hence, $gx = x$. i.e.

$$\begin{aligned}
 0_A &\preceq S(fx, fx, x) \\
 &= S(fx, fx, gx) \\
 &\preceq a^*S(x, x, x)a \\
 &= 0_A.
 \end{aligned}$$

we have $S(fx, fx, x) = 0_A$, which implies $fx = x$.

Uniqueness:

Suppose that $y(\neq)x \in X$, is another fixed point of f . Then,

$$\begin{aligned}
 0_A &\preceq S(x, x, y) \\
 &= S(fx, fx, fy) \\
 &\preceq a^*S(x, x, y)a
 \end{aligned}$$

which together with $\|a\| < 1$ yields that

$$\begin{aligned}
 0 \leq \|S(x, x, y)\| &\leq \|a^*S(fx, fx, fy)a\| \\
 &\leq \|a^*\| \|S(x, x, y)\| \|a\| \\
 &\leq \|a\|^2 \|S(x, x, y)\| \\
 &< \|S(x, x, y)\|.
 \end{aligned}$$

Thus, $\|S(x, x, y)\| = 0$ and $S(x, x, y) = 0_A$, which gives $x = y$. Hence f and g have a unique common fixed point in X . $\square$

Corollary 3.2. Let $(X, \mathcal{A}, S)$ be a complete C^* -algebra-valued S -metric space. Suppose that the mapping $f : X \rightarrow X$ satisfies

$$(4) \quad \|S(fx, fx, fy)\| \leq \|a\| \|S(x, x, y)\|$$

where $a \in \mathcal{A}'_+$ with $\|a\| < 1$, for all $x, y \in X$. Then there exist a unique fixed point in X .

Theorem 3.3. Let $(X, \mathcal{A}, S)$ be a complete C^* -algebra-valued S -metric space. Suppose that two mappings $f, g : X \rightarrow X$ satisfy

$$(5) \quad S(fx, fx, gy) \preceq a^*S(gx, gx, gy)a$$

where $a \in \mathcal{A}$ with $\|a\| < 1$, for all $x, y \in X$. If $R(T)$ is contained in $R(S)$ and $R(S)$ is complete in X , then f and g have a unique point of coincidence in X . Furthermore, if f and g are weakly compatible, f and g have a unique common fixed point in X .

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $gx_1 = fx_0$, which can be done since $R(T) \subseteq R(S)$. Let $x_2 \in X$ such that $gx_2 = fx_1$. Repeating the process, we get a sequences $\{x_n\} \subseteq X$ satisfying $gx_n = fx_{n-1}$. Then,

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx_n) &\preceq S(fx_n, fx_n, fx_{n-1}) \\ &\preceq a^* S(gx_n, gx_n, gx_{n-1})a \\ &\vdots \\ &\preceq (a^*)^n S(gx_1, gx_1, gx_0)(a)^n, \end{aligned}$$

which shows that $\{gx_n\}$ is a Cauchy sequence in $R(S)$. Since $R(S)$ is complete in X , there exists $w \in X$ such that $\lim_{n \rightarrow \infty} gx_n = gw$.

$$\begin{aligned} S(gx_n, gx_n, fw) &= S(fx_{n-1}, fx_{n-1}, fw) \\ &\preceq a^* S(gx_{n-1}, gx_{n-1}, gw)a \end{aligned}$$

From $\lim_{n \rightarrow \infty} gx_n = gw$ and by Lemma 2.6, we have $a^* S(gx_{n-1}, gx_{n-1}, gw)a \rightarrow 0_A$ then $\lim_{n \rightarrow \infty} gx_n = fw$. It follows from Lemma 2.8, $gw = fw$. If there is a point w in X such that $fw = gw$, equation (5) shows

$$\begin{aligned} S(gw, gw, gz) &= a^* S(fw, fw, fz)a \\ &\preceq a^* S(gw, gw, gz)a \end{aligned}$$

so we have $gw = gz$. Hence, f and g have a unique point of coincidence in X . It follows from Lemma 2.12, that f and g have a unique common fixed point in X . $\square$

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